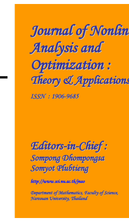
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol.7, No.1, (2016), 55-74
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



COMMON SOLUTIONS TO SOME SYSTEMS OF VECTOR EQUILIBRIUM PROBLEMS AND COMMON FIXED POINT PROBLEMS IN BANACH SPACE

MOHAMMAD FARID*AND K.R. KAZMI**

*Department of Mathematics, Unaizah College of Engineering, Al-Qassim University 51911, Saudi Arabia

**Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India

ABSTRACT. In this paper, we study some properties of set of solutions of a new generalized mixed vector equilibrium problem in Banach space. Further, we introduce an iterative method based on hybrid method and convex approximation method for finding a common element to the set of solutions of a system of unrelated generalized mixed vector equilibrium problems and the set of solutions of common fixed point problems for the two families of generalized asymptotically quasi ϕ-nonexpansive mappings in Banach space. Furthermore, we obtain a strong convergence theorem for the sequences generated by the proposed iterative scheme. Finally, we derive some consequences from our main result. The results presented in this paper extended and unify many of the previously known results in this area.

KEYWORDS : System of unrelated generalized mixed vector equilibrium problem; Common fixed-point problem; Generalized asymptotically quasi ϕ-nonexpansive mappings; Iterative schemes.
AMS Subject Classification: 49J30, 47H10, 47H17, 90C99.

1. INTRODUCTION

Throughout the paper unless otherwise stated, let E be a real Banach space with its dual space E^* and let $\langle .,. \rangle$ denote the duality pairing between E and E^* and $\|.\|$ denote the norm of E as well as of E^*. Let C be a nonempty, closed and convex subset of E and let 2^E denote the set of all nonempty subsets of E. Let Y be an ordered Banach space and let P be a pointed, proper, closed and convex cone of Y with $\text{int}P \neq \emptyset$.

In 1994, Blum and Oettli [3] introduced and studied the following equilibrium problem (in short, EP): Find $x \in C$ such that

$$F(x,y) \geq 0, \ \forall y \in C, \tag{1.1}$$

* *Corresponding author.*
Email address : mohdfrd55@gmail.com(Mohammad Farid).
Article history : Received December 18, 2015 Accepted March 20, 2016.

where $F : C \times C \longrightarrow \mathbb{R}$ is a bifunction.

The EP(1.1) includes variational inequality problems, optimization problems, Nash equilibrium problems, saddle point problems, fixed point problems, complementary problems as special cases. In other words, EP(1.1) is a unified model for several problems arising in science, engineering, optimization, economics, etc.

In the last two decades, EP(1.1) has been generalized and extensively studied in many directions due to its importance; See for example [9, 11, 17] and references therein, for the literature on the existence of solution of the various generalizations of EP(1.1). Some iterative methods have been studied for solving various classes of equilibrium problems, see for example [4, 7, 12, 13, 14, 18, 25, 26, 27] and references therein.

In this paper, we introduce and study the following generalized mixed vector equilibrium problem (in short, GMVEP). Let $F : C \times C \longrightarrow Y$ and $\psi : C \times C \longrightarrow Y$ be nonlinear bimappings and let $A : C \longrightarrow B(E, Y)$, where $B(E, Y)$ is a Banach space of all continuous linear operators from E into Y, be a nonlinear mapping, then GMVEP is to find $x^* \in C$ such that

$$F(x^*, x) + \langle x - x^*, Ax^* \rangle + \psi(x, x^*) - \psi(x^*, x^*) \in P, \ \forall x \in C. \tag{1.2}$$

The solution set of GMVEP(1.2) is denoted by Sol(GMVEP(1.2)).

Example 1.1. Let $E = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y \rangle = xy, \ \forall x, y \in \mathbb{R}$. Let $Y = \mathbb{R}$, then $P = [0, +\infty)$ and let $C = [0, 2]$. Let F and ψ be defined by $F(x, y) = x^2 - 3y - xy$ and $\psi(x, y) = x^2 - y^2 + 2y, \ \forall x, y \in C$, and $A(x) = x + 2, \ \forall x \in C$, respectively, then it is easy to observe that $\mathrm{Sol(GMVEP(1.2))} = [1, 2] \neq \emptyset$.

If $\psi = 0$ and $A = 0$, then GMVEP(1.2) reduces to the strong vector equilibrium problem (in short, SVEP): Find $x^* \in C$ such that

$$F(x^*, x) \in P, \ \forall x \in C, \tag{1.3}$$

which has been studied by Kazmi and Khan [15]. It is well known that the vector equilibrium problem provides a unified model of several problems, for example, vector optimization, vector variational inequality, vector complementary problem and vector saddle point problem [11, 17]. In recent years, the vector equilibrium problem has been intensively studied by many authors, see for example [9, 11, 15, 16, 17, 23] and the references therein.

If $Y = \mathbb{R}$, then $P = [0, +\infty)$ and hence GMVEP(1.2) reduces to the following generalized mixed equilibrium problem (in short, GMEP): Find $x \in C$ such that

$$F(x^*, x) + \langle x - x^*, Ax^* \rangle + \psi(x, x^*) - \psi(x^*, x^*) \geq 0, \ \forall x \in C, \tag{1.4}$$

where $\psi : C \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function. The solution set of GMEP(1.4) is denoted by $\mathrm{Sol(GMEP(1.4))}$. The GMEP(1.4) with $\psi(x, x^*) = \psi(x), \ \forall x, x^* \in C$ has been studied by Ceng and Yao [4] in Hilbert space.

If $Y = \mathbb{R}$ and $F = 0$, then $P = [0, +\infty)$ and hence GMVEP(1.2) reduces to the following generalized variational inequality problem (in short, GVIP): Find $x \in C$ such that

$$\langle x - x^*, Ax^* \rangle + \psi(x, x^*) - \psi(x^*, x^*) \geq 0, \ \forall x \in C, \tag{1.5}$$

where $\psi : C \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function.

Further, we consider the following new system of generalized mixed vector equilibrium problems, which we call the system of unrelated generalized mixed vector equilibrium problems (in short, SUGMVEP): For each $i = 1, 2, 3, ..., N$, let K_i be a nonempty, closed and convex set in E with $K = \cap_{i=1}^{N} K_i \neq \emptyset$; let $F_i : K_i \times K_i \longrightarrow Y$ be a bimapping, $\psi_i : K_i \times K_i \longrightarrow \mathbb{R}$ be a bifunction and $A_i : K_i \longrightarrow B(E, Y)$ be a nonlinear mapping. Then SUGMVEP is to find $x^* \in \cap_{i=1}^{N} K_i$ such that

$$F_i(x^*, y_i) + \langle y_i - x^*, A_i x^* \rangle + \psi_i(y_i, x^*) - \psi_i(x^*, x^*) \in P, \ \forall y_i \in K_i, \ i = 1, 2, 3, ..., N. \quad (1.6)$$

We denote by Sol$(\mathrm{GMVEP}(F_i, A_i, \psi_i, K_i))$, the set of solution of GMVEP(1.2) corresponding to the mappings F_i, A_i, ψ_i and K_i. Then the set of solutions of SUGMVEP(1.6) is given by $\cap_{i=1}^{N} \mathrm{Sol}(\mathrm{GMVEP}(F_i, A_i, \psi_i, K_i))$.

If $Y = \mathbb{R}$, then SUGMVEP(1.6) reduces to the system of unrelated generalized mixed equilibrium problems (SUGMEP) of finding $x^* \in \cap_{i=1}^{N} K_i$ such that

$$F_i(x^*, y_i) + \langle y_i - x^*, A_i x^* \rangle + \psi_i(y_i, x^*) - \psi_i(x^*, x^*) \geq 0, \ \forall y_i \in K_i, \ i = 1, 2, 3, ..., N, \quad (1.7)$$

which appears to be new.

If $E = H$, Hilbert space, $\psi_i = 0$ and $Y = \mathbb{R}$ for all i, then SUGMVEP(1.6) reduces to the system of unrelated mixed equilibrium problems introduced and studied by Djafari-Rouhani, Kazmi and Rizvi [8].

Further if $E = H$, $A_i = 0$, $\psi_i = 0$ and $Y = \mathbb{R}$ for all i, then SUGMVEP(1.6) reduces to the system of unrelated variational inequality problems considered and studied by Censor et al. [5] for set-valued version of mappings A_i.

We also observe that if $F_i = 0$, $\psi_i = 0$, $Y = \mathbb{R}$ and $E = H$ for all i, then SUGMVEP(1.6) reduces to the problem of finding a point $x \in \cap_{i=1}^{N} K_i$ which is well known convex feasibility problem. If the set K_i are fixed sets of family of operators $S_i : H \longrightarrow H$ then the convex feasibility problem is the the common fixed points problem (in short, CFPP).

Next, we recall a mapping $T : C \longrightarrow C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$.

The fixed point problem (in short, FPP) for a nonexpansive mapping T is to:

$$\text{Find } x \in C \text{ such that } x \in \mathrm{Fix}(T), \quad (1.8)$$

where Fix(T) is the fixed point set of the nonexpansive mapping T. It is well known that if Fix$(T) \neq \emptyset$, then Fix(T) is closed and convex.

Let $U(E) = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1 \ \forall x, y \in U(E)$ with $x \neq y$. It is said to be *uniformly convex* if for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U(E)$, $\|x - y\| \geq \epsilon$ implies $\frac{\|x+y\|}{2} \leq 1 - \delta$. The space E is said to be *smooth* if the limit $\lim_{t \longrightarrow 0} \frac{\|x+ty\| - \|x\|}{t}$ exists $\forall x, y \in U(E)$. It is also said to be *uniformly smooth* if the limit exists uniformly for $x, y \in U(E)$.

The *normalized duality mapping* $J : E \longrightarrow 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \ \forall x \in E.$$

It is well known that if E^* is strictly convex, then J is single valued and demicontinuous, i.e., if $x_n \longrightarrow x$ then $Jx_n \rightharpoonup Jx$.

It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that E is uniformly smooth if and only if E^* is uniformly convex. Recall that E enjoys the Kadec-Klee property if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \longrightarrow \|x\|$ then $\|x_n - x\| \longrightarrow 0$ as $n \longrightarrow \infty$. It is well known that if E is a uniformly convex Banach space then E enjoys the Kadec-Klee property.

Let E be a smooth Banach space. The *Lyapunov functional* $\phi : E \times E \longrightarrow \mathbb{R}_+$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \ \forall x, y \in E.$$

Observe that in a Hilbert space H, $\phi(x, y) = \|x - y\|^2$, $\forall x, y \in H$. Alber [1] recently introduced a generalized projection operator $\prod_C$ in a Banach space E which is an analogue of the metric projection P_C in Hilbert space. Recall that the generalized projection $\prod_C : E \longrightarrow C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\prod_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x).$$

The existence and uniqueness of the operator $\prod_C$ follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J. In Hilbert space, $\prod_C = P_C$. It is obvious from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle,$$

and

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \ \forall x, y \in E. \quad (1.9)$$

Note that if E is a reflexive, strictly convex and smooth Banach space, then $\phi(x, y) = 0$ if and only if $x = y$.

Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty subset of E.

Definition 1.2. A mapping $T : C \longrightarrow C$ is said to be:

(i) *asymptotically regular* on C if for any bounded subset K of C,

$$\limsup_{n \longrightarrow \infty} \{\|T^{n+1}x - T^n x\| : x \in K\} = 0;$$

(ii) *closed* if for any sequence $\{x_n\} \subset C$ such that $\lim_{n \longrightarrow \infty} x_n = x_0$ and $\lim_{n \longrightarrow \infty} Tx_n = y_0$, then $Tx_0 = y_0$.

Definition 1.3. [27] Let $T : C \longrightarrow C$ be a mapping. A point $p \in C$ is said to be an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}$ which converges weakly to p so that $\lim_{n \longrightarrow \infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{\mathrm{Fix}}(T)$.

Definition 1.4. A mapping $T : C \longrightarrow C$ is said to be:

(i) [27] *relatively nonexpansive* if

$$\widehat{\mathrm{Fix}}(T) = \mathrm{Fix}(T) \neq \emptyset, \ \phi(p, Tx) \leq \phi(p, x), \ \forall x \in C, \ \forall p \in \mathrm{Fix}(T);$$

(ii) *relatively asymptotically nonexpansive* if

$$\widehat{\mathrm{Fix}}(T) = \mathrm{Fix}(T) \neq \emptyset,\ \phi(p, T^n x) \leq (1+\mu_n)\phi(p,x),\ \forall x \in C,\ \forall p \in \mathrm{Fix}(T),\ \forall n \geq 1,$$

where $\mu_n \subset [0,\infty)$ is a sequence such that $\mu_n \longrightarrow 0$ as $n \longrightarrow \infty$;

(iii) [29] *quasi ϕ-nonexpansive* if

$$\mathrm{Fix}(T) \neq \emptyset,\ \phi(p, Tx) \leq \phi(p,x),\ \forall x \in C,\ \forall p \in \mathrm{Fix}(T);$$

(iv) [29, 21] *asymptotically quasi ϕ-nonexpansive* if there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \longrightarrow 0$ as $n \longrightarrow \infty$ such that

$$\mathrm{Fix}(T) \neq \emptyset,\ \phi(p, T^n x) \leq (1+\mu_n)\phi(p,x),\ \forall x \in C,\ \forall p \in \mathrm{Fix}(T),\ \forall n \geq 1;$$

(v) [22] *generalized asymptotically quasi ϕ-nonexpansive* if $\mathrm{Fix}(T) \neq \emptyset$ and there exist two nonnegative sequences $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \longrightarrow 0$ and $\xi_n \subset [0,\infty)$ with $\xi_n \longrightarrow 0$ as $n \longrightarrow \infty$ such that

$$\phi(p, T^n x) \leq (1+\mu_n)\phi(p,x) + \xi_n,\ \forall x \in C,\ \forall p \in \mathrm{Fix}(T),\ \forall n \geq 1.$$

In 2007, Tada and Takahashi [25] and Takahashi and Takahashi [26] proved weak and strong convergence theorems for finding a common solution of EP(1.1) and FPP(1.8) of a nonexpansive mapping in a Hilbert space. For further related work, see Ceng and Yao [4] and Shan and Huang [23].

In 2009, Takahashi and Zembayashi [27] proved weak and strong convergence theorems for finding a common solution of EP(1.1) and FPP(1.8) of a relatively nonexpansive mapping in real Banach space. For further related work, see Kazmi and Farid [16] and the references therein. Recently, Qin and Agarwal [21] proved a strong convergence to common fixed points of the pair of asymptotic quasi ϕ-nonexpansive mappings in Banach space. Later Qin *et al.* [22] proved a strong convergence to common fixed points of a family of generalized asymptotically quasi ϕ-nonexpansive mappings. Very recently, Song and Chen [24] proved a strong convergence to common element of the set of solutions of mixed equilibrium problem and set of fixed points of FPP(1.8) of a generalized asymptotically quasi ϕ-nonexpansive mapping in Banach space.

Motivated by the work of Qin *et al.* [21, 22], Song and Chen [24], Shan and Huang [23], and by the ongoing research in this direction, we study the existence and properties of solution of a new generalized mixed vector equilibrium problem in Banach space. Further, we introduce an iterative method based on hybrid method and convex approximation method for finding a common element to the set of solutions of SUGMVEP(1.6) and the set of solutions of common fixed point problems for the two families of two generalized asymptotically quasi ϕ-nonexpansive mappings in Banach space. Furthermore, we obtain a strong convergence theorem for the sequences generated by the proposed iterative scheme. Finally, we derive some consequences from our main result. The results presented in this paper extended and unify many of the previously known results in this area, see for instance [16, 24].

2. PRELIMINARIES AND NOTATIONS

We recall some concepts and results which are needed in sequel.

Lemma 2.1. [1] *Let E be a smooth, strictly convex and reflexive Banach space, and C be a nonempty closed and convex subset of E. Then the following conclusions hold:*

(i) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y),\ \forall x \in C, y \in E$;

(ii) *Let $x \in E$ and $z \in C$ then*

$$z = \Pi_C(x) \Leftrightarrow \langle z - y, Jx - Jz \rangle \geq 0,\ \forall y \in C.$$

Lemma 2.2. [20] *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E and let T be a relatively nonexpansive mapping from C into itself. Then $\mathrm{Fix}(T)$ is closed and convex.*

Lemma 2.3. [6] *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \longrightarrow [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \gamma \|z\|^2 - \mu\gamma g(\|y - z\|),$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$, where $B_r(0) = \{z \in E : \|z\| \leq r\}$.

Definition 2.4. [19, 28] Let X and Y be two Hausdorff topological spaces and let D be a nonempty, convex subset of X and P be a pointed, proper, closed and convex cone of Y with $\mathrm{int}P \neq \emptyset$. Let 0 be the zero point of Y, $\mathbb{U}(0)$ be the neighborhood set of 0, $\mathbb{U}(x_0)$ be the neighborhood set of x_0 and $f : D \longrightarrow Y$ be a mapping.

(i) If, for any $V \in \mathbb{U}(0)$ in Y, there exists $U \in \mathbb{U}(x_0)$ such that

$$f(x) \in f(x_0) + V + P,\ \forall x \in U \cap D,$$

then f is called *upper P-continuous* on x_0. If f is *upper P-continuous* for all $x \in D$, then f is called *upper P-continuous* on D;

(ii) If, for any $V \in \mathbb{U}(0)$ in Y, there exists $U \in \mathbb{U}(x_0)$ such that

$$f(x) \in f(x_0) + V - P,\ \forall x \in U \cap D,$$

then f is called *lower P-continuous* on x_0. If f is *lower P-continuous* for all $x \in D$, then f is called *lower P-continuous* on D;

(iii) If, for any $x, y \in D$ and $t \in [0, 1]$, the mapping f satisfies

$$f(x) \in f(tx + (1-t)y) + P \text{ or } f(y) \in f(tx + (1-t)y) + P,$$

then f is called *proper P-quasiconvex*;

(iv) If, for any $x, y \in D$ and $t \in [0, 1]$, the mapping f satisfies

$$tf(x) + (1-t)f(y) \in f(tx + (1-t)y) + P,$$

then f is called *P-convex*.

Lemma 2.5. [10] *Let X and Y be two real Hausdorff topological spaces, D be a nonempty, compact and convex subset of X and P be a pointed, proper, closed and convex cone of Y with $\mathrm{int}P \neq \emptyset$. Assume that $g : D \times D \longrightarrow Y$ and $\Phi : D \longrightarrow Y$ are two nonlinear mappings. Suppose that g and Φ satisfy*

(i) *$g(x, x) \in P,\ \forall x \in D$;*

(ii) *Φ is upper P-continuous on D;*

(iii) *$g(., y)$ is lower P-continuous, $\forall x \in D$;*

(iv) *$g(x, .) + \Phi(.)$ is proper P-quasiconvex, $\forall x \in D$.*

Then there exists a point $x \in D$ satisfies

$$G(x, y) \in P \setminus \{0\},\ \forall y \in D,$$

where

$$G(x,y) = g(x,y) + \Phi(y) - \Phi(x),\ \forall x, y \in D.$$

Let $F, \psi : C \times C \longrightarrow Y$ be two mappings and $A : C \longrightarrow B(E,Y)$ be a nonlinear mapping. For any $z \in E$, define a mapping $G_z : C \times C \longrightarrow Y$ as follows:

$$G_z(x,y) = F(x,y) + \langle y - x, Ax\rangle + \psi(y,x) - \psi(x,x) + \frac{e}{r}\langle y - x, Jx - Jz\rangle, \quad (2.1)$$

where r is a positive real number and $e \in \text{int}P$.

Assumption 2.6. *Let G_z, F, ψ satisfy the following conditions:*

(i) *For all $x \in C$, $F(x,x) \in P$;*
(ii) *F is P-monotone, i.e., $F(x,y) + F(y,x) \in -P,\ \forall x, y \in C$;*
(iii) *$F(.,y)$ is continuous, $\forall y \in C$;*
(iv) *$F(x,.)$ is weakly continuous and P-convex, i.e.,*

$$tF(x,y_1) + (1-t)F(x,y_2) \in F(x, ty_1 + (1-t)y_2) + P,\ \forall x, y_1, y_2 \in C,\ \forall t \in [0,1];$$

(v) *$G_z(.,y)$ is lower P-continuous, $\forall y \in C$ and $z \in E$;*
(vi) *$\psi(.,y)$ is P-convex and weakly continuous;*
(vii) *$G_z(x,.)$ is proper P-quasiconvex, $\forall x \in C$ and $z \in E$;*
(viii) *ψ is P-skew symmetric, i.e.,*

$$\psi(x,x) - \psi(x,y) - \psi(y,x) + \psi(y,y) \in P,\ \forall x, y \in C.$$

Remark 2.7. P-skew-symmetric bifunctions are natural extensions of skew-symmetric bifunctions. The skew-symmetric bifunctions have the properties that can be considered ananalogous to the monotonicity of the gradient and the non-negativity of the second derivative for convex functions. For the properties and applications of the skew-symmetric bifunctions, we refer the reader to [2].

3. MAIN RESULTS

For any $r > 0$, define a mapping $T_r : E \longrightarrow C$ as follows:

$$T_r(z) = \{x \in C : F(x,y) + \langle y - x, Ax\rangle + \psi(y,x) - \psi(x,x) + \frac{e}{r}\langle y - x, Jx - Jz\rangle \in P,\ \forall y \in C\}, \quad (3.1)$$

where $e \in \text{int}P$.

Now, we study some properties of set of solutions of GMVEP(1.2) and the mapping T_r.

Theorem 3.1. *Let E be a uniformly smooth and strictly convex Banach space and let C be a nonempty, compact and convex subset of E. Assume that P is a pointed, proper, closed and convex cone of a real order Banach space Y with $\text{int}P \neq \emptyset$. Let $G_z : C \times C \longrightarrow Y$ be defined by (2.1). Let $F, \psi : C \times C \longrightarrow Y$ and G_z satisfy Assumption 2.6 and $A : C \longrightarrow B(E,Y)$ be a continuous and P-monotone mapping. Let $T_r : E \longrightarrow C$ be defined by (3.1). Then the following conclusions hold:*

(i) *$T_r(z) \neq \emptyset,\ \forall z \in E$;*
(ii) *T_r is single-valued;*
(iii) *T_r is firmly nonexpansive type mapping, i.e., for all $z_1, z_2 \in E$,*

$$\langle T_r z_1 - T_r z_2, JT_r z_1 - JT_r z_2\rangle \leq \langle T_r z_1 - T_r z_2, Jz_1 - Jz_2\rangle;$$

(iv) *Fix(T_r) =Sol(GMVEP(1.2));*
(v) *Sol(GMVEP(1.2)) is closed and convex.*

Proof. (i) Let $g(x, y) = G_z(x, y)$ and $\Phi(y) = 0$ for all $x, y \in C$ and $z \in E$. It is easy to observe that $g(x, y)$ and $\Phi(y)$ satisfy all the conditions of Lemma 2.5 Then there exists a point $x \in C$ such that

$$G_z(x, y) + \Phi(y) - \Phi(x) \in P, \ \forall y \in C,$$

and thus $T_r(z) \neq \emptyset, \ \forall z \in E$.

(ii) For each $z \in E$, $T_r(z) \neq \emptyset$, let $x_1, x_2 \in T_r(z)$. Then

$$F(x_1, y)+\langle y-x_1, Ax_1\rangle+\psi(y, x_1)-\psi(x_1, x_1)+\frac{e}{r}\langle y-x_1, Jx_1-Jz\rangle \in P, \ \forall y \in C, \quad (3.2)$$

and

$$F(x_2, y)+\langle y-x_2, Ax_2\rangle+\psi(y, x_2)-\psi(x_2, x_2)+\frac{e}{r}\langle y-x_2, Jx_2-Jz\rangle \in P, \ \forall y \in C. \quad (3.3)$$

Letting $y = x_2$ in (3.2) and $y = x_1$ in (3.3), and then adding, we have

$$\begin{aligned} &F(x_1, x_2) + F(x_2, x_1) + \langle x_2 - x_1, Ax_1 - Ax_2\rangle \\ &+\psi(x_2, x_1) - \psi(x_1, x_1) + \psi(x_1, x_2) - \psi(x_2, x_2) + \tfrac{e}{r}\langle x_2 - x_1, Jx_1 - Jx_2\rangle \in P. \end{aligned}$$

Since F is P-monotone, A is P-monotone, i.e., $\langle x_2 - x_1, Ax_1 - Ax_2\rangle \in -P$ and ψ is P-skew symmetric, then we have

$$\frac{e}{r}\langle x_2 - x_1, Jx_1 - Jx_2\rangle \in P.$$

Since $e \in \text{int}P$, $r > 0$ and P is closed and convex cone, we have

$$\frac{1}{r}\langle x_2 - x_1, Jx_1 - Jx_2\rangle \geq 0.$$

Since E is strictly convex, the preceding inequality implies $x_1 = x_2$. Hence T_r is single-valued.

(iii) For any $z_1, z_2 \in E$, let $x_1 = T_r(z_1)$ and $x_2 = T_r(z_2)$. Then

$$F(x_1, y) + \langle y - x_1, Ax_1\rangle + \psi(y, x_1) - \psi(x_1, x_1) + \frac{e}{r}\langle y - x_1, Jx_1 - Jz_1\rangle \in P, \ \forall y \in C, \quad (3.4)$$

and

$$F(x_2, y) + \langle y - x_2, Ax_2\rangle + \psi(y, x_2) - \psi(x_2, x_2) + \frac{e}{r}\langle y - x_2, Jx_2 - Jz_2\rangle \in P, \ \forall y \in C. \quad (3.5)$$

Letting $y = x_2$ in (3.4) and $y = x_1$ in (3.5), and then adding, we have

$$\begin{aligned} &F(x_1, x_2) + F(x_2, x_1) + \langle x_2 - x_1, Ax_1 - Ax_2\rangle \\ &+\psi(x_2, x_1) - \psi(x_1, x_1) + \psi(x_1, x_2) - \psi(x_2, x_2) + \tfrac{e}{r}\langle x_2 - x_1, Jx_1 - Jx_2 - Jz_1 + Jz_2\rangle \in P. \end{aligned}$$

By using the monotoncity of F, A and the properties of ψ and P, we have

$$\frac{1}{r}\langle x_2 - x_1, Jx_1 - Jx_2 - Jz_1 - Jz_2\rangle \geq 0.$$

Hence, we have

$$\langle x_2 - x_1, Jx_1 - Jx_2\rangle + \langle x_2 - x_1, Jz_2 - Jz_1\rangle \geq 0,$$

or,

$$\langle x_1 - x_2, Jx_1 - Jx_2\rangle \leq \langle x_1 - x_2, Jz_1 - Jz_2\rangle,$$

i.e.,

$$\langle T_r(z_1) - T_r(z_2), JT_r(z_1) - JT_r(z_2)\rangle \leq \langle T_r(z_1) - T_r(z_2), Jz_1 - Jz_2\rangle. \quad (3.6)$$

Thus T_r is firmly nonexpansive-type mapping.

(iv) Let $x \in \mathrm{Fix}(T_r)$. Then

$$F(x,y) + \langle y-x, Ax\rangle + \psi(y,x) - \psi(x,x) + \frac{e}{r}\langle y-x, Jx - Jx\rangle \in P,\ \forall y \in C,$$

and so

$$F(x,y) + \langle y-x, Ax\rangle + \psi(y,x) - \psi(x,x) \in P,\ \forall y \in C.$$

Thus $x \in \mathrm{Sol}(\mathrm{GMVEP}(1.2))$.

Let $x \in \mathrm{Sol}(\mathrm{GMVEP}(1.2))$. Then

$$F(x,y) + \langle y-x, Ax\rangle + \psi(y,x) - \psi(x,x) \in P,\ \forall y \in C,$$

and so

$$F(x,y) + \langle y-x, Ax\rangle + \psi(y,x) - \psi(x,x) + \frac{e}{r}\langle y-x, Jx - Jx\rangle \in P,\ \forall y \in C.$$

Hence $x \in \mathrm{Fix}(T_r)$. Thus $\mathrm{Fix}(T_r) = \mathrm{Sol}(\mathrm{GMVEP}(1.2))$.

(v) By the definition of ϕ, we have

$$\begin{aligned} \phi(T_r(z_1),T_r(z_2)) + \phi(T_r(z_1),T_r(z_2)) &= 2\|T_r(z_1)\|^2 - 2\langle T_r(z_1), JT_r(z_2)\rangle \\ &\quad -2\langle T_r(z_2), JT_r(z_1)\rangle + 2\|T_r(z_2)\|^2 \\ &= 2\langle T_r(z_1), JT_r(z_1) - JT_r(z_2)\rangle \\ &\quad +2\langle T_r(z_2), JT_r(z_2) - JT_r(z_1)\rangle \\ &= 2\langle T_r(z_1) - T_r(z_2), JT_r(z_1) - JT_r(z_2)\rangle, \end{aligned}$$

and

$\phi(T_r(z_1), z_2) + \phi(T_r(z_2), z_1) - \phi(T_r(z_1), z_1) - \phi(T_r(z_2), z_2)$

$$\begin{aligned} &= \|T_r(z_1)\|^2 - 2\langle T_r(z_1), Jz_2\rangle + \|z_2\|^2 + \|T_r(z_2)\|^2 + \|z_1\|^2 \\ &\quad -2\langle T_r(z_2), Jz_1\rangle - \|T_r(z_2)\|^2 + 2\langle T_r(z_2), Jz_2\rangle - \|z_2\|^2 \\ &\quad -\|T_r(z_1)\|^2 + 2\langle T_r(z_1), Jz_1\rangle - \|z_1\|^2 \\ &= 2\langle T_r(z_1), Jz_1 - Jz_2\rangle - 2\langle T_r(z_2), Jz_1 - Jz_2\rangle \\ &= 2\langle T_r(z_1) - T_r(z_2), Jz_1 - Jz_2\rangle. \end{aligned}$$

Thus, it follows from (3.6) and the preceding two equations that

$$\phi(T_r(z_1),T_r(z_2)) + \phi(T_r(z_2),T_r(z_1)) \le \phi(T_r z_1, z_2) + \phi(T_r z_2, z_1) - \phi(T_r(z_1), z_1) - \phi(T_r(z_2), z_2).$$

Hence, for $z_1, z_2 \in C$, we have

$$\phi(T_r(z_1),T_r(z_2)) + \phi(T_r(z_2),T_r(z_1)) \le \phi(T_r z_1, z_2) + \phi(T_r z_2, z_1).$$

Taking $z_2 = u \in \mathrm{Fix}(T_r)$, we have

$$\phi(u, T_r z_1) \le \phi(u, z_1).$$

Next, we show that $\widehat{\mathrm{Fix}}(T_r) = \mathrm{Sol}(\mathrm{GMVEP}(1.2))$. Indeed, let $p \in \widehat{\mathrm{Fix}}(T_r)$. Then there exists a sequence $\{z_n\} \subset E$ such that $z_n \rightharpoonup p$ and $\lim_{n\longrightarrow\infty} \|z_n - T_r z_n\| = 0$. Moreover, we get $T_r z_n \rightharpoonup p$. Hence, we have $p \in C$. Since J is uniformly continuous on bounded sets, we have

$$\lim_{n\longrightarrow\infty} \frac{\|Jz_n - JT_r z_n\|}{r} = 0,\ r > 0. \tag{3.7}$$

From the definition of T_r, we have, $\forall y \in C$,

$$F(T_r z_n, y) + \langle y - T_r z_n, AT_r z_n\rangle + \psi(y, T_r z_n) - \psi(T_r z_n, T_r z_n) + \frac{e}{r}\langle y - T_r z_n, JT_r z_n - Jz_n\rangle \in P,$$

$$\begin{aligned} 0 \in\ & F(y, T_r z_n) - \langle y - T_r z_n, AT_r z_n\rangle - (\psi(y, T_r z_n) - \psi(T_r z_n, T_r z_n)) \\ & -\tfrac{e}{r}\langle y - T_r z_n, JT_r z_n - Jz_n\rangle + P,\ \forall y \in C. \end{aligned}$$

Let $y_t = (1-t)p + ty,\ \forall t \in (0,1]$. Since $y \in C$ and $p \in C$, we get $y_t \in C$ and hence

$$\begin{array}{rcl} 0 & \in & F(y, T_r z_n) - \langle y_t - T_r z_n, AT_r z_n\rangle - (\psi(y_t, T_r z_n) - \psi(T_r z_n, T_r z_n)) \\ & & -\frac{e}{r}\langle y_t - T_r z_n, JT_r z_n - Jz_n\rangle + P, \\ & = & F(y_t, T_r z_n) - \langle y_t - T_r z_n, AT_r z_n\rangle - (\psi(y_t, T_r z_n) - \psi(T_r z_n, T_r z_n)) \\ & & -e\langle y_t - T_r z_n, \frac{JT_r z_n - Jz_n}{r}\rangle + P. \\ 0 & \in & F(y_t, p) - \langle y_t - p, Ap\rangle - \psi(y_t, p) + \psi(p,p) + P. \end{array} \tag{3.8}$$

It follows from Assumption 2.6 (i), (iv) and (vi) that

$$\begin{array}{rcl} tF(y_t, y) + (1-t)F(y_t, p) + t\psi(y,p) + (1-t)\psi(p,p) - \psi(y_t, p) & \in & F(y_t, y_t) + \psi(y_t, p) \\ & & -\psi(y_t, p) + P \\ & \in & P. \end{array}$$

Now,

$$\begin{array}{l} -t[F(y_t, y) + \psi(y,p) - \psi(y_t,p) - \langle y_t - p, Ap\rangle] - \\ (1-t)[F(y_t,p) + \psi(p,p) - \psi(y_t,p) - \langle y_t - p, Ap\rangle] \in -P + \langle y_t - p, Ap\rangle \\ -t[F(y_t, y) + \psi(y,p) - \psi(y_t,p) - \langle y_t - p, Ap\rangle] - (1-t)P \in -P + \langle y_t - p, Ap\rangle,\ (\text{using } (3.7)) \\ -t[F(y_t, y) + \psi(y,p) - \psi(y_t,p) - \langle y_t - p, Ap\rangle] \in -P + (1-t)P + \langle y_t - p, Ap\rangle \\ -t[F(y_t, y) + \psi(y,p) - \psi(y_t,p) - \langle y_t - p, Ap\rangle] \in -tP + \langle y_t - p, Ap\rangle \\ \in -tP + \langle ty + (1-t)p - p, Ap\rangle\ \ \in -tP + t\langle y - p, Ap\rangle \end{array}$$

$$F(y_t, y) + \psi(y,p) - \psi(y_t,p) - \langle y_t - p, Ap\rangle \in P - \langle y - p, Ap\rangle.$$

Letting $t \longrightarrow 0_+$, we have

$$F(p,y) + \langle y - p, Ap\rangle + \psi(y,p) - \psi(p,p) \in P.$$

Thus $p \in \mathrm{Sol}(\mathrm{GMVEP}(1.2))$. So, we get $\mathrm{Fix}(T_r) = \mathrm{Sol}(\mathrm{GMVEP}(1.2)) = \widehat{\mathrm{Fix}}(T_r)$. Therefore T_r is a relatively nonexpansive mapping. Further, it follows from Lemma 2.2 that $\mathrm{Sol}(\mathrm{GMVEP}(1.2)) = \mathrm{Fix}(T_r)$ is closed and convex. This completes the proof. □

Next, we have the following consequence of Theorem 3.1

Lemma 3.2. *Let $E,\ C,\ F,\ \psi,\ G_z$ be the same as in Theorem 3.1 and let $r > 0$. Then, for $x \in E$ and $q \in \mathrm{Fix}(T_r)$, we have*

$$\phi(q, T_r x) + \phi(T_r x, x) \le \phi(q, x).$$

We prove a strong convergence theorem for finding a common element to the set of solutions of SUGMVEP(1.6) and set of fixed points of common fixed point problems of two families of generalized asymptotically quasi ϕ-nonexpansive mappings in Banach space.

Theorem 3.3. *Let E be a uniformly smooth and strictly convex Banach space such that E has Kadec-Klee property. For each $i \in I := \{1, 2, 3, ..., N\}$, let K_i be a nonempty, compact and convex subset of E such that $K = \cap_{i=1}^N K_i \neq \emptyset$. Assume that P is a pointed, proper, closed and convex cone of a real ordered Banach space Y with $\mathrm{int}P \neq \emptyset$. Let for each i, the mappings $F_i, \psi_i : K_i \times K_i \longrightarrow Y$ satisfy Assumption 2.6 and $A_i : K_i \longrightarrow B(E, Y)$ be continuous and P-monotone mapping. For each fixed i, let $S_i, T_i : K_i \longrightarrow E$ be closed, asymptotically regular and generalized asymptotically quasi ϕ-nonexpansive mappings with the sequences $\{\eta_{n,i}\}$, $\{\varsigma_{n,i}\}$ and $\{\zeta_{n,i}\}$, $\{\xi_{n,i}\}$ such that $\Gamma := (\cap_{i=1}^N \mathrm{Fix}(S_i)) \cap (\cap_{i=1}^N \mathrm{Fix}(T_i)) \cap (\cap_{i=1}^N \mathrm{Sol}(\mathrm{GMVEP}(F_i, A_i, \psi_i, K_i))) \neq \emptyset$. Assume that, for each fixed i, the sequences $\{r_{n,i}\} \subset [a, \infty)$ for some $a > 0$, $\{\alpha_{n,i}\} \subseteq (0,1)$ and $\{\beta_{n,i}^j\} \subseteq (0,1)$ $(j = 1,2,3)$ be such that*

(i) $\beta^1_{n,i} + \beta^2_{n,i} + \beta^3_{n,i} = 1$;
(ii) $\liminf_{n\longrightarrow\infty} \beta^1_{n,i}\beta^2_{n,i} > 0$ *and* $\liminf_{n\longrightarrow\infty} \beta^1_{n,i}\beta^3_{n,i} > 0$;
(iii) $\limsup_{n\longrightarrow\infty} \alpha_{n,i} < 1$;
(iv) $\liminf_{n\longrightarrow\infty} r_{n,i} > 0$.

Let $\{x_n\}$ *be a sequence generated by the iterative scheme:*

$$\begin{array}{l} x_0 \in E, \\ C_{1,i} =: K_i,\ C_1 = \cap_{i=1}^N C_{1,i} = K, \\ x_1 = \prod_{C_1} x_0, \\ y_{n,i} = J^{-1}(\alpha_{n,i}Jx_n + (1-\alpha_{n,i})Jz_{n,i}), \\ z_{n,i} = J^{-1}(\beta^1_{n,i}Jx_n + \beta^2_{n,i}JT_i^n x_n + \beta^3_{n,i}JS_i^n x_n), \\ u_{n,i} = T_{r_{n,i}}(y_{n,i}) \\ C_{n+1,i} = \{v \in C_{n,i} : \phi(v, u_{n,i}) \le \phi(v, x_n) + \delta_{n,i}M_n + \mu_{n,i}\}, \\ C_{n+1} = \cap_{i\in I} C_{n+1,i}, \\ x_{n+1} = \prod_{C_{n+1}} x_0, \text{ for every } n \in N \cup \{0\}, \end{array}$$

where $M_n = \sup\{\phi(p, x_n) : p \in \Gamma\}$; $e \in \mathrm{int}P$; $J : E \longrightarrow E^*$ *is the normalized duality mapping with its inverse* J^{-1}; $\delta_{n,i} = \beta^2_{n,i}\zeta_{n,i} + \beta^3_{n,i}\eta_{n,i}$ *and* $\mu_{n,i} = \xi_{n,i}\beta^2_{n,i} + \varsigma_{n,i}\beta^3_{n,i}$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\prod_\Gamma x_0$.

Proof. First, we show that C_n is closed and convex for every $n \ge 1$. It suffices to show that for each $i \in I$, $C_{n,i}$ is closed and convex for every $n \ge 1$. This can be proved by induction on n. In fact, for $n = 1$, $C_{1,i} = K_i$ is closed and convex for each $i \in I$. Assume that $C_{n,i}$ is closed and convex for some $n \ge 1$ and for each $i \in I$. For $v \in C_{n+1,i}$,

$$\phi(v, u_{n,i}) \le \phi(v, x_n) + \delta_{n,i}M_n + \mu_{n,i},$$

which is equivalent to

$$2\langle v, Jx_n - Ju_{n,i}\rangle \le \|x_n\|^2 - \|u_{n,i}\|^2 + \delta_{n,i}M_n + \mu_{n,i}.$$

It is easy to see that $C_{n+1,i}$ is closed and convex for each $i \in I$. Then, for all $n \ge 1$, $C_{n,i}$ is closed and convex for each $i \in I$. Consequently, $C_n = \cap_{i=1}^N C_{n,i}$ is closed and convex for all $n \ge 1$. This shows that $\prod_{C_{n+1}} x_0$ is well defined.

Next, we prove $\Gamma \subset C_n$ for all $n \ge 1$. It suffices to show that for each $i \in I$, $\Gamma \subset C_{n,i}$. Indeed, $\Gamma \subset C_{1,i} = K_i$ is obvious. Suppose $\Gamma \subset C_{k,i}$ for some $k \ge 1$. Then, for $\forall w \in \Gamma \subset C_{k,i}$, we have

$$\begin{array}{rcl} \phi(w, z_{k,i}) & = & \phi(w, J^{-1}(\beta^1_{k,i}Jx_k + \beta^2_{k,i}JT_i^k x_k + \beta^3_{k,i}JS_i^k x_k)) \\ & \le & \|w\|^2 - 2\langle w, \beta^1_{k,i}Jx_k + \beta^2_{k,i}JT_i^k x_k + \beta^3_{k,i}JS_i^k x_k\rangle \\ & & +\|\beta^1_{k,i}Jx_k + \beta^2_{k,i}JT_i^k x_k + \beta^3_{k,i}JS_i^k x_k\|^2 \\ & \le & \|w\|^2 - 2\beta^1_{k,i}\langle w, Jx_k\rangle - 2\beta^2_{k,i}\langle w, JT_i^k x_k\rangle - 2\beta^3_{k,i}\langle w, JS_i^k x_k\rangle \\ & & +\beta^1_{k,i}\|x_k\|^2 + \beta^2_{k,i}\|T_i^k x_k\|^2 + \beta^3_{k,i}\|S_i x_k\|^2 \\ & = & \beta^1_{k,i}\phi(w, x_k) + \beta^2_{k,i}\phi(w, T_i^k x_k) + \beta^3_{k,i}\phi(w, S_i^k x_k) \\ & \le & \beta^1_{k,i}\phi(w, x^k) + \beta^2_{k,i}(1+\zeta_{k,i})\phi(w, x_k) + \xi_{k,i}\beta^2_{k,i} + \beta^3_{k,i}(1+\eta_{k,i})\phi(w, x_k) + \varsigma_{k,i}\beta^3_{k,i} \\ & \le & \phi(w, x_k) + \delta_{k,i}\phi(w, x_k) + \mu_{k,i}. \end{array} \quad (3.9)$$

Further, it follows from Theorem 3.1 that $u_{n,i} = T_{r_{n,i}} y_{n,i}$ for all $n \in N \cup \{0\}$, and $T_{r_{n,i}}$ is relatively nonexpansive. Therefore

$$\begin{aligned}
\phi(w, u_{k,i}) &= \phi(w, T_{r_{k,i}} y_{k,i}) \\
&\le \phi(w, y_{k,i}) \\
&= \phi(w, J^{-1}(\alpha_{k,i} J x_k + (1-\alpha_{k,i}) J z_{k,i})) \\
&= \|w\|^2 - 2\langle w, \alpha_{k,i} J x_k + (1-\alpha_{k,i}) J z_{k,i}\rangle + \|\alpha_{k,i} J x_k + (1-\alpha_{k,i}) J z_{k,i}\|^2 \\
&= \|w\|^2 - 2\alpha_{k,i}\langle w, J x_k\rangle - 2(1-\alpha_{k,i})\langle w, J z_{k,i}\rangle + \alpha_{k,i}\|x_k\|^2 + (1-\alpha_{k,i})\|z_{k,i}\|^2 \\
&= \alpha_{k,i}\phi(w, x_k) + (1-\alpha_{k,i})\phi(w, z_{k,i}) \\
&\le \alpha_{k,i}\phi(w, x_k) + (1-\alpha_{k,i})\phi(w, x_k) - (1-\alpha_{k,i})(\beta^2_{k,i}\zeta_{k,i} + \beta^3_{k,i}\eta_{k,i})\phi(w, x_k) \\
&\quad + (1-\alpha_{k,i})(\xi_{k,i}\beta^2_{k,i} + \varsigma_{k,i}\beta^3_{k,i}) \\
&\le \phi(w, x_k) + (1-\alpha_{k,i})[(\beta^2_{k,i}\zeta_{k,i} + \beta^3_{k,i}\eta_{k,i})\phi(w, x_k) + (\xi_{k,i}\beta^2_{k,i} + \varsigma_{k,i}\beta^3_{k,i})] \\
&\le \phi(w, x_k) + (\beta^2_{k,i}\zeta_{k,i} + \beta^3_{k,i}\eta_{k,i})\phi(w, x_k) + (\xi_{k,i}\beta^2_{k,i} + \varsigma_{k,i}\beta^3_{k,i}) \\
&\le \phi(w, x_k) + \delta_{k,i} M_k + \mu_{k,i}.
\end{aligned} \tag{3.10}$$

This shows that $w \in C_{k+1,i}$. That is, $\Gamma \subset C_{n,i}$, for all $n \ge 1$ and each $i \in I$. Therefore, $w \in C_n = \cap_{i \in I} C_{n,i}$ for all $n \ge 1$.

From Lemma 2.1, we have

$$\begin{aligned}
\phi(x_n, x_0) &= \phi(\Pi_{C_n} x_0, x_0) \\
&\le \phi(w, x_0) - \phi(w, x_n) \\
&\le \phi(w, x_0), \text{ for each } w \in \Gamma \subset C_n \text{ and for each } n \ge 1.
\end{aligned}$$

Therefore, the sequence $\{\phi(x_n, x_0)\}$ is bounded. It follows from (1.9) that the sequence $\{x_n\}$ is also bounded. Since E is reflexive, without loss of generality, we may assume that $x_n \rightharpoonup p$ as $n \longrightarrow \infty$. Since $C_j \subset C_n$ for $j \ge n$, we have $x_j \in C_n$ for $j \ge n$. Since C_n is closed and convex, $p \in C_n$ for all $n \ge 1$. Hence $p \in \cap_{n=1}^{\infty} C_n$. Since $\phi(x_n, x_0) \le \phi(x_{n+1}, x_0) \le \phi(p, x_0)$, we have

$$\phi(p, x_0) \le \liminf_{n \longrightarrow \infty} \phi(x_n, x_0) \le \limsup_{n \longrightarrow \infty} \phi(x_n, x_0) \le \phi(p, x_0),$$

which implies that $\phi(x_n, x_0) \longrightarrow \phi(p, x_0)$ as $n \longrightarrow \infty$. Hence $\|x_n\| \longrightarrow \|p\|$. By the Kadec-Klee property of E, we have $x_n \longrightarrow p \in C$ as $n \longrightarrow \infty$.

Since $x_n \longrightarrow p$ and $J : E \longrightarrow E^*$ is demicontinuous, we have $J x_n \rightharpoonup Jp \in E^*$. Note that

$$|\|J x_n\| - \|Jp\|| = |\|x_n\| - \|p\|| \le \|x_n - p\|.$$

This implies that $\|J x_n\| \longrightarrow \|Jp\|$. Since E is uniformly smooth, E^* is uniformly convex Banach space and hence it enjoys the Kadec-Klee property, we see that

$$\lim_{n \longrightarrow \infty} \|J x_n - Jp\| = 0. \tag{3.11}$$

On the other hand, since $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0), \text{ for all } n \ge 1.$$

Therefore, $\{\phi(x_n, x_0)\}$ is non-decreasing. Further, it follows that the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of C_n, one has that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m \ge n$. It follows that

$$\begin{aligned}
\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\
&\le \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\
&= \phi(x_m, x_0) - \phi(x_n, x_0).
\end{aligned} \tag{3.12}$$

Letting $m, n \longrightarrow \infty$ in (3.12), we have $\phi(x_m, x_n) \longrightarrow 0$.

Next, we show that $p \in (\cap_{i=1}^{n} \text{Fix}(S_i)) \cap (\cap_{i=1}^{n} \text{Fix}(T_i))$. By taking $m = n+1$ in (3.12), we have

$$\lim_{n \longrightarrow \infty} \phi(x_{n+1}, x_n) = 0. \tag{3.13}$$

Notice that $x_{n+1} \in C_{n+1}$, from the definition of C_n, for every $i \in I$, we have

$$\phi(x_{n+1}, u_{n,i}) \leq \phi(x_{n+1}, x_n) + \delta_{n,i} M_n + \mu_{n,i}. \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\lim_{n \longrightarrow \infty} \phi(x_{n+1}, u_{n,i}) = 0. \tag{3.15}$$

It follows from (3.15) and inequality $0 \leq (\|x_{n+1}\| - \|u_{n,i}\|)^2 \leq \phi(x_{n+1}, u_{n,i})$ that $\|u_{n,i}\| \longrightarrow \|p\|$ and consequently, we have $\|Ju_{n,i}\| \longrightarrow \|Jp\|$. This implies that $\{J(u_{n,i})\}$ is bounded. Since E is reflexive, E^* is also reflexive. So, we may assume that $J(u_{n,i}) \rightharpoonup f_{0,i} \in E^*$.

On the other hand, in view of the reflexivity of E, we have $J(E) = E^*$, which means that for $f_{0,i} \in E^*$, there exists $e_i \in E$, such that $Je_i = f_{0,i}$. Using weakly lower semi-continuity of $\|.\|^2$, we have

$$\begin{aligned}
\liminf_{n \longrightarrow \infty} \phi(x_{n+1}, u_{n,i}) &= \liminf_{n \longrightarrow \infty} (\|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_{n,i}\rangle + \|u_{n,i}\|^2) \\
&= \liminf_{n \longrightarrow \infty} (\|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_{n,i}\rangle + \|Ju_{n,i}\|^2) \\
&\geq \|p\|^2 - 2\langle p, f_{0,i}\rangle + \|f_{0,i}\|^2 \\
&= \|p\|^2 - 2\langle p, Je_i\rangle + \|Je_i\|^2 \\
&= \phi(p, e_i).
\end{aligned}$$

It follows from (3.15) that $\phi(p, e_i) = 0$. Hence $p = e_i$, which implies that $f_{0,i} = Jp$. Hence $Ju_{n,i} \rightharpoonup Jp \in E^*$. Since $\|Ju_{n,i}\| \longrightarrow \|Jp\|$ and by the Kadec-Klee property of E^*, we have

$$\|Ju_{n,i} - Jp\| \longrightarrow 0, \ \forall i \in I. \tag{3.16}$$

Since $J^{-1} : E^* \longrightarrow E$ is demi-continuous, therefore $u_{n,i} \rightharpoonup p$. Since $\|u_{n,i}\| \longrightarrow \|p\|$ and using the Kadec-Klee property of E^*, we have

$$u_{n,i} \longrightarrow p, \ \text{as } n \longrightarrow \infty \ \forall i \in I. \tag{3.17}$$

Hence,

$$\lim_{n \longrightarrow \infty} \|x_n - u_{n,i}\| = 0, \ \forall i \in I. \tag{3.18}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \longrightarrow \infty} \|Jx_n - Ju_{n,i}\| = 0, \ \forall i \in I. \tag{3.19}$$

Now,

$$\begin{aligned}
\phi(w, x_n) - \phi(w, u_{n,i}) &= \|x_n\|^2 - \|u_{n,i}\|^2 - 2\langle w, Jx_n - Ju_{n,i}\rangle \\
&\leq \|x_n - u_{n,i}\|(\|x_n\| + \|u_{n,i}\|) + 2\|w\|\|Jx_n - Ju_{n,i}\|.
\end{aligned}$$

It follows from (3.18) and (3.19) that

$$\phi(w, x_n) - \phi(w, u_{n,i}) \longrightarrow 0, \ \text{as } n \longrightarrow \infty. \tag{3.20}$$

From $u_{n,i} = T_{r_{n,i}} y_{n,i}$ and Lemma 3.2, we have

$$\begin{aligned}
\phi(u_{n,i}, y_{n,i}) &= \phi(T_{r_{n,i}} y_{n,i}, y_{n,i}) \\
&\leq \phi(w, y_{n,i}) - \phi(w, u_{n,i}) \\
&\leq \alpha_{n,i}\phi(w, x_n) + (1 - \alpha_{n,i})\phi(w, z_{n,i}) - \phi(w, u_{n,i}) \\
&\leq \alpha_{n,i}\phi(w, x_n) + (1 - \alpha_{n,i})[\phi(w, x_n) + \delta_{n,i} M_n + \mu_{n,i} \\
&\quad - \beta_{n,i}^1 \beta_{n,i}^3 g(\|Jx_n - JS_i^n x_n\|)] - \phi(w, u_{n,i}) \\
&\leq \phi(w, x_n) - \phi(w, u_{n,i}) + (1 - \alpha_{n,i})[\delta_{n,i} M_n + \mu_{n,i}].
\end{aligned} \tag{3.21}$$

Using (3.20) and the restrictions on the sequences in above inequality, we have

$$\lim_{n\longrightarrow\infty}\phi(u_{n,i},y_{n,i})=0. \tag{3.22}$$

Since $0\leq(\|u_{n,i}\|-\|y_{n,i})\|)^2\leq\phi(u_{n,i},y_{n,i})$, it follows from (3.17) that $\|y_{n,i}\|\longrightarrow\|p\|$ and consequently $\|Jy_{n,i}\|\longrightarrow\|Jp\|$. This implies that $\{Jy_{n,i}\}$ is bounded. Since E is reflexive, E^* is also reflexive, so we may assume that $J(y_{n,i})\rightharpoonup h_{0,i}\in E^*$.

On the other hand, in view of the reflexivity of E, we have $J(E)=E^*$, which means that for $h_{0,i}\in E^*$, there exists $d_i\in E$, such that $Jd_i=h_{0,i}$. Using lower semi-continuity of $\|.\|^2$, we have

$$\begin{aligned}\liminf_{n\longrightarrow\infty}\phi(u_{n,i},y_{n,i}) &= \liminf_{n\longrightarrow\infty}(\|u_{n,i}\|^2-2\langle u_{n,i},Jy_{n,i}\rangle+\|y_{n,i}\|^2)\\ &= \liminf_{n\longrightarrow\infty}(\|u_{n,i}\|^2-2\langle u_{n,i},Jy_{n,i}\rangle+\|Jy_{n,i}\|^2)\\ &\geq \|p\|^2-2\langle p,h_{0,i}\rangle+\|h_{0,i}\|^2\\ &= \|p\|^2-2\langle p,Jd_i\rangle+\|d_i\|^2\\ &= \phi(p,d_i).\end{aligned}$$

It follows from (3.22) that $\phi(p,d_i)=0$. Hence $p=d_i$, which implies that $h_{0,i}=Jp$. Hence $Jy_{n,i}\rightharpoonup Jp\in E^*$. Since $Jy_{n,i}\rightharpoonup Jp$, $\|Jy_{n,i}\|\longrightarrow\|Jp\|$ and Kadec-Klee property of E^*, we have

$$\|Jy_{n,i}-Jp\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty. \tag{3.23}$$

Since $J^{-1}:E^*\longrightarrow E$ is demi-continuous, we have $y_{n,i}\rightharpoonup p$. Further, since $\|y_{n,i}\|\longrightarrow\|p\|$ and E has the Kadec-Klee property, we have

$$\|y_{n,i}-p\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty. \tag{3.24}$$

It follows from (3.18) and (3.24) that

$$\|y_{n,i}-u_{n,i}\|\leq\|u_{n,i}-p\|+\|y_{n,i}-p\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty.$$

Since J is uniformly norm-to- norm continuous on bounded sets, we have

$$\|Jy_{n,i}-Ju_{n,i}\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty. \tag{3.25}$$

It follows from (3.19) and (3.25) that

$$\|Jx_n-Jy_{n,i}\|\leq\|Jx_n-Ju_{n,i}\|+\|Ju_{n,i}-Jy_{n,i}\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty. \tag{3.26}$$

From iterative scheme, (3.26) and $\limsup_{n\longrightarrow\infty}\alpha_{n,i}<1$, it follows that

$$\|Jz_{n,i}-Jx_n\|\leq\frac{1}{(1-\alpha_{n,i})}\|Jx_n-Jy_{n,i}\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty. \tag{3.27}$$

Hence,

$$\|Jz_{n,i}-Jp\|\leq\|Jz_{n,i}-Jx_n\|+\|Jx_n-Jp\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty. \tag{3.28}$$

Since $J^{-1}:E^*\longrightarrow E$ is demi-continuous, we have $z_{n,i}\rightharpoonup p$. From (3.28), we have that

$$|\|z_{n,i}\|-\|p\||=|\|Jz_{n.i}\|-\|Jp\||\leq\|Jz_{n,i}-Jp\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty.$$

This implies that $\|z_{n,i}\|\longrightarrow\|p\|$. Since E enjoys the Kadec-Klee property, we see that

$$\lim_{n\longrightarrow\infty}\|z_{n,i}-p\|=0. \tag{3.29}$$

Further,

$$\|z_{n,i}-x_n\|\leq\|z_{n,i}-p\|+\|x_n-p\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty. \tag{3.30}$$

Now,

$$\begin{aligned}\phi(w,x_n)-\phi(w,z_{n,i}) &= \|x_n\|^2-\|z_{n,i}\|^2-2\langle w,Jx_n-Jz_{n,i}\rangle\\ &\leq \|x_n-z_{n,i}\|(\|x_n\|+\|z_{n,i}\|)+2\|w\|\|Jx_n-Jz_{n,i}\|.\end{aligned}$$

It follows from (3.27) and (3.30) that

$$\phi(w, x_n) - \phi(w, z_{n,i}) \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{3.31}$$

Let $r = \max\{\sup_{n\geq 1}\{\|x_n\|\}, \sup_{n\geq 1}\{\|T_i^n x_n\|\}, \sup_{n\geq 1}\{\|S_i^n x_n\|\}\}$. Since E is uniformly smooth, then E^* is uniformly convex. In the light of Lemma 2.3, we have, for any fixed $w \in \mathrm{Fix}(T_i) \cap \mathrm{Fix}(S_i)$,

$$\begin{aligned}
\phi(w, z_{n,i}) &= \phi(w, J^{-1}(\beta_{n,i}^1 Jx_n + \beta_{n,i}^2 JT_i^n x_n + \beta_{n,i}^3 JS_i^n x_n)) \\
&= \|w\|^2 - 2\langle w, \beta_{n,i}^1 Jx_n + \beta_{n,i}^2 JT_i^n x_n + \beta_{n,i}^3 JS_i^n x_n\rangle \\
&\quad + \|\beta_{n,i}^1 Jx_n + \beta_{n,i}^2 JT_i^n x_n + \beta_{n,i}^3 JS_i^n x_n\|^2 \\
&\leq \|w\|^2 - 2\beta_{n,i}^1\langle w, Jx_n\rangle - 2\beta_{n,i}^2\langle w, JT_i^n x_n\rangle - 2\beta_{n,i}^3\langle w, JS_i^n x_n\rangle \\
&\quad + \beta_{n,i}^1\|x_n\|^2 + \beta_{n,i}^2\|T_i^n x_n\|^2 + \beta_{n,i}^3\|S_i^n x_n\|^2 - \beta_{n,i}^1\beta_{n,i}^3 g(\|Jx_n - JS_i^n x_n\|) \\
&= \beta_{n,i}^1\phi(w, x_n) + \beta_{n,i}^2(1 - \zeta_{n,i})\phi(w, x_n) + \beta_{n,i}^2\xi_{n,i} \\
&\quad + \beta_{n,i}^3(1 + \eta_{n,i})\phi(w, x_n) + \beta_{n,i}^3\varsigma_{n,i} - \beta_{n,i}^1\beta_{n,i}^3 g(\|Jx_n - JS_i^n x_n\|) \\
&= \phi(w, x_n) + (\beta_{n,i}^2\zeta_{n,i} + \beta_{n,i}^3\eta_{n,i})\phi(w, x_n) \\
&\quad + \beta_{n,i}^2\xi_{n,i} + \beta_{n,i}^3\varsigma_{n,i} - \beta_{n,i}^1\beta_{n,i}^3 g(\|Jx_n - JS_i^n x_n\|) \\
&= \phi(w, x_n) + \delta_{n,i}\phi(w, x_n) + \mu_{n,i} - \beta_{n,i}^1\beta_{n,i}^3 g(\|Jx_n - JS_i^n x_n\|) \\
&\leq \phi(w, x_n) + \delta_{n,i}M_n + \mu_{n,i} - \beta_{n,i}^1\beta_{n,i}^3 g(\|Jx_n - JS_i^n x_n\|).
\end{aligned}$$

This implies that

$$\beta_{n,i}^1\beta_{n,i}^3 g(\|Jx_n - JS_i^n x_n\|) \leq \phi(w, x_n) - \phi(w, z_{n,i}) + \delta_{n,i}M_n + \mu_{n,i}. \tag{3.32}$$

Taking limit on both sides of (3.32) and using $\liminf_{n\longrightarrow\infty} \beta_{n,i}^1\beta_{n,i}^3 > 0$ and (3.31), we have

$$g(\|Jx_n - JS_i^n x_n\|) \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{3.33}$$

Therefore, using Lemma 2.3, (3.33) implies that

$$\|Jx_n - JS_i^n x_n\| \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{3.34}$$

Next,

$$\|JS_i^n x_n - Jp\| \leq \|JS_i^n x_n - Jx_n\| + \|Jx_n - Jp\|.$$

Using (3.34) and (3.11) in above inequality, we get

$$\lim_{n\longrightarrow\infty} \|JS_i^n x_n - Jp\| = 0. \tag{3.35}$$

Since J^{-1} is demicontinuous, it follows that $S_i^n x_n \rightharpoonup p$ for each fixed i.

Using $|\|S_i^n x_n\| - \|p\|| = |\|JS_i^n x_n\| - \|Jp\|| \leq \|JS_i^n x_n - Jp\|$ with (3.35), we get $\|S_i^n x_n\| \longrightarrow \|p\|$ for each fixed i as $n \longrightarrow \infty$. Since E enjoys the Kadec-Klee property, we obtain

$$\lim_{n\longrightarrow\infty} \|S_i^n x_n - p\| = 0. \tag{3.36}$$

Similarly, we obtain

$$\lim_{n\longrightarrow\infty} \|T_i^n x_n - p\| = 0. \tag{3.37}$$

Notice that

$$\|S_i^{n+1} x_n - p\| \leq \|S_i^{n+1} x_n - S_i^n x_n\| + \|S_i^n x_n - p\|. \tag{3.38}$$

Since S_i^n is asymptotic regular, then from (3.36) and (3.38), we have

$$\lim_{n\longrightarrow\infty} \|S_i^{n+1} x_n - p\| = 0,$$

that is, $S_i S_i^n x_n - p \longrightarrow 0$ as $n \longrightarrow \infty$. It follows from the closedness of S_i that $S_i p = p$, for each fixed i. Hence $p \in \cap_{i=1}^n \mathrm{Fix}(S_i)$. In a similar way, we can obtain $p \in \cap_{i=1}^n \mathrm{Fix}(T_i)$. Hence $p \in (\cap_{i=1}^n \mathrm{Fix}(T_i)) \cap (\cap_{i=1}^n \mathrm{Fix}(S_i))$.

Next, we prove $p \in \cap_{i=1}^N \mathrm{Sol}(\mathrm{GMVEP}(F_i, A_i, \psi_i, K_i))$.

It follows from (3.25) and $\liminf_{n\longrightarrow\infty} r_{n,i} > 0$ that

$$\lim_{n\longrightarrow\infty} \frac{\|Jy_{n,i} - JT_{r_{n,i}}y_{n,i}\|}{r_{n,i}} = 0.$$

Since $u_{n,i} = T_{r_{n,i}}y_{n,i}$, we have

$$\begin{aligned} F(T_{r_{n,i}}y_{n,i}, y_i) + \langle y_i - T_{r_{n,i}}y_{n,i}, AT_{r_{n,i}}y_{n,i}\rangle &+ \psi(y_i, T_{r_{n,i}}y_{n,i}) - \psi(T_{r_{n,i}}y_{n,i}, T_{r_{n,i}}y_{n,i}) \\ &+ \tfrac{e}{r_{n,i}}\langle y - T_{r_{n,i}}y_{n,i}, JT_{r_{n,i}}y_{n,i} - Jy_{n,i}\rangle \in P,\ \forall y_i \in K_i, \end{aligned}$$

$$\begin{aligned} 0 \in\ & F(y_i, T_{r_{n,i}}y_{n,i}) - \langle y_i - T_{r_{n,i}}y_{n,i}, AT_{r_{n,i}}y_{n,i}\rangle - (\psi(y_i, T_{r_{n,i}}y_{n,i}) - \psi(T_{r_{n,i}}y_{n,i}, T_{r_{n,i}}y_{n,i})) \\ & -\tfrac{e}{r_{n,i}}\langle y_i - T_{r_{n,i}}y_{n,i}, JT_{r_{n,i}}y_{n,i} - Jy_{n,i}\rangle + P,\ \forall y_i \in K_i. \end{aligned}$$

Let $y_{i,t} = (1-t)p + ty_i,\ \forall t \in (0,1]$. Since $y_i \in K_i$ and $p \in K_i$, we get $y_{i,t} \in K_i$ and hence

$$\begin{aligned} 0 &\in F(y_i, T_{r_{n,i}}y_{n,i}) - \langle y_{i,t} - T_{r_{n,i}}y_{n,i}, AT_{r_{n,i}}y_{n,i}\rangle - (\psi(y_{i,t}, T_{r_{n,i}}y_{n,i}) - \psi(T_{r_{n,i}}y_{n,i}, T_{r_{n,i}}y_{n,i})) \\ &\quad -\tfrac{e}{r_{n,i}}\langle y_{i,t} - T_{r_{n,i}}y_{n,i}, JT_{r_{n,i}}y_{n,i} - Jy_{n,i}\rangle + P, \\ &= F(y_{i,t}, T_{r_{n,i}}y_{n,i}) - \langle y_{i,t} - T_{r_{n,i}}y_{n,i}, AT_{r_{n,i}}y_{n,i}\rangle - (\psi(y_{i,t}, T_{r_{n,i}}y_{n,i}) - \psi(T_{r_{n,i}}y_{n,i}, T_{r_{n,i}}y_{n,i})) \\ &\quad -e\langle y_{i,t} - T_{r_{n,i}}y_{n,i}, \tfrac{JT_{r_{n,i}}y_{n,i} - Jy_{n,i}}{r_{n,i}}\rangle + P. \\ 0 &\in F(y_{i,t}, p) - \langle y_{i,t} - p, Ap\rangle - \psi(y_{i,t}, p) + \psi(p,p) + P. \end{aligned}$$

It follows from Assumptions 2.6 (i), (iv) and (vi) that

$$\begin{aligned} tF(y_{i,t}, y_i) + (1-t)F(y_t, p) + t\psi(y_i, p) + (1-t)\psi(p,p) - \psi(y_{i,t}, p) &\in F(y_{i,t}, y_t) + \psi(y_{i,t}, p) \\ &\quad -\psi(y_{i,t}, p) + P \\ &\in P. \end{aligned}$$

Now,

$-t[F(y_{i,t}, y_i) + \psi(y_i, p) - \psi(y_{i,t}, p) - \langle y_{i,t} - p, Ap\rangle] -$
$(1-t)[F(y_{i,t}, p) + \psi(p,p) - \psi(y_{i,t}, p) - \langle y_{i,t} - p, Ap\rangle] \in -P + \langle y_{i,t} - p, Ap\rangle$
$-t[F(y_{i,t}, y_i) + \psi(y_i, p) - \psi(y_{i,t}, p) - \langle y_{i,t} - p, Ap\rangle] - (1-t)P \in -P + \langle y_{i,t} - p, Ap\rangle$, (using (3.7))
$-t[F(y_{i,t}, y_i) + \psi(y_i, p) - \psi(y_{i,t}, p) - \langle y_{i,t} - p, Ap\rangle] \in -P + (1-t)P + \langle y_{i,t} - p, Ap\rangle$
$-t[F(y_{i,t}, y_i) + \psi(y_i, p) - \psi(y_{i,t}, p) - \langle y_{i,t} - p, Ap\rangle] \in -tP + \langle y_{i,t} - p, Ap\rangle$
$\in -tP + \langle ty_i + (1-t)p - p, Ap\rangle \ \in -tP + t\langle y_i - p, Ap\rangle$

$$F(y_{i,t}, y_i) + \psi(y_i, p) - \psi(y_{i,t}, p) - \langle y_{i,t} - p, Ap\rangle \in P - \langle y_i - p, Ap\rangle.$$

Letting $t \longrightarrow 0_+$, we have

$$F(p, y_i) + \psi(y_i, p) - \psi(p,p) - \langle p - p, Ap\rangle + \langle y_i - p, Ap\rangle \in P$$

$$F(p, y_i) + \langle y_i - p, Ap\rangle + \psi(y_i, p) - \psi(p,p) \in P.$$

Thus $p \in \mathrm{Sol}(\mathrm{GMVEP}(F_i, A_i, \psi_i, K_i))$. Hence $p \in \Gamma$.

Finally, we prove that $p = \prod_\Gamma x_0$. Since $\Gamma \subset C_{n+1}$ and $x_{n+1} = \prod_{C_{n+1}} x_0$, we have

$$\langle x_{n+1} - q, Jx_0 - Jx_{n+1}\rangle \geq 0,\ \forall q \in \Gamma. \tag{3.39}$$

By taking the limit in (3.39), we have

$$\langle p - q, Jx_0 - Jp\rangle \geq 0,\ \forall q \in \Gamma.$$

Hence, in view of Lemma 2.1, we see that $p = \prod_\Gamma x_0$. This completes the proof. □

Finally we give some consequences of Theorem 3.3.

Corollary 3.4. *Let E be a uniformly smooth and strictly convex Banach space such that E has Kadec-Klee property. Let K be a nonempty, compact and convex subset of E. Assume that P is a pointed, proper, closed and convex cone of a real ordered Banach space Y with $\mathrm{int}P \neq \emptyset$. Let the mappings $F, \psi : K \times K \longrightarrow Y$ satisfy Assumption 2.6 and $A : K \longrightarrow B(E, Y)$ be continuous and P-monotone mapping. Let $S, T : K \longrightarrow E$ be closed, asymptotically regular and generalized asymptotically quasi ϕ-nonexpansive mappings with the sequences $\{\eta_n\}$, $\{\varsigma_n\}$ and $\{\zeta_n\}$, $\{\xi_n\}$ such that $\Gamma := \mathrm{Fix}(S) \cap (\mathrm{Fix}(T)) \cap (\mathrm{Sol}(\mathrm{GMVEP}(1.2))) \neq \emptyset$. Assume that, the sequences $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\{\alpha_n\} \subseteq (0, 1)$ and $\{\beta_n^j\} \subseteq (0, 1)$ $(j = 1, 2, 3)$ be such that*

(i) $\beta_n^1 + \beta_n^2 + \beta_n^3 = 1$;
(ii) $\liminf_{n\longrightarrow\infty} \beta_n^1\beta_n^2 > 0$ *and* $\liminf_{n\longrightarrow\infty} \beta_n^1\beta_n^3 > 0$;
(iii) $\limsup_{n\longrightarrow\infty} \alpha_n < 1$;
(iv) $\liminf_{n\longrightarrow\infty} r_n > 0$.

Let $\{x_n\}$ be a sequence generated by the iterative scheme:

$$\begin{aligned}
&x_0 \in E,\\
&x_1 = \textstyle\prod_{C_1} x_0,\\
&y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n),\\
&z_n = J^{-1}(\beta_n^1 Jx_n + \beta_n^2 JT^n x_n + \beta_n^3 JS^n x_n),\\
&u_n = T_{r_n}(y_n)\\
&C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n) + \delta_n M_n + \mu_n\},\\
&x_{n+1} = \textstyle\prod_{C_{n+1}} x_0, \text{ for every } n \in N \cup \{0\},
\end{aligned}$$

where $M_n = \sup\{\phi(p, x_n) : p \in \Gamma\}$; $e \in \mathrm{int}P$; $J : E \longrightarrow E^*$ is the normalized duality mapping with its inverse J^{-1}; $\delta_n = \beta_n^2\zeta_n + \beta_n^3\eta_n$ and $\mu_n = \xi_n\beta_n^2 + \varsigma_n\beta_n^3$.
Then the sequence $\{x_n\}$ converges strongly to $\prod_\Gamma x_0$.

Proof. The proof follows by taking $i = 1$ in Theorem 3.3. □

Corollary 3.5. *Let E be a uniformly smooth and strictly convex Banach space such that E has Kadec-Klee property. For each $i \in I := \{1, 2, 3, ..., N\}$, let K_i be a nonempty, compact and convex subset of E such that $K = \cap_{i=1}^N K_i \neq \emptyset$. Let for each i, the mappings $F_i, \psi_i : K_i \times K_i \longrightarrow \mathbb{R}$ satisfy Assumption 2.6 and $A_i : K_i \longrightarrow E^*$ be continuous and monotone mapping. For each fixed i, let $S_i, T_i : K_i \longrightarrow E$ be closed, asymptotically regular and generalized asymptotically quasi ϕ-nonexpansive mappings with the sequences $\{\eta_{n,i}\}$, $\{\varsigma_{n,i}\}$ and $\{\zeta_{n,i}\}$, $\{\xi_{n,i}\}$ such that $\Gamma := (\cap_{i=1}^N \mathrm{Fix}(S_i)) \cap (\cap_{i=1}^N \mathrm{Fix}(T_i)) \cap (\cap_{i=1}^N \mathrm{Sol}(\mathrm{SUGMEP}(1.7))) \neq \emptyset$. Assume that, for each fixed i, the sequences $\{r_{n,i}\} \subset [a, \infty)$ for some $a > 0$, $\{\alpha_{n,i}\} \subseteq (0, 1)$ and $\{\beta_{n,i}^j\} \subseteq (0, 1)$ $(j = 1, 2, 3)$ be such that*

(i) $\beta_{n,i}^1 + \beta_{n,i}^2 + \beta_{n,i}^3 = 1$;
(ii) $\liminf_{n\longrightarrow\infty} \beta_{n,i}^1\beta_{n,i}^2 > 0$ *and* $\liminf_{n\longrightarrow\infty} \beta_{n,i}^1\beta_{n,i}^3 > 0$;
(iii) $\limsup_{n\longrightarrow\infty} \alpha_{n,i} < 1$;
(iv) $\liminf_{n\longrightarrow\infty} r_{n,i} > 0$.

Let $\{x_n\}$ be a sequence generated by the iterative scheme:

$$\begin{aligned}
&x_0 \in E,\\
&C_{1,i} =: K_i,\ C_1 = \cap_{i=1}^{N} C_{1,i} = K,\\
&x_1 = \textstyle\prod_{C_1} x_0,\\
&y_{n,i} = J^{-1}(\alpha_{n,i} J x_n + (1-\alpha_{n,i}) J z_{n,i}),\\
&z_{n,i} = J^{-1}(\beta^1_{n,i} J x_n + \beta^2_{n,i} J T_i^n x_n + \beta^3_{n,i} J S_i^n x_n),\\
&u_{n,i} = T_{r_{n,i}}(y_{n,i})\\
&C_{n+1,i} = \{v \in C_{n,i} : \phi(v, u_{n,i}) \le \phi(v, x_n) + \delta_{n,i} M_n + \mu_{n,i}\},\\
&C_{n+1} = \cap_{i\in I} C_{n+1,i},\\
&x_{n+1} = \textstyle\prod_{C_{n+1}} x_0,\ \text{for every } n \in N \cup \{0\},
\end{aligned}$$

where $M_n = \sup\{\phi(p, x_n) : p \in \Gamma\}$; $e \in \mathrm{int}P$; $J : E \longrightarrow E^*$ is the normalized duality mapping with its inverse J^{-1}; $\delta_{n,i} = \beta^2_{n,i}\zeta_{n,i} + \beta^3_{n,i}\eta_{n,i}$ and $\mu_{n,i} = \xi_{n,i}\beta^2_{n,i} + \varsigma_{n,i}\beta^3_{n,i}$. Then the sequence $\{x_n\}$ converges strongly to $\prod_\Gamma x_0$.

Proof. The proof follows by taking $Y = \mathbb{R},\ P = [0,\infty)$ in Theorem 3.3. □

Corollary 3.6. *Let E be a uniformly smooth and strictly convex Banach space such that E has Kadec-Klee property. Let K be a nonempty, compact and convex subset of E. Let the mappings $F, \psi : K \times K \longrightarrow \mathbb{R}$ satisfy Assumption 2.6 and $A : K \longrightarrow E^*$ be continuous and monotone mapping. Let $S, T : K \longrightarrow E$ be closed, asymptotically regular and generalized asymptotically quasi ϕ-nonexpansive mappings with the sequences $\{\eta_n\}$, $\{\varsigma_n\}$ and $\{\zeta_n\}$, $\{\xi_n\}$ such that $\Gamma := \mathrm{Fix}(S) \cap (\mathrm{Fix}(T)) \cap (\mathrm{Sol}(\mathrm{GMEP}(1.4))) \neq \emptyset$. Assume that, the sequences $\{r_n\} \subset [a,\infty)$ for some $a > 0$, $\{\alpha_n\} \subseteq (0,1)$ and $\{\beta^j_n\} \subseteq (0,1)$ $(j = 1,2,3)$ be such that*

- (i) $\beta^1_n + \beta^2_n + \beta^3_n = 1$;
- (ii) $\liminf_{n\longrightarrow\infty} \beta^1_n\beta^2_n > 0$ *and* $\liminf_{n\longrightarrow\infty} \beta^1_n\beta^3_n > 0$;
- (iii) $\limsup_{n\longrightarrow\infty} \alpha_n < 1$;
- (iv) $\liminf_{n\longrightarrow\infty} r_n > 0$.

Let $\{x_n\}$ be a sequence generated by the iterative scheme:

$$\begin{aligned}
&x_0 \in E,\\
&x_1 = \textstyle\prod_{C_1} x_0,\\
&y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n),\\
&z_n = J^{-1}(\beta^1_n J x_n + \beta^2_n J T^n x_n + \beta^3_n J S^n x_n),\\
&u_n = T_{r_n}(y_n)\\
&C_{n+1} = \{v \in C_n : \phi(v, u_n) \le \phi(v, x_n) + \delta_n M_n + \mu_n\},\\
&x_{n+1} = \textstyle\prod_{C_{n+1}} x_0,\ \text{for every } n \in N \cup \{0\},
\end{aligned}$$

where $M_n = \sup\{\phi(p, x_n) : p \in \Gamma\}$; $e \in \mathrm{int}P$; $J : E \longrightarrow E^*$ is the normalized duality mapping with its inverse J^{-1}; $\delta_n = \beta^2_n\zeta_n + \beta^3_n\eta_n$ and $\mu_n = \xi_n\beta^2_n + \varsigma_n\beta^3_n$. Then the sequence $\{x_n\}$ converges strongly to $\prod_\Gamma x_0$.

Proof. The proof follows by taking $Y = \mathbb{R},\ P = [0,\infty),\ i = 1$ in Theorem 3.3. □

The following corollary is similar to Qin and Agrawal [21].

Corollary 3.7. *Let E be a uniformly smooth and strictly convex Banach space such that E has Kadec-Klee property. Let K be a nonempty, compact and convex subset of E. Let $S, T : K \longrightarrow E$ be closed, asymptotically regular and generalized asymptotically quasi ϕ-nonexpansive mappings with the sequences $\{\eta_n\}$, $\{\varsigma_n\}$ and $\{\zeta_n\}$, $\{\xi_n\}$ such that $\Gamma := \mathrm{Fix}(S) \cap (\mathrm{Fix}(T)) \neq \emptyset$. Assume that, the sequences $\{r_n\} \subset [a,\infty)$ for some $a > 0$, $\{\alpha_n\} \subseteq (0,1)$ and $\{\beta^j_n\} \subseteq (0,1)$ $(j = 1,2,3)$ be such that*

- (i) $\beta^1_n + \beta^2_n + \beta^3_n = 1$;

(ii) $\liminf_{n\longrightarrow\infty} \beta_n^1\beta_n^2 > 0$ *and* $\liminf_{n\longrightarrow\infty} \beta_n^1\beta_n^3 > 0$;
(iii) $\limsup_{n\longrightarrow\infty} \alpha_n < 1$.

Let $\{x_n\}$ *be a sequence generated by the iterative scheme:*

$$\begin{aligned}
&x_0 \in E,\\
&x_1 = \textstyle\prod_{C_1} x_0,\\
&y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n),\\
&z_n = J^{-1}(\beta_n^1 Jx_n + \beta_n^2 JT^n x_n + \beta_n^3 JS^n x_n),\\
&C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n) + \delta_n M_n + \mu_n\},\\
&x_{n+1} = \textstyle\prod_{C_{n+1}} x_0, \text{ for every } n \in N \cup \{0\},
\end{aligned}$$

where $M_n = \sup\{\phi(p, x_n) : p \in \Gamma\}$; $e \in \text{int}P$; $J : E \longrightarrow E^*$ *is the normalized duality mapping with its inverse* J^{-1}; $\delta_n = \beta_n^2\zeta_n + \beta_n^3\eta_n$ *and* $\mu_n = \xi_n\beta_n^2 + \varsigma_n\beta_n^3$.
Then the sequence $\{x_n\}$ *converges strongly to* $\prod_\Gamma x_0$.

Proof. The proof follows by taking $Y = \mathbb{R},\ P = [0,\infty),\ i = 1,$ and $F = 0,\ A = 0,\ \phi = 0$ in Theorem 3.3. □

References

1. Y. I. Alber, Metric and generalized projection operators in Banach spaces: Properties and applications, pp. 15-50. In: A.G. Kartosatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996.
2. A. S. Antipin, Iterative gradient predictor type methods for computing fixed point of external mappings, in: J. Guddat, H.Th. Jonden, F. Nizicka, G. Still, F. Twitt (Eds.) Parametric Optimization and related topics IV[C], Peter Lang, Frankfurt Main, 1997, pp.11-24.
3. E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Stud. 63 (1994), 123-145.
4. L. C. Ceng and J. C. Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems, J. Comput. Appl. Math. 214 (2008), 186-201.
5. Y. Censor, A. Gibali, S. Reich and S. Sabach, Common solutions to variational inequalities, Set-Valued Var. Anal. 20 (2012), 229-247.
6. Y. J. Cho, H. Y. Zhou and G. Guo, Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings , Comput. Math. Appl. 47 (2004), 707-717.
7. P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (2005), 117-136.
8. B. Djafari-Rouhani, K. R. Kazmi and S. H. Rizvi, A hybrid-extragradient convex approximation method for a system of unrelated mixed equilibrium problems, Transactions on mathematical programming and applications, 1(2013) Number18, 82-95.
9. F. Giannessi, Vector variational inequalities and vector equilibria. Mathematical theories. Nonconvex Optimization and its Applications, vol. 38, Kluwer Academic Publishers, Dordrecht, 2000.
10. X. H. Gong and H. M. Yue, Existence of efficient solutions and strong solutions for vector equilibrium problems, J. Nanchang Univ. 32 (2008), 1-5.
11. K. R. Kazmi, On vector equilibrium problem, Proc. Indian Acad. Sci. (Math. Sci.) 110 (2000), 213-223.
12. K. R. Kazmi, A. Khaliq and A. Raouf, Iterative approximation of solution of generalized mixed set-valued variational inequality problem, Math. Inequal. Appl. 10 (2007), 677-691.
13. K. R. Kazmi and F. A. Khan, Existence and iterative approximation of solutions of generalized mixed equilibrium problems, Comput. Math. Appl. 56(2008), 1314-1321.
14. K. R. Kazmi, F. A. Khan and Mohammad Shahzad, Two step algorithm for solving regularized generalized mixed variational inequality problem, Bull. Korean Math. Soc. 47 (2010), 675-685.
15. K. R. Kazmi and S. A. Khan, Existence of solutions to a generalized system, J. Optim. Theory Appl. 142 (2009), 355-361.
16. K. R. Kazmi and Mohammad Farid, Some iterative schemes for generalized vector equilibrium problems and relatively nonexpansive mappings in Banach spaces, Mathematical Sciences (2013), 7:19.
17. K. R. Kazmi and A. Raouf, A class of operator equilibrium problems, J. Math. Anal. Appl. 308 (2005), 554-564.

18. K. R. Kazmi and S. H. Rizvi, A hybrid extragradient method for approximating the common solutions of a variational inequality, a system of variational inequalities, a mixed equilibrium problem and a fixed point problem, Appl. Math. Comput. 218 (2012), 5439-5452.
19. D. T. Luc, Theory of Vector Optimization Lecture Notes in Economics and Mathematical Systems, 319, Springer-Verlag, Berlin, 1989.
20. S. Matsushita and W. Takahashi, Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces, Fixed Point Theory Appl. 2004 (2004), 37-47.
21. X. Qin and R. P. Agrawal, Shrinking projection methods for a pair of asymptptically quasi-ϕ-nonexpansive mappings, Numer. Funct. Anal. Optim.: 31 (2010), 1072-1089. doi:10.1080/01630563.2010.501643.
22. X. Qin, R. P. Agrawal, S. Y. Cho and S. M. Kang, Convergence of algorithms for fixed points of generalized asymptptically quasi-ϕ-nonexpansive mappings with applications, Fixed Point Theory Appl. Optim.: 2012 2012:58, doi:10.1080/01630563.2010.501643.
23. S. Q. Shan and N. J. Huang, An iterative method for generalized mixed vector equilibrium problems and fixed point of nonexpansive mappings and variational inequalities, Taiwanese J. Math. 16(2012) Number 5, 1681-1705.
24. J. Song and M. Chen, On generalized asymptotically quasi-ϕ-nonexpansive mappings and a Ky Fan inequality, Fixed Point Theory Appl. 2013. doi:10.1186/1687-1812-2013-237.
25. A. Tada and W. Takahashi, Strong convergence theorem for an equilibrium problem and a nonexpansive mapping, pp. 609-617. In: W. Takahashi, T. Tanaka (Eds.), Nonlinear Analysis and Convex Analysis, Yokohama, Yokohama, 2006.
26. S. Takahashi and W. Takahashi, Viscosity approximation method for equilibrium problems and fixed point problems in Hilbert space, J. Math. Anal. Appl. 331 (2007), 506-515.
27. W. Takahashi and K. Zembayashi, Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces, Nonlinear Anal. 70 (2009), 45-57.
28. N. X. Tan, On the existence of solution of quasivariational inclusion problems, J. Optim. Theory Appl. 123 (2004), 619-638.
29. H. Zhou, G. Gao and B. Tan, Convergence theorems of a modified hybrid algorithm for a family of quasi-ϕ-asymptptically nonexpansive mappings, J. Appl. Math. Comput. 32 (2010), 453-464. doi:10.1007/s12190-009-0263-4.