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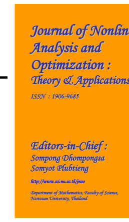

FIXED POINT THEOREMS FOR FUNDAMENTALLY NONEXPANSIVE MAPPINGS IN CAT(k) SPACES

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ABSTRACT. In this paper, we obtain fixed point theorems and $\Delta-$convergence theorems for fundamentally nonexpansive mappings on CAT(k) spaces with $k > 0$. Our results extend and improve some results of Salahifard et al. [3], and many others.

KEYWORDS : CAT(k) space; Fixed point; $\Delta-$convergence; Generalized nonexpansive mapping.
AMS Subject Classification: 47H09; 47H10.

1. INTRODUCTION

A mapping T on a subset E of Banach space X is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in E$. We denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in E \ : \ Tx = x\}$. A mapping T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - z\| \leq \|x - z\|$ for all $x \in E$ and $z \in F(T)$. In 2008, Suzuki [1] introduced condition (C) as follows:
A mapping T on a subset E of Banach space X is said to satisfy the condition (C) (or Suzuki's generalized nonexpansive) on E if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \text{ implies } \|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in E$. Moreover, he obtained some interesting fixed point theorems and convergence theorems for such mappings. In 2012, Nanjaras et al. [2] extend Suzuki's results on fixed point theorems and $\triangle-$convergence theorems for such mappings in CAT(0) spaces. In 2013, Salahifard et al. [3] introduced fundamentally nonexpansive mapping which generalizes the Suzuki's generalized nonexpansive mapping and proved some fixed point theorems for this kind of mappings in CAT(0) spaces.

Fixed point theory in CAT(k) spaces was first studied by Kirk [4, 5]. His works were followed by a series of new works by many authors, mainly focusing on CAT(0)

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spaces (see e.g., [2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]). Since any CAT(k) space is a CAT(k') space for $k' \geq k$, all results for CAT(0) spaces immediately apply to any CAT(k) space with $k \leq 0$. However, there are only a few articles that contain fixed point results in the setting of CAT(k) spaces with $k > 0$. In this paper, we extend the results of Salahifard et al. [3] to the general setting of CAT(k) spaces with $k > 0$.

2. PRELIMINARIES AND NOTATIONS

Let (X, ρ) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $\rho\left(c(t), c(t')\right) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $\rho(x, y) = l$. The image $c\left([0, l]\right)$ of c is called a *geodesic segment* joining x and y. Write $c(\alpha 0 + (1-\alpha)l) = \alpha x \oplus (1-\alpha)y$ for $\alpha \in (0, 1)$. When it is unique this geodesic segment is denoted by $[x, y]$. This means that $z \in [x, y]$ if and only if there exists $\alpha \in [0, 1]$ such that

$$\rho(x, z) = (1-\alpha)\rho(x, y) \text{ and } \rho(y, z) = \alpha\rho(x, y).$$

In this case, we write $z = \alpha x \oplus (1-\alpha)y$. Let D be a positive constant. A metric space (X, ρ) is said to be a *geodesic space* (*D-geodesic space*) if every two points of X (every two points of distance smaller than D) are joined by a geodesic, and X is said to be *uniquely geodesic* (*D-uniquely geodesic*) if there is exactly one geodesic joining x and y for each $x, y \in X$ (for $x, y \in X$ with $\rho(x, y) < D$). A subset E of X is said to be *convex* if E includes every geodesic segment joining any two of its points. If this condition holds for any two points in E with distance smaller than D, E is said to be *D-convex*. The set E is said to be *bounded* if

$$\operatorname{diam}(E) := \sup\{\rho(x, y) : x, y \in E\} < \infty.$$

Now we introduce the model spaces M_k^n, for more details on these spaces the reader is referred to [21]. Let $n \in \mathbb{N}$. We denote by $\mathbb{E}^n$ the metric space $\mathbb{R}^n$ endowed with the usual Euclidean distance. We denote by $(\cdot|\cdot)$ the Euclidean scalar product in $\mathbb{R}^n$, that is,

$$(x|y) = x_1y_1 + ... + x_ny_n \text{ where } x = (x_1, ..., x_n),\ y = (y_1, ..., y_n).$$

Let $\mathbb{S}^n$ denote the *$n-$dimensional sphere* defined by

$$\mathbb{S}^n = \{x = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : (x|x) = 1\},$$

with metric $d_{\mathbb{S}^n}(x, y) = \arccos(x|y),\ x, y \in \mathbb{S}^n$.

Let $\mathbb{E}^{n,1}$ denote the vector space $\mathbb{R}^{n+1}$ endowed with the symmetric bilinear form which associates to vectors $u = (u_1, ..., u_{n+1})$ and $v = (v_1, ..., v_{n+1})$, the real number $\langle u|v\rangle$ is defined by

$$\langle u|v\rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_iv_i.$$

Let $\mathbb{H}^n$ denote the *hyperbolic $n-$space* defined by

$$\mathbb{H}^n = \{u = (u_1, ..., u_{n+1}) \in \mathbb{E}^{n,1} : \langle u|u\rangle = -1, u_{n+1} > 0\},$$

with metric $d_{\mathbb{H}^n}$ such that

$$\cosh d_{\mathbb{H}^n}(x, y) = -\langle x|y\rangle,\ x, y \in \mathbb{H}^n.$$

Definition 2.1. Given $k \in \mathbb{R}$, we denote by M_k^n the following metric spaces:
(i) if $k = 0$ then M_0^n is the Euclidean space $\mathbb{E}^n$;
(ii) if $k > 0$ then M_k^n is obtained from the spherical space $\mathbb{S}^n$ by multiplying the distance function by the constant $1/\sqrt{k}$;
(iii) if $k < 0$ then M_k^n is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the distance.

A *geodesic triangle* $\triangle(x, y, z)$ in a geodesic space (X, ρ) consists of three points x, y, z in X (the *vertices* of $\triangle$) and three geodesic segments between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for a geodesic triangle $\triangle(x, y, z)$ in (X, ρ) is a triangle $\overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$ in M_k^2 such that

$$\rho(x, y) = d_{M_k^2}(\overline{x}, \overline{y}),\ \rho(y, z) = d_{M_k^2}(\overline{y}, \overline{z}) \text{ and } \rho(z, x) = d_{M_k^2}(\overline{z}, \overline{x}).$$

If $k \leq 0$ then such a comparison triangle always exists in M_k^2. If $k > 0$ then such a triangle exists whenever $\rho(x, y) + \rho(y, z) + \rho(z, x) < 2D_k$, where $D_k = \pi/\sqrt{k}$. A point $\overline{p} \in [\overline{x}, \overline{y}]$ is called a *comparison point* for $p \in [x, y]$ if $\rho(x, p) = d_{M_k^2}(\overline{x}, \overline{p})$.

A geodesic triangle $\triangle(x, y, z)$ in X is said to satisfy the CAT(k) inequality if for any $p, q \in \triangle(x, y, z)$ and for their comparison points $\overline{p}, \overline{q} \in \overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$, one has

$$\rho(p, q) \leq d_{M_k^2}(\overline{p}, \overline{q}).$$

Definition 2.2. If $k \leq 0$, then X is called a CAT(k) space if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(k) inequality.

If $k > 0$, then X is called a CAT(k) space if X is D_k-geodesic and any geodesic triangle $\triangle(x, y, z)$ in X with $\rho(x, y) + \rho(y, z) + \rho(z, x) < 2D_k$ satisfies the CAT(k) inequality.

In a CAT(0) space (X, ρ), if $x, y, z \in X$ then the CAT(0) inequality implies

$$\rho^2\left(x, \frac{1}{2}y \oplus \frac{1}{2}z\right) \leq \frac{1}{2}\rho^2(x, y) + \frac{1}{2}\rho^2(x, z) - \frac{1}{4}\rho^2(y, z). \tag{CN}$$

This is the *(CN) inequality* of Bruhat and Tits [22]. This inequality is extended by Dhompongsa and Panyanak [19] as

$$\rho^2(x, (1-\alpha)y \oplus \alpha z) \leq (1-\alpha)\rho^2(x, y) + \alpha\rho^2(x, z) - \alpha(1-\alpha)\rho^2(y, z) \tag{CN*}$$

for all $\alpha \in [0, 1]$ and $x, y, z \in X$. In fact, if X is a geodesic space then the following statements are equivalent:
(i) X is a CAT(0) space;
(ii) X satisfies (CN);
(iii) X satisfies (CN*).

Let $R \in (0, 2]$. Recall that a geodesic space (X, ρ) is said to be $R-$convex for R (see [23]) if for any three points $x, y, z \in X$, we have

$$\rho^2(x, (1-\alpha)y \oplus \alpha z) \leq (1-\alpha)\rho^2(x, y) + \alpha\rho^2(x, z) - \frac{R}{2}\alpha(1-\alpha)\rho^2(y, z). \tag{1}$$

It follows from (CN*) that a geodesic space (X, ρ) is a CAT(0) space if and only if (X, ρ) is $R-$convex for $R = 2$. The following lemma is a consequence of Proposition 3.1 in [23].

Lemma 2.3. *Let $k > 0$ and (X, ρ) be a CAT(k) space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0, \pi/2)$. Then (X, ρ) is $R-$convex for $R = (\pi - 2\varepsilon)\tan(\varepsilon)$.*

The following lemma is also needed.

Lemma 2.4. ([21, p.176]) *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0, \pi/2)$. Then*

$$\rho(x, \alpha y \oplus (1-\alpha)z) \leq \alpha\rho(x, y) + (1-\alpha)\rho(x, z).$$

for all $x, y, z \in X$ and $\alpha \in [0, 1]$.

Let $\{x_n\}$ be a bounded sequence in a CAT(k) space (X, ρ). For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \longrightarrow \infty} \rho(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from Proposition 4.1 of [11] that in a CAT(k) space X with diameter smaller than $\frac{\pi}{2\sqrt{k}}$, $A(\{x_n\})$ consists of exactly one point. We now give the concept of $\triangle-$convergence and collect some of its basic properties.

Definition 2.5. ([9], [24]) A sequence $\{x_n\}$ in X is said to $\Delta-$*converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and call x the $\Delta-$limit of $\{x_n\}$.

Lemma 2.6. *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0, \pi/2)$. Then the following statements hold:*

(i) [11, Corollary 4.4] *every sequence in X has a $\Delta-$convergence subsequence;*

(ii) [11, Proposition 4.5] *if $\{x_n\} \subset X$ and $\Delta-\lim_n x_n = x$, then $x \in \cap_{k=1}^{\infty} \overline{conv}\{x_k, x_{k+1}, ...\}$, where $\overline{conv}(A) = \cap\{B : B \supseteq A$ and B is closed and convex$\}$.*

By the uniqueness of asymptotic centers, we can obtain the following lemma (cf. [19, Lemma 2.8]).

Lemma 2.7. *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0, \pi/2)$. If $\{x_n\}$ is a sequence in X with $A(\{x_n\}) = \{x\}$ and let $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{\rho(x_n, u)\}$ converges, then $x = u$.*

Definition 2.8. [3] Let E be a nonempty subset of a CAT(k) space (X, ρ). A mapping $T : E \longrightarrow E$ is said to be *fundamentally nonexpansive* if

$$\rho(T^2(x), T(y)) \leq \rho(T(x), y),$$

for all $x, y \in E$.

Proposition 2.9. [3] *Every mapping which satisfies condition (C) is fundamentally nonexpansive, but the inverse is not true.*

3. MAIN RESULTS

Let X be a uniformly convex Banach space, K be a nonempty closed convex subset of X.

In this section, we prove our main theorems.

Lemma 3.1. *Let E be a nonempty subset of a CAT(k) space (X, ρ), and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping. Then*

$$\rho(x, T(y)) \leq 3\rho(T(x), x) + \rho(x, y),$$

for all $x, y \in E$.

Proof. Since T is fundamentally nonexpansive, we have

$$\begin{aligned}\rho(x,T(y)) &\leq \rho(x,T(x))+\rho(T(x),T^2(x))+\rho(T^2(x),T(y))\\ &\leq 2\rho(x,T(x))+\rho(T(x),y)\\ &\leq 3\rho(x,T(x))+\rho(x,y).\end{aligned}$$

This completes the proof. □

The following lemma is a consequence of Lemma 3.4 of [16].

Lemma 3.2. *Let $k>0$ and (X,ρ) be a CAT(k) space such that* $\operatorname{diam}(X)\leq\frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon\in(0,\pi/2)$ and let $\{z_n\}$ and $\{w_n\}$ be two sequences in X. Let $\{\beta_n\}$ be a sequence in $[0,1]$ such that $0<\liminf_n\beta_n\leq\limsup_n\beta_n<1$. Suppose that $z_{n+1}=\beta_n z_n+(1-\beta_n)w_n$ for all $n\in\mathbb{N}$ and $\limsup_n(\rho(w_{n+1},w_n)-\rho(z_{n+1},z_n))\leq 0$. Then $\lim_n\rho(w_n,z_n)=0$.*

Lemma 3.3. *Let $k>0$ and (X,ρ) be a complete CAT(k) space with* $\operatorname{diam}(X)\leq\frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon\in(0,\pi/2)$. Let E be a nonempty closed convex subset of X, and $T:E\longrightarrow E$ be a fundamentally nonexpansive mapping. Define a sequence $\{x_n\}$ by $x_1\in E$ and $x_{n+1}=\alpha_nT(x_n)\oplus(1-\alpha_n)x_n$ for all $n\in\mathbb{N}$ where $\{\alpha_n\}\subset[0,1]$ such that $0<\liminf_n\alpha_n\leq\limsup_n\alpha_n<1$. Then $\lim_{n\longrightarrow\infty}\rho(T(x_n),x_n)=0$.*

Proof. Since T is fundamentally nonexpansive, we have

$$\rho(T(x_{n+1}),T(x_n))=\alpha_n\rho(T^2(x_n),T(x_n))\leq\alpha_n\rho(T(x_n),x_n)=\rho(x_{n+1},x_n)$$

for all $n\in\mathbb{N}$ and hence

$$\rho(T(x_{n+1}),T(x_n))\leq\rho(x_{n+1},x_n).$$

This implies that

$$\lim\sup_{n\longrightarrow\infty}(\rho(Tx_{n+1},Tx_n)-\rho(x_{n+1},x_n))\leq 0.$$

So, by Lemma 3.2, we have

$$\lim_{n\longrightarrow\infty}\rho(T(x_n),x_n)=0.$$

This completes the proof. □

Theorem 3.1. *Let $k>0$ and (X,ρ) be a complete CAT(k) space with* $\operatorname{diam}(X)\leq\frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon\in(0,\pi/2)$. Let E be a nonempty closed convex subset of X, and $T:E\longrightarrow E$ be a fundamentally nonexpansive mapping. Then $F(T)$ is nonempty.*

Proof. Define a sequence $\{x_n\}$ by $x_1\in E$ and $x_{n+1}=\frac{1}{2}Tx_n\oplus\frac{1}{2}x_n$ for all $n\in\mathbb{N}$. Suppose that $A(\{x_n\})=\{z\}$. Then by Lemma 2.6, $z\in E$. By Lemma 3.3, we have $\lim_n\rho(T(x_n),x_n)=0$ and by Lemma 3.1,

$$\rho(x_n,T(z))\leq 3\rho(T(x_n),x_n)+\rho(x_n,z).$$

Taking the limit superior on both sides in the above inequality, we obtain

$$\limsup_{n\longrightarrow\infty}d(x_n,T(z))\leq\limsup_{n\longrightarrow\infty}d(x_n,z).$$

Since $A(\{x_n\})=\{z\}$, it must be the case that $z=T(z)$. □

The following corollary shows that how we derive a result for CAT(0) spaces from Theorem 3.1.

Corollary 3.4. *Let (X, ρ) be a complete CAT(0) space, E be a nonempty bounded closed convex subset of X, and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping. Then $F(T)$ is nonempty.*

Proof. It well known that every convex subset of a CAT(0) space, equipped with the included metric, is a CAT(0) space (cf. [21]). Then (E, ρ) is a CAT(0) space and hence it is a CAT(k) space for all $k > 0$. Notice also that E is $R-$convex for $R = 2$. Since E is bounded, we can choose $\varepsilon \in (0, \pi/2)$ and $k > 0$ so that $\text{diam}(E) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$. The conclusion follows from Theorem 3.1. □

Theorem 3.2. *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with $\text{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$. Let E be a nonempty closed convex subset of X, and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping and $F(T) \neq \emptyset$. Then $F(T)$ is closed and convex.*

Proof. We can prove by following the steps of the Theorem 4.1 of [3]. □

Theorem 3.3. *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with $\text{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$. Let E be a nonempty closed convex subset of X, and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping. Let $\{x_n\}$ be a sequence in E with $\lim_n \rho(T(x_n), x_n) = 0$ and $\Delta - \lim_n x_n = z$. Then $z \in E$ and $z = T(z)$.*

Proof. Since $\Delta - \lim_n x_n = z$, by Lemma 2.6, we have $z \in E$. It follows from Lemma 3.1 that

$$\rho(x_n, T(z)) \leq 3\rho(T(x_n), x_n) + \rho(x_n, z).$$

Taking the limit superior on both sides in the above inequality, we obtain

$$\limsup_{n\longrightarrow\infty} \rho(x_n, T(z)) \leq \limsup_{n\longrightarrow\infty} \rho(x_n, z).$$

By the uniqueness of asymptotic center, we obtain $z = T(z)$. □

The following corollary shows that how we derive a result for CAT(0) spaces from Theorem 3.3.

Corollary 3.5. *Let (X, ρ) be a complete CAT(0) space, E be a nonempty bounded closed convex subset of X, and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping. Let $\{x_n\}$ be a sequence in E with $\lim_n \rho(T(x_n), x_n) = 0$ and $\Delta - \lim_n x_n = z$. Then $z \in E$ and $z = T(z)$.*

Proof. It well known that every convex subset of a CAT(0) space, equipped with the included metric, is a CAT(0) space (cf. [21]). Then (E, ρ) is a CAT(0) space and hence it is a CAT(k) space for all $k > 0$. Notice also that E is $R-$convex for $R = 2$. Since E is bounded, we can choose $\varepsilon \in (0, \pi/2)$ and $k > 0$ so that $\text{diam}(E) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$. The conclusion follows from Theorem 3.3. □

Lemma 3.6. *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with $\text{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$. Let E be a nonempty closed convex subset of X, and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping. Suppose $\{x_n\}$ is a sequence in E such that $\lim_n \rho(T(x_n), x_n) = 0$ and $\{\rho(x_n, v)\}$ converges for all $v \in F(T)$, then $\omega_w(x_n) \subset F(T)$. Here $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

Proof. Let $u \in \omega_w(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.6, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in E$. By Theorem 3.3, $v \in F(T)$. By Lemma 2.7, $u = v$. This shows that $\omega_w(x_n) \subset F(T)$. Next, we show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subset F(T)$, we have $\{\rho(x_n, u)\}$ converges. Again, by Lemma 2.7, $x = u$. This completes the proof. □

Theorem 3.4. *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with* $\text{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0, \pi/2)$. Let E be a nonempty closed convex subset of X, and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping. Define a sequence $\{x_n\}$ by $x_1 \in E$ and $x_{n+1} = \alpha_n T(x_n) \oplus (1-\alpha_n)x_n$ for all $n \in \mathbb{N}$ where $\{\alpha_n\} \subset [0, 1]$ such that $0 < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$. Then $\{x_n\}$ $\Delta-$ converges to a fixed point of T.*

Proof. By Lemma 3.3, we have $\lim_n \rho(T(x_n), x_n) = 0$. By Theorem 3.1, $F(T)$ is nonempty. Given $z \in F(T)$, by Lemma 3.1 we have

$$\rho(T(x_n), z) \leq 3\rho(T(z), z) + \rho(x_n, z) \leq \rho(x_n, z).$$

This implies that

$$\begin{aligned}\rho(x_{n+1}, z) &= \rho(\alpha_n T(x_n) \oplus (1-\alpha_n)x_n, z)\\ &\leq \alpha_n \rho(T(x_n), z) + (1-\alpha_n)\rho(x_n, z)\\ &\leq \rho(x_n, z).\end{aligned}$$

That is

$$\rho(x_{n+1}, z) \leq \rho(x_n, z). \tag{2}$$

Thus $\{\rho(x_n, z)\}$ is bounded and decreasing for all $z \in F(T)$, and so it is convergent. By Lemma 3.6, $\omega_w(x_n)$ consists of exactly one point and is contained in $F(T)$. This show that $\{x_n\}$ $\Delta-$ converges to an element of $F(T)$. □

Theorem 3.5. *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with* $\text{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0, \pi/2)$. Let E be a nonempty compact convex subset of X, and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping. Define a sequence $\{x_n\}$ by $x_1 \in E$ and $x_{n+1} = \alpha_n T(x_n) \oplus (1-\alpha_n)x_n$ for all $n \in \mathbb{N}$ where $\{\alpha_n\} \subset [0, 1]$ such that $0 < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 3.3, we have $\lim_{n\longrightarrow\infty} \rho(T(x_n), x_n) = 0$. Since E is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_k x_{n_k} = z$ for some $z \in E$. It follows from Lemma 3.1 that

$$\rho(x_{n_k}, T(z)) \leq 3\rho(T(x_{n_k}), x_{n_k}) + \rho(x_{n_k}, z) \text{ for all } k \in \mathbb{N}.$$

Letting $k \longrightarrow \infty$, we have $\{x_{n_k}\}$ converges to $T(z)$. This implies that $z = T(z)$, that is $z \in F(T)$. Following the proof of Theorem 3.4, we obtain $\lim_n \rho(x_n, z)$ exists for all $z \in F(T)$, it must be the case that $\lim_n \rho(x_n, z) = 0$. Therefore we obtain the desired result. □

Recall that a mapping $T : E \longrightarrow E$ is said to satisfy condition (I) if there exists a nondecreasing function $f : [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $\rho(x, T(x)) \geq f(\rho(x, F(T)))$ for all $x \in E$, where $\rho(x, F(T)) = \inf_{z\in F(T)} \rho(x, z)$.

Theorem 3.6. *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with* $\mathrm{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0, \pi/2)$. Let E be a nonempty closed convex subset of X, and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping satisfies the condition (I). Define a sequence $\{x_n\}$ by $x_1 \in E$ and $x_{n+1} = \alpha_n T(x_n) \oplus (1-\alpha_n)x_n$ for all $n \in \mathbb{N}$ where $\{\alpha_n\} \subset [0,1]$ such that $0 < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By condition (I), we have

$$f\left(\rho\left(x_n, F(T)\right)\right) \leq \rho(x_n, T(x_n)) \text{ for all } n \in \mathbb{N}.$$

It follows from Lemma 3.3 that

$$\lim_{n \longrightarrow \infty} f\left(\rho\left(x_n, F(T)\right)\right) = 0.$$

This implies that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\rho(x_{n_k}, z_k) \leq \frac{1}{2^k} \text{ for all } k \in \mathbb{N}. \tag{3}$$

Where $\{z_k\} \subset F(T)$. By (2), we have

$$\rho(x_{n_{k+1}}, z_k) \leq \rho(x_{n_k}, z_k) \leq \frac{1}{2^k}.$$

Hence

$$\begin{aligned} \rho(z_{k+1}, z_k) &\leq \rho(z_{k+1}, x_{n_{k+1}}) + \rho(x_{n_{k+1}}, z_k) \\ &\leq \frac{1}{2^{(k+1)}} + \frac{1}{2^k} < \frac{1}{2^{k-1}} \longrightarrow 0 \text{ as } k \longrightarrow \infty. \end{aligned}$$

This shows that $\{z_k\}$ is a Cauchy in $F(T)$. Since $F(T)$ is closed in X, there exists a point z in $F(T)$ such that $\lim_{k \longrightarrow \infty} z_k = z$. It follows from (3) that $\lim_{k \longrightarrow \infty} x_k = z$. Since $\lim_{n \longrightarrow \infty} \rho(x_n, z)$ exists, it must be the case that $\lim_{n \longrightarrow \infty} \rho(x_n, z) = 0$. Therefore we obtain the desired result. □

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