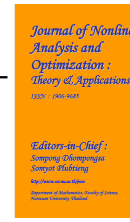

Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol.7, No.1, (2016), 41-54
ISSN : 1906-9685
http://www.math.sci.nu.ac.th

Q-HYPERCONVEXITY IN QUASI-CONE METRIC SPACES AND FIXED POINT THEOREMS.

YAÉ ULRICH GABA*

Department of Mathematics and Applied Mathematics, University of Cape Town, South Africa.

ABSTRACT.
In this article, we introduce the concept of a q-hyperconvexity for quasi-cone metric spaces and generalise some fixed point theorems that we take from [8] and [11]. Mainly, we give some fixed point results for a pair of maps of Jungck type. Moreover we prove that quasi-cone metric spaces are topological spaces and we give a characterization of bounded sets in such spaces.

KEYWORDS :quasi-cone metric space; bicomplete space; q-Hyperconvexity.
AMS Subject Classification: 47H03, 47H10

1. INTRODUCTION AND PRELIMINARIES

Cone metric spaces were introduced in [7] and many fixed point results concerning mappings in such spaces have been established. Basically, cone metric spaces are defined by substituting, in the definition of a metric, the real line by a real Banach space that we endow with a partial order. In [14], Fawzia et al. discussed the newly introduced notion of quasi-cone metric spaces and proved some fixed point results for mappings on such spaces. Recently in [8], E. F. Kazeem et al defined a new type of completeness for quasi-cone metric spaces and the first related fixed point results. Quasi-cone metric spaces are just an asymmetric version of cone metric spaces which generalize the latter. Therefore, in light of what is done in [11], we also introduce a concept of convexity for quasi-cone metric spaces that we call q-hyperconvexity and discuss related properties. Most of the results follow the classical ones, but generalize them.

For all the key and recent results concerning cone metric spaces, the reader is advised to read [2, 5, 7, 8, 13].

* *Corresponding author.*
Email address : gabayae2@gmail.com.
Article history : Received June 11, 2014 Accepted August 14, 2015.

Definition 1.1. Let E be a real Banach space with norm $\|.\|$ and P be a subset of E. Then P is called a cone if and only if

(i) P is closed, nonempty and $P \neq \{\theta\}$, where θ is the zero vector in E;
(ii) for any $a, b \geq 0$, and $x, y \in P$, we have $ax + by \in P$;
(iii) for $x \in P$, if $-x \in P$, then $x = \theta$.

Given a cone P in a Banach space E, we define on E a partial order $\preceq$ with respect to P by

$$x \preceq y \Longleftrightarrow y - x \in P.$$

We also write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in Int(P)$ (where $Int(P)$ designates the interior of P).
The cone P is called **normal** if there is a number $C > 0$, such that for all $x, y \in E$, we have

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq C\|y\|.$$

The least positive number satisfying this inequality is called the **normal constant** of P. Therefore, we shall then say that P is a K-normal cone to indicate the fact that the normal constant is K.

Definition 1.2. Let E be a Banach space, P a cone on E and $\preceq$ the partial order defined by P. A subset $F \subset E$ is said to be **bounded from above with respect to** P if there exists $e \in E$ such that for all $f \in F,\ f \preceq e$.

Definition 1.3. (Compare [4]) A cone P is said to be **minihedral** if $x \vee y := \sup\{x, y\}$ exists for all $x, y \in E$ and **strongly minihedral** if every subset of E which is bounded from above with respect to P has a supremum.

Lemma 1.4. *(Compare[7, 12]) Let (X, q) be a quasi-cone metric space over a cone P. Then we have;*

a) *$Int(P) + Int(P) \subset Int(P)$ and $\lambda Int(P) \subset Int(P)$ for any positive real number λ.*
b) *For any given $c \gg \theta$ and $c_0 \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} c_0 \ll c$.*
c) *If (a_n) and (b_n) are sequences in E such that $a_n \longrightarrow a$, $b_n \longrightarrow b$ and $a_n \preceq b_n$ for all $n \geq 1$, then $a \preceq b$.*

Definition 1.5. (Compare[8]) Let X be a nonempty set. Suppose the mapping $q : X \times X \longrightarrow E$ satisfies

(q1) $\theta \preceq q(x, y)$ for all $x, y \in X$;
(q2) $q(x, y) = \theta = q(y, x)$ if and only if $x = y$;
(q3) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$. Then, q is called a **quasi-cone metric** on X, and (X, q) is called a **quasi-cone metric space**.
Moreover, if q satisfies
(q4) $q(x, y) = q(y, x)$ for all $x, y \in X$; then (X, q) is called a **cone metric space** in the sense of [7].

Remark 1.6. Let q be a quasi-cone metric on X, then the map q^{-1} defined by $q^{-1}(x, y) = q(y, x)$ whenever $x, y \in X$ is also a quasi-cone metric on X, called the **conjugate** of q. It can also be denoted by q^t or $\bar{q}$.

Definition 1.7. (Compare [8]) A sequence (x_n) in a quasi-cone metric space (X, q) is called

(i) Q-**Cauchy** or **bi-Cauchy** if for every $c \in X$ with $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall\, n, m \geq n_0 \quad q(x_n, x_m) \ll c;$$

(ii) **left(resp. right) Cauchy** if for every $c \in X$ with $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall\, n, m : n_0 \leq m \leq n \quad q(x_m, x_n) \ll c \text{ (resp. } q(x_n, x_m) \ll c \text{).}$$

Remark 1.8. A sequence is Q-Cauchy if and only if it is both left and right Cauchy.

Definition 1.9.

(i) In a quasi-cone metric space (X, q), we say that the sequence (x_n) **left converges** to $x \in X$ if for every $c \in E$ with $\theta \ll c$ there exists N such that for all $n > N$, $q(x_n, x) \ll c$.
(ii) Similarly, in a quasi-cone metric space (X, q), we say that a sequence (x_n) **right converges** to $x \in X$ if for every $c \in E$ with $\theta \ll c$ there exists N such that for all $n > N$, $q(x, x_n) \ll c$.
(iii) Finally, in a quasi-cone metric space (X, q), we say that the sequence (x_n) **converges** to $x \in X$ if for every $c \in E$ with $\theta \ll c$ there exists N such that for all $n > N$, $q(x_n, x) \ll c$ and $q(x, x_n) \ll c$.

Definition 1.10. (Compare [8]) A quasi-cone metric space (X, q) is called

(i) **left complete** (resp. **right complete**) if every left Cauchy(resp. right Cauchy) sequence in X left(resp. right) converges.
(ii) **bicomplete** if every Q-Cauchy sequence converges.

Remark 1.11. A quasi-cone metric space (X, q) is bicomplete if and only if it is left complete and right complete.

Definition 1.12. (Compare [8]) Let (X, q) be a quasi-cone metric space. A function $f : X \longrightarrow X$ is said to be **lipschitzian** if there exists some $\kappa \in \mathbb{R}$ such that

$$q(f(x), f(y)) \preceq \kappa q(x, y) \qquad \forall\, x, y \in X.$$

The smallest constant which satisfies the above inequality is called the **Lipschitz constant** of f and is denoted $Lip(f)$. In particular f is said to be **contractive** if $Lip(f) \in [0, 1)$ and **expansive** if $Lip(f) = 1$.

Definition 1.13. (Compare [2]) Let f and g be self maps on a set X. If $w = fx = gx$ for some $x \in X$, then x is called a **coincidence point** of f and g, and w is called the **point of coincidence** of f and g.

Definition 1.14. Let f and g be self maps on a nonempty set X. We say that f and g are **weakly compatible** if they commute at their coincidence point, that is there exists $x_0 \in X$ such that $fx_0 = gx_0$ then $gfx_0 = fgx_0$.

We also give the following proposition that we take from [2] by omitting the proof.

Proposition 1.15. *(Compare [2]) Let f and g be weakly compatible self maps on a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Lemma 1.16. *(Compare [8]) Let (X, q) be a quasi-cone metric space, P be a K-normal cone. Let (x_n) be a sequence in X. Then (x_n) converges to x if and only if $q(x_n, x) \longrightarrow \theta$ $(n \longrightarrow \infty)$ and $q(x, x_n) \longrightarrow \theta$ $(n \longrightarrow \infty)$.*

Remark 1.17. In fact, a sequence (x_n) left-converges (resp. right-converges) to x if and only if $q(x_n, x) \longrightarrow \theta$ (resp $q(x, x_n) \longrightarrow \theta$) $(n \longrightarrow \infty)$.

Lemma 1.18. *(Compare [8]) Let (X,q) be a quasi-cone metric space and (x_n) be a sequence in X. If (x_n) converges to x, then (x_n) is a bi-Cauchy sequence.*

Lemma 1.19. *(Compare [8]) Let (X,q) be a quasi-cone metric space, P be a K-normal cone and (x_n) be a sequence in X. Then (x_n) is a bi-Cauchy sequence if and only if $q(x_n,x_m) \longrightarrow \theta$ as $n,m \longrightarrow \infty$.*

---

2. Common fixed points results

Lemma 2.1. *Let (X,q) be a quasi-cone metric space, P be a K-normal cone. Let (x_n) be a sequence in X. If (x_n) converges to x and (x_n) converges to y, then $x=y$. That is the limit of (x_n) is unique.*

Proof. For any $n \geq 0$, $q(x,y) \preceq q(x,x_n)+q(x_n,y)$ which entails that $q(x,y)=\theta$. Similarly $q(y,x)=\theta$, hence by property $(q2)$ $x=y$.

□

Theorem 2.2. *Let (X,q) be a quasi-cone metric space, P a K-normal cone. Suppose that mappings $f,g : X \longrightarrow X$ satisfy the contractive condition*

$$q(fx,fy) \preceq k\; q(gx,gy) \qquad \textit{for all } \; x,y \in X,$$

where $k \in (0,1)$. If the range of g contains the range of f and $g(X)$ is bicomplete, then f and g have a unique point of coincidence. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Take an arbitrary $x_0 \in X$. Choose a point x_1 in X such that $f(x_0)=g(x_1)$. This can be done, since $f(X) \subseteq g(X)$. Iterating this process, once x_n is chosen in X, we can obtain x_{n+1} in X such that $f(x_n)=g(x_{n+1})$. Then

$$\begin{aligned} q(gx_n,gx_{n+1}) &= q(fx_{n-1},fx_n) \preceq kq(gx_{n-1},gx_n) \\ &\preceq k^2 q(gx_{n-2},gx_{n-1}) \preceq \ldots \preceq \; k^n q(gx_0,gx_1). \end{aligned}$$

i.e.

$$q(gx_n,gx_{n+1}) \preceq \; k^n q(gx_0,gx_1).$$

Similarly,

$$q(gx_{n+1},gx_n) \preceq \; k^n q(gx_1,gx_0).$$

So for $n<m$,

$$\begin{aligned} q(gx_n,gx_m) &\preceq q(gx_n,gx_{n+1})+q(gx_{n+1},gx_{n+2})+\ldots+q(gx_{m-1},gx_m) \\ &\preceq (k^n+k^{n+1}+\ldots+k^{m-1})q(gx_0,gx_1) \preceq \frac{k^n}{1-k}q(gx_0,gx_1). \end{aligned}$$

It entails that $\|q(gx_n,gx_m)\| \leq K\frac{k^n}{1-k}\|q(gx_0,gx_1)\| \longrightarrow 0$ as $n \longrightarrow \infty$.
Similarly for $n>m$

$$q(gx_n,gx_m) \preceq \frac{k^m}{1-k}q(gx_1,gx_0).$$

It entails that $\|q(gx_n,gx_m)\| \leq K\frac{k^m}{1-k}\|q(gx_1,gx_0)\| \longrightarrow 0$ as $m \longrightarrow \infty$. Hence (gx_n) is a bi-Cauchy sequence. Since $g(X)$ is bicomplete, there exists $x^* \in g(X)$ such that (gx_n) converges to x^*. In other words, there is a $p^* \in X$ such that (gx_n) converges to $g(p^*)=x^*$.
Moreover since

$$q(gx_n,fp^*) = q(fx_{n-1},fp^*) \preceq kq(gx_{n-1},gp^*),$$

we get that

$$\|q(gx_n, fp^*)\| \leq Kk\|q(gx_{n-1}, gp^*)\| \longrightarrow 0, \text{ as } n \longrightarrow \infty,$$

hence $q(gx_n, fp^*) \longrightarrow \theta$ as $n \longrightarrow \infty$. In the same way, we establish that $q(fp^*, gx_n) \longrightarrow \theta$ as $n \longrightarrow \infty$, to then conclude that $gx_n \longrightarrow fp^*$. The uniqueness of the limit implies that $fp^* = gp^*$. We finish the proof by showing that f and g have a unique point of coincidence. For this, assume $z^* \in X$ is a point such that $fz^* = gz^*$. Now

$$q(gz^*, gp^*) = q(fz^*, fp^*) \preceq kq(gz^*, gp^*),$$

which gives $q(gz^*, gp^*) = \theta$. On the other hand, by the same reasoning, it also clear that $q(gp^*, gz^*) = \theta$. By property $(q2)$, $gz^* = gp^*$. From Proposition 1.15, f and g have a unique common fixed point. □

Corollary 2.3. *Taking $g = I$(Identity map) in Theorem 2.2, we obtain Theorem 4.1 of* [8].

Remark 2.4. Let (X, q) be a bicomplete quasi-cone metric space, P be a K-normal cone. Assume $f, g : X \longrightarrow X$ satisfy the contractive condition

$$q(f^n x, f^n y) \preceq kq(gx, gy), \quad \text{for all } \ x, y \in X,$$

for some positive integer, where $k \in [0, 1)$ is a constant. If the range of g contains the range of f^n and $g(X)$ is a bicomplete quasi-cone metric space, then by the theorem above, there exists a unique $x^* \in X$ such that $f^n x^* = gx^* = x^*$. Observe that $f^n(fx^*) = f(f^n x^*) = fx^*$, so fx^* is also a fixed point of f^n. If f and g commute at x^*, i.e. $fgx^* = gfx^*$, then f and g have a unique common fixed point. This is easily seen by observing that $gfx^* = fgx^* = fx^*$.

Theorem 2.5. *Let (X, q) be a quasi-cone metric space, P a K-normal cone. Suppose that mappings $f, g : X \longrightarrow X$ satisfy the contractive condition*

$$q(fx, fy) \preceq k\left[q(fx, gy) + q(gx, fy)\right] \qquad \text{for all } \ x, y \in X,$$

where $k \in \left(0, \frac{1}{2}\right)$. If the range of g contains the range of f and $g(X)$ is bicomplete , then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Take an arbitrary $x_0 \in X$. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since $f(X) \subseteq g(X)$. Iterating this process, once x_n is chosen in X, we can obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} q(gx_n, gx_{n+1}) &= q(fx_{n-1}, fx_n) \preceq k[q(fx_{n-1}, gx_n) + q(gx_{n-1}, fx_n)] \\ &\preceq kq(gx_{n-1}, gx_{n+1}) \\ &\preceq k[q(gx_{n-1}, gx_n) + q(gx_n, gx_{n+1})], \end{aligned}$$

which entails that

$$q(gx_n, gx_{n+1}) \preceq \frac{k}{1-k} q(gx_{n-1}, gx_n).$$

Similarly,

$$q(gx_{n+1}, gx_n) \preceq \frac{k}{1-k} q(gx_n, gx_{n-1}).$$

So for $n < m$,

$$q(gx_n, gx_m) \preceq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \ldots + q(gx_{m-1}, gx_m)$$

$$\preceq (h^n + h^{n+1} + \ldots + h^{m-1})q(gx_0, gx_1) \preceq \frac{h^n}{1-h}q(gx_0, gx_1),$$

where $h = \frac{k}{1-k}$. It entails that $\|q(gx_n, gx_m)\| \leq K\frac{h^n}{1-h}\|q(gx_0, gx_1)\| \longrightarrow 0$ as $n \longrightarrow \infty$.
Similarly for $n > m$,

$$q(gx_n, gx_m) \preceq \frac{h^m}{1-h}q(gx_1, gx_0).$$

It entails that $\|q(gx_n, gx_m)\| \leq K\frac{h^m}{1-h}\|q(gx_1, gx_0)\| \longrightarrow 0$ as $m \longrightarrow \infty$. Hence (gx_n) is a bi-Cauchy sequence. Since $g(X)$ is bicomplete, there exists $x^* \in g(X)$ such that (gx_n) converges to x^*. In other words, there is a $p^* \in X$ such that (gx_n) converges to $g(p^*) = x^*$.
Moreover since

$$q(gx_n, fp^*) = q(fx_{n-1}, fp^*) \preceq k[q(fx_{n-1}, gp^*) + q(gx_{n-1}, fp^*)],$$

we get that

$$q(gp^*, fp^*) \preceq kq(gp^*, fp^*)$$

which implies that $q(gp^*, fp^*) = \theta$.
In the same way, we establish that $q(fp^*, gp^*) = \theta$, to then conclude that $fp^* = gp^*$.
We finish the proof by showing that f and g have a unique point of coincidence. For this, assume $z^* \in X$ is a point such that $fz^* = gz^*$. Now

$$q(gz^*, gp^*) = q(fz^*, fp^*) \preceq k[q(fz^*, gp^*) + q(gz^*, fp^*)] \preceq 2kq(gz^*, gp^*),$$

which gives $q(gz^*, gp^*) = \theta$. On the other hand, by the same reasoning, it also clear that $q(gp^*, gz^*) = \theta$. By property $(q2)$, $gz^* = gp^*$. From Proposition 1.15, f and g have a unique common fixed point. □

The results for this section therefore summarise in the following way.

Theorem 2.6. *Let (X, q) be a quasi-cone metric space, P a K-normal cone. Suppose that mappings $f, g : X \longrightarrow X$ satisfy the contractive condition: for all $x, y \in X$*

$$q(fx, fy) \preceq Aq(gx, gy) + B[q(gx, fx) + q(gy, fy)] + C[q(fx, gy) + q(gx, fy)] \tag{2.1}$$

where A, B, C are non-negative real numbers with $A + 2B + 2C < 1$. If the range of $f(X) \subseteq g(X)$ and f or $g(X)$ is bicomplete , then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. The proof follows the same idea as in Theorems 2.2 and 2.5. We shall just give here the main points. Take an arbitrary $x_0 \in X$. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since $f(X) \subseteq g(X)$. Iterating this process, once x_n is chosen in X, we can obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1}), n = 0, 1, \cdots$.

It is then very easy to derive that for $n < m$ we have

$$q(gx_n, gx_m) \preceq \frac{h^n}{1-h}q(gx_0, gx_1)$$

and for for $n > m$ we have

$$q(gx_n, gx_m) \preceq \frac{h^m}{1-h}q(gx_0, gx_1)$$

where $h = \frac{A+B+C}{1-B-C}$. Hence (gx_n) is a bi-Cauchy sequence. Since $g(X)$ is bicomplete, there exists $x^* \in g(X)$ such that (gx_n) converges to x^*. In other words, there is a $p^* \in X$ such that (gx_n) converges to $g(p^*) = x^*$. Moreover, observe that

$$q(gx_n, fp^*) = q(fx_{n-1}, fp^*) \quad \text{and} \quad q(fp^*, gx_n) = q(fp^*, fx_{n-1})$$

and using the condition (2.1), we obtain that $q(gz^*, gp^*) = \theta = q(gp^*, gz^*)$, i.e. f and g have a point of coincidence. Again by use of condition (2.1), we conclude that this point of coincidence is unique. Finally, from Proposition 1.15, since f and g are weakly compatible, then f and g have a unique common fixed point. □

Example 2.7. Let $X = \mathbb{R}$, $E = \mathbb{R}^2$, $q(x,y) = (x * y, \alpha(x * y))$, $\alpha > 0$ where $x * y = \max\{x - y, 0\}$, whenever $x, y \in \mathbb{R}$ and $P = \{(x,y) : x \geq 0, y \geq 0\}$, $f(x) = 2x^2 + 4x + 3$ and $g(x) = 3x^2 + 6x + 4$. Then it easy to see that

$$f(X) = g(X) = [1, \infty) \text{ is bicomplete.}$$

All the conditions of Theorem 2.6 are satisfied for

$$A \in \left[\frac{2}{3}, 1\right), B = C = 0.$$

The unique point of coincidence here is $1 = f(-1) = g(-1)$.

However, since $fg(-1) = 9 \neq 13 = gf(-1)$, f and g are not weakly compatible and therefore fail to have a common fixed point. This shows the importance of Proposition 1.15.

But if we modify f and g in the following way $f(x) = 2x^2 + 4x + 1$ and $g(x) = 3x^2 + 6x + 2$, then again all the conditions of Theorem 2.6 are satisfied, f and g become weakly compatible and we obtain a unique point of coincidence and a unique common fixed point $-1 = f(-1) = g(-1)$.

The next example explain how crucial the condition $f(X) \subseteq g(X)$ is in the statement of the theorems.

Example 2.8. Let $X = [0, \infty)$, $E = \mathbb{R}^2$, $q(x,y) = (x * y, e(x * y))$, where $x * y = \max\{x - y, 0\}$, whenever $x, y \in \mathbb{R}$ and $P = \{(x,y) : x \geq 0, y \geq 0\}$, $f(x) = e^x$ and $g(x) = e^{x+1}$. Then it easy to see that

$$f(X) = (0, \infty) \nsubseteq g(X) = [e, \infty).$$

$$\begin{aligned} q(fx, fy) &= (e^x * e^y, e^{x+1} * e^{y+1}) \\ &= \frac{1}{e}(e^{x+1} * e^{y+1}, e^{x+2} * e^{y+2}) \\ &= \frac{1}{e} q(gx, gy). \end{aligned}$$

All the conditions of Theorem 2.6 except $f(X) \subseteq g(X)$ are satisfied for

$$A = \frac{1}{e}, B = C = 0.$$

But f and g do not have a point of coincidence.

---

sectionMore fixed point results We begin with the following lemmas.

Lemma 2.9. *Let (X, q) be a quasi-cone metric space, P be a K-normal cone. Let (y_n) be a sequence in X. If (y_n) satisfies*

$$q(y_n, y_{n+1}) \leq \lambda q(y_{n-1}, y_n) \tag{2.2}$$

for some $\lambda > 0$ with $\lambda < 1$. Then (y_n) is left Cauchy.

Proof. Let $m < n \in \mathbb{N}$. From the condition $(q3)$ in the definition of a quasi-cone metric , we can write:

$$\begin{aligned} q(y_m, y_n) &\preceq q(y_m, y_{m+1}) + q(y_{m+1}, y_n) \\ &\preceq q(y_m, y_{m+1}) + q(y_{m+1}, y_{m+2}) + q(y_{m+2}, y_n) \\ &\vdots \\ &\preceq q(y_m, y_{m+1}) + q(y_{m+1}, y_{m+2}) + \cdots + q(y_{n-2}, y_{n-1}) + q(y_{n-1}, y_n). \end{aligned}$$

From (2.2) the above becomes

$$\begin{aligned} q(y_m, y_n) &\preceq (\lambda^m + \lambda^{m+1} + \cdots + \lambda^{n-1}) q(y_0, y_1) \\ &\preceq \frac{\lambda^m}{1-\lambda} q(y_0, y_1). \end{aligned}$$

It entails that $\|q(y_m, y_n)\| \leq K \frac{\lambda^m}{1-\lambda} \|q(y_0, y_1)\| \longrightarrow 0$ as $m \longrightarrow \infty$. It follows that (y_n) is left Cauchy.

□

Similarly,

Lemma 2.10. *Let (X, q) be a quasi-cone metric space, P be a K-normal cone. Let (y_n) be a sequence in X. If (y_n) satisfies*

$$q(y_{n+1}, y_n) \preceq \lambda q(y_n, y_{n-1}) \tag{2.3}$$

for some $\lambda > 0$ with $\lambda < 1$. Then (y_n) is right Cauchy.

Theorem 2.11. *Let (X, q) be a quasi-cone metric space, P a K-normal cone. Suppose that mappings $f, g : X \longrightarrow X$ satisfy the contractive condition*

$$q(fx, fy) \preceq \lambda q(gx, gy) + \gamma q(fx, gy) \text{ for all } x, y \in X. \tag{2.4}$$

where λ, γ are positive constant such that $\lambda + 2\gamma < 1$. If the range of g contains the range of f and $g(X)$ is a bicomplete quasi-cone metric space, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Take an arbitrary $x_0 \in X$. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since $f(X) \subset g(X)$. Iterating this process, once x_n is chosen in X, we can obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} q(gx_n, gx_{n+1}) = q(fx_{n-1}, fx_n) &\preceq \lambda q(gx_{n-1}, gx_n) + \gamma q(fx_{n-1}, gx_n) \\ &\preceq \lambda q(gx_{n-1}, gx_n). \end{aligned}$$

Therefore (gx_n) is a left Cauchy sequence. In a similar manner, we establish that (gx_n) is also a right Cauchy sequence. Hence (gx_n) is a bi-Cauchy sequence. Since $g(X)$ is bicomplete, there exists $x^* \in g(X)$ such that (gx_n) converges to x^*. In other words, there is a $p^* \in X$ such that (gx_n) converges to $g(p^*) = x^*$.
Moreover since

$$q(gx_n, fp^*) = q(fx_{n-1}, fp^*) \preceq \lambda q(gx_{n-1}, gp^*) + \gamma q(fx_{n-1}, gp^*)$$

we get that $q(gp^*, fp^*) = \theta$. On the other hand, by the same reasoning, it is also clear that $q(fp^*, gp^*) = \theta$. By property $(q2)$, $fp^* = gp^*$.
We finish the proof by showing that f and g have a unique point of coincidence. For this, assume $z^* \in X$ is a point such that $fz^* = gz^*$. Now

$$q(gz^*, gp^*) = q(fz^*, fp^*) \preceq \lambda q(gz^*, gp^*) + \gamma q(fz^*, gp^*) \preceq (\lambda + \gamma) q(gz^*, gp^*),$$

which gives $q(gz^*, gp^*) = \theta$. On the other hand, by the same reasoning, it also clear that $q(gp^*, gz^*) = \theta$. By property $(q2)$, $gz^* = gp^*$. From Proposition 1.15, f and g have a unique common fixed point. □

Corollary 2.12. *Let (X, q) be a quasi-cone metric space, P a K-normal cone. Suppose that mappings $f, g : X \longrightarrow X$ satisfy the contractive condition*

$$q(fx, fy) \preceq \alpha[q(gx, gy) + q(fx, gy)] \text{ for all } x, y \in X. \tag{2.5}$$

where $\alpha \in \left(0, \frac{1}{3}\right)$. If the range of g contains the range of f and $g(X)$ is a bicomplete quasi-cone metric space, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Theorem 2.13. *Let (X, q) be a quasi-cone metric space, P a K-normal cone. Suppose that mappings $f, g : X \longrightarrow X$ satisfy the contractive condition*

$$q(fx, fy) \preceq \lambda q(gx, gy) + \gamma q(gx, fy) \text{ for all } x, y \in X. \tag{2.6}$$

where λ, γ are positive constant such that $\lambda + 2\gamma < 1$. If the range of g contains the range of f and $g(X)$ is a bicomplete quasi-cone metric space, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Corollary 2.14. *Let (X, q) be a quasi-cone metric space, P a K-normal cone. Suppose that mappings $f, g : X \longrightarrow X$ satisfy the contractive condition*

$$q(fx, fy) \preceq \alpha[q(gx, gy) + q(gx, fy)] \text{ for all } x, y \in X. \tag{2.7}$$

where $\alpha \in \left(0, \frac{1}{3}\right)$. If the range of g contains the range of f and $g(X)$ is a bicomplete quasi-cone metric space, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

In the following section, we shall establish some topological properties of quasi-cone metric spaces.

---

sectionTopology on quasi-cone metric spaces

We begin by stating the following lemma.

Lemma 2.15. *Let (X, q) be a quasi-cone metric space. For each $c \in E$ with $c \gg \theta$, there exists $\sigma > 0$ such that $x \ll c$ whenever $\|x\| < \sigma,\ x \in E$.*

Proof. Since $c \gg \theta$, then $c \in Int(P)$. Hence, find $\sigma > 0$ such that

$$\{x \in E : \|x - c\| < \sigma\} \subset Int(P).$$

Now if $\|x\| < \sigma$ then $\|(c - x) - c\| = \| - x\| = \|x\| < \sigma$ and $(c - x) \in Int(P)$. □

Lemma 2.16. *Let (X, q) be a quasi-cone metric space over a cone P. Then for each $c_1, c_2 \in Int(P)$, there exists $c \in Int(P)$ such that $c_1 - c \in Int(P)$ and $c_2 - c \in Int(P)$.*

Proof. Since $c_2 \gg \theta$, then by Lemma 2.15, we pick $\delta > 0$ such that $\|x\| < \delta$ implies that $x \ll c_2$. Choose n_0 such that $\frac{1}{n_0} < \frac{\delta}{\|c_1\|}$. then $\|c\| = \frac{\|c_1\|}{n_0} < \delta$ and hence, $c \ll c_2$. But it is also clear that $c \gg \theta$ and $c \ll c_1$. □

Proposition 2.17. *Every quasi-cone metric space is a topological space.*

Proof. For $c \gg \theta$, and $x \in (X, q)$ let

$$B_q(x, c) = \{y \in X : q(x, y) \ll c\}$$

and

$$\mathcal{B} = \{B_q(x, c) : x \in X, c \ll \theta\}.$$

Then the collection

$$\mathcal{T} = \{U \subset X : \forall x \in U, \exists c \gg \theta, B_q(x, c) \subset U\}$$

is a topology on X. Indeed,

(i) $\emptyset$ and X belong to $\mathcal{T}$,

(ii) let $U, V \in \mathcal{T}$ and let $x \in U \cap V$. Then there exist $c_1 \gg \theta$ and $c_2 \gg \theta$ such that $B_q(x, c_1) \subset U$ and $B(x, c_2) \subset V$.

By Lemma 2.16, there exists $c \in Int(P)$ such that $c_1 - c \in Int(P)$ and $c_2 - c \in Int(P)$. Then, it is clear that $x \in B_q(x, c) \subset U \cap V$, hence $U \cap V \in \mathcal{T}$.

(iii) let $(U_\alpha)_\alpha$ be a family of sets from $\mathcal{T}$. We consider $x \in \bigcup_\alpha U_\alpha$. There exists α_0 such that $x \in U_{\alpha_0}$. Hence, find $c \gg \theta$ such that

$$x \in B_q(x, c) \subset U_{\alpha_0} \subset \bigcup_\alpha U_\alpha.$$

That is $\bigcup_\alpha U_\alpha \in \mathcal{T}$.

This completes the proof.

□

Definition 2.18. Let (X, q) be a quasi-cone metric space. Then a subset $F \subset X$ is called **up bounded**(resp. **bounded**) if there exists $c \gg \theta$ such that $q(x, y) \preceq c$ for all $x, y \in F$ (resp. $\delta(F) = \sup\{q(x, y) : x, y \in F\}$ exists in E).

Lemma 2.19. *Let (X, q) be a quasi-cone metric space. Then a subset $F \subset X$ is up bounded if and only if there exist $x \in X, c_1, c_2 \in Int(P)$ such that $F \subset C_q(x, c_1) \cap C_{q^{-1}}(x, c_2)$ where $C_q(x, c) = \{y \in X : q(x, y) \preceq c\}$ for any $x \in X$ and $c \gg \theta$.*

When the cone P is strongly minihedral, we have the following characterization.

Proposition 2.20. *Let (X, q) be a quasi-cone metric space over a Banach space with a K-normal strongly minihedral cone P. Then a subset $F \subset X$ is bounded if and only if the quantity $\delta'(F) = \sup_{x,y\in F} \|q(x, y)\| < \infty$.*

Proof. Suppose that F is bounded. For all $x, y \in F$, $q(x, y) \preceq \delta(F)$, which entails that $\|q(x, y)\| \leq K\|\delta(F)\| < \infty$.
Conversely, assume that $\delta'(F) = \sup_{x,y\in F} \|q(x, y)\| = M < \infty$, and fix some $c_1 \gg \theta$. By Lemma 2.15, we know that we can find $\delta > 0$ such that $\|z\| < \delta$ implies that $c_1 \gg z$. For any $x, y \in F$, we set $c_{x,y} = \frac{\delta q(x,y)}{2\|q(x,y)\|}$. Since $\|c_{x,y}\| = \delta/2 < \delta$, we have that $c_1 - c_{x,y} \in Int(P)$. By setting $\alpha_{x,y} = \frac{2\|q(x,y)\|}{\delta}$, it is then clear that $\alpha_{x,y}(c_1 - c_{x,y}) \in Int(P)$, i.e $\alpha_{x,y}c_1 - q(x, y) \in Int(P)$. In other words $q(x, y) \ll \alpha_{x,y}c_1 \preceq \frac{2M}{\delta}c_1 \in Int(P)$, from which we derive that $q(x, y) \preceq \frac{2M}{\delta}c_1$. Since P is minihedral, then F is bounded. □

We conclude this section by the following lemma.

Lemma 2.21. *Every quasi-cone metric space (X, q) is first countable.*

Proof. Let $p \in X$. Fix $c \in Int(P)$. We show that $\mathcal{B}_p = \{B_q(p, \frac{1}{n}c) : n \in \mathbb{N}\}$ is a local base at p. Let U be an open set containing p. There exists $c_1 \in Int(P)$ such that $B_q(p, c_1) \subset U$. We know by Lemma 1.4 that we can find $n_0 \in \mathbb{N}$ such that $\frac{c}{n_0} \ll c_1$. Hence $B_q(p, \frac{c}{n_0}) \subset B_q(p, c_1)$. This completes the proof.
□

---

3. Concept of convexity

In this section, we introduce a concept of convexity in quasi-cone metric spaces. Most of the results here are a generalization of some similar known notions in quasi-pseudometric spaces (see [10] and [11]).

Definition 3.1. Let E be a real Banach space, P be a cone on E and $\preceq$ the partial order with respect to P. An element $x \in E$ is said to be a **nonnegative vector** if $\theta \preceq x$ and a **positive vector** if $\theta \prec x$. Hence P is the set of all nonnegative elements. We shall use the following notations:

- $[\theta, \longrightarrow[:= P = \{x \in E : \theta \preceq x\}$ and;
- $]\theta, \longrightarrow[:= \{x \in E : \theta \prec x\}$.

Definition 3.2. A quasi-cone metric space (X, q) will be called **Isbell-convex** or **q-hyperconvex** provided that for each family $(x_i)_{i\in I}$ of points of X and families of nonnegative vectors $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ the following condition holds:

$$\text{If } q(x_i, x_j) \preceq r_i + s_j \text{ whenever } i, j \in I, \text{ then } \bigcap_{i\in I}(C_q(x_i, r_i) \cap C_{q^t}(x_i, s_i)) \neq \emptyset.$$

Definition 3.3. A quasi-cone metric space (X, q) will be called **metrically convex** if for any point $x, y \in X$ and nonnegative vectors r and s such that $q(x, y) \preceq r + s$, there exists $z \in X$ such that $q(x, z) \preceq r$ and $q(z, y) \preceq s$.

The following examples are basic but not trivial, and therefore, important.

Example 3.4. Let $X = \mathbb{R},\ E = \mathbb{R}^2,\ q(x, y) = (x * y, x * y)$, where the operation $*$ is defined by $x*y = \max\{x-y, 0\}$, whenever $x, y \in \mathbb{R}$ and $P = \{(x, y) : x \geq 0, y \geq 0\}$. Then $(\mathbb{R}, q)$ is metrically convex.

Example 3.5. Let $X = \mathbb{R},\ E = \mathbb{R}^2,\ q(x, y) = (x\sharp y, x\sharp y)$, where the operation $\sharp$ is defined by $x\sharp y = x - y$, if $x \geq y$ and $x\sharp y = 1$ otherwise, whenever $x, y \in \mathbb{R}$ and $P = \{(x, y) : x \geq 0, y \geq 0\}$. Then $(\mathbb{R}, q)$ is not metrically convex. Indeed, we have

$$q\left(\frac{1}{2}, 1\right) = (1, 1) \preceq \left(\frac{1}{2} + \frac{1}{2}, \frac{1}{2} + \frac{1}{2}\right).$$

But there is no $z \in \mathbb{R}$ such that $q\left(\frac{1}{2}, z\right) \preceq \left(\frac{1}{2}, \frac{1}{2}\right)$ and $q\left(z, 1\right) \preceq \left(\frac{1}{2}, \frac{1}{2}\right)$, since such a z would satisfy $z \leq \frac{1}{2}$ and $z \geq 1$.

Definition 3.6. Let (X, q) be a quasi-cone metric space. A family of balls $(C_q(x_i, r_i), C_{q^t}(x_i, s_i))_{i\in I}$ with $x_i \in X$ and $r_i, s_i \in [\theta, \longrightarrow[$ whenever $i \in I$ is said to have the **mixed binary intersection property** if for all indices $i, j \in I$, $(C_q(x_i, r_i) \cap C_{q^t}(x_j, s_j) \neq \emptyset$.

Definition 3.7. A quasi-cone metric space (X, q) will be called **Isbell-complete** if every family of balls $(C_q(x_i, r_i), C_{q^t}(x_i, s_i))_{i\in I}$ with $x_i \in X$ and $r_i, s_i \in [\theta, \longrightarrow[$ whenever $i \in I$ having the mixed binary intersection property satisfies $\bigcap_{i\in I}(C_q(x_i, r_i)\cap C_{q^t}(x_i, s_i)) \neq \emptyset$.

Corollary 3.8. *If (X, q) is an Isbell-convex (resp. Isbell-complete, metrically convex) quasi-cone metric space, then (X, q^t) is Isbell-convex(resp. Isbell-complete, metrically convex).*

Proposition 3.9. *A quasi-cone metric space (X, q) is Isbell-convex if and only if it is metrically convex and Isbell-complete.*

Proof. Suppose that (X, q) is Isbell-convex. Let $x_1, x_2 \in X,\ r_1, r_2 \in [\theta, \longrightarrow[$ such that $q(x_1, x_2) \preceq r_1 + s_2$. Then set $r_2 = s_1 = q(x_2, x_1)$. By Isbell-convexity, there exists $x \in C_q(x_1, r_1) \cap C_{q^t}(x_2, s_2)$, hence (X, q) is metrically convex.
Consider now a family of balls $(C_q(x_i, r_i), C_{q^t}(x_i, s_i))_{i\in I}$ that have the mixed binary intersection property. Thus $q(x_i, x_j) \preceq r_i + s_j$ whenever $i, j \in I$. By Isbell-convexity, there exists $x \in \bigcap_{i\in I}(C_q(x_i, r_i) \cap C_{q^t}(x_i, s_i))$. So (X, q) is Isbell-complete.
For the converse, assume that (X, q) is metrically convex and Isbell-complete. Consider a family $(x_i)_{i\in I}$ of points of X and families $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ of nonnegative vectors such that $q(x_i, x_j) \preceq r_i + s_j$ whenever $i, j \in I$. Since (X, q) is metrically convex, then $(C_q(x_i, r_i), C_{q^t}(x_i, s_i))_{i\in I}$ has the mixed binary intersection property.
Therefore, there exists $x \in \bigcap_{i\in I}(C_q(x_i, r_i) \cap C_{q^t}(x_i, s_i))$ by Isbell-completeness. Thus, (X, q) is Isbell-convex. □

Example 3.10. Let $X = \mathbb{R},\ E = \mathbb{R}^2,\ q(x, y) = (x*y, x*y)$, where the operation $*$ is defined by $x*y = \max\{x-y, 0\}$, whenever $x, y \in \mathbb{R}$ and $P = \{(x, y) : x \geq 0, y \geq 0\}$. Then $(\mathbb{R}, q)$ is q-hyperconvex.

We first observe that, for any $\varepsilon = (\varepsilon_1, \varepsilon_2) \in P$, $C_q(x, \varepsilon) = [x - \tilde{\varepsilon}, \infty)$ and $C_{q^{-1}}(x, \varepsilon) = (-\infty, x + \tilde{\varepsilon}]$ whenever $x \in \mathbb{R}$ where $\tilde{\varepsilon} = \min\{\varepsilon_1, \varepsilon_2\}$. Let $(x_i)_{i\in I}$ be a family of points in $\mathbb{R}$ and $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be families of nonnegative vectors with

$r_i = (r_i^1, r_i^2)$ and $s_i = (s_i^1, s_i^2)$ such that $q(x_i, x_j) \preceq r_i + s_j$, whenever $i, j \in I$. We recall quickly that $\tilde{r_i} = \min\{r_i^1, r_i^2\}$ and $\tilde{s_i} = \min\{s_i^1, s_i^2\}$. Suppose that

$$\bigcap_{i\in G} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)) = \emptyset \text{ for some finite subset } G \text{ of } I.$$

We can assume that G is nonempty. It follows that

$$\max\{x_i - \tilde{r_i} : i \in G\} > \min\{x_i + \tilde{s_i} : i \in G\}.$$

Therefore, there are $i_0, j_0 \in G$ such that $x_{i_0} - \tilde{r_{i_0}} > x_{j_0} - \tilde{s_{i_0}}$, that is $C_q(x_{i_0}, r_{i_0}) \cap C_{q^{-1}}(x_{j_0}, s_{i_0}) = \emptyset$. In particular, $x_{i_0} > x_{j_0}$.

Thus $r_{i_0} + s_{j_0} \prec q(x_{i_0}, x_{j_0}) = (x_{i_0} - x_{j_0}, x_{i_0} - x_{j_0})$ —a contradiction. We conclude that

$$\bigcap_{i\in G} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)) \neq \emptyset \text{ whenever } G \text{ is a finite subset of } I.$$

Since for any $i \in I$,$C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)$ is compact with respect to the standard topology on $\mathbb{R}$, we conclude that

$$\bigcap_{i\in G} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)) \neq \emptyset.$$

Hence $(\mathbb{R}, q)$ is q-hyperconvex.

Example 3.11. Let $X = \mathbb{R},\ E = \mathbb{R}^2,\ q(x, y) = (|x - y|, |x - y|)$, whenever $x, y \in \mathbb{R}$ and $P = \{(x, y) : x \geq 0, y \geq 0\}$. Then $(\mathbb{R}, q)$ is not q-hyperconvex.

Indeed, for any $i \in [0, 1]$, set $r_i = \left(\frac{1}{4}, \frac{1}{4}\right)$ and $s_i = \left(\frac{3}{4}, \frac{3}{4}\right)$. Then for any $i, j \in [0, 1]$, $q(x, y) \preceq (1, 1) = r_i + s_i$. But

$$\begin{aligned}\bigcap_{i\in[0,1]} (C_q(i, r_i) \cap C_{q^{-1}}(i, s_i)) &\subseteq C_q\left(0, \frac{1}{4}\right) \bigcap C_{q^{-1}}\left(1, \frac{1}{4}\right)\\ &= \left(\left[-\frac{1}{4}, \frac{1}{4}\right] \times \left[-\frac{1}{4}, \frac{1}{4}\right]\right) \bigcap \left(\left[\frac{3}{4}, \frac{5}{4}\right] \times \left[\frac{3}{4}, \frac{5}{4}\right]\right)\\ &= \emptyset.\end{aligned}$$

Proposition 3.12. *Let (X, q) be a q-hyperconvex quasi-cone metric space. Let $(x_i)_{i\in I}$ be a family of points in X and let $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be families of nonnegative vectors such that $q(x_i, x_j) \preceq r_i + s_j$.*

The set $D = \underset{i\in I}{\cap} (C_q(x_i, r_i) \cap C_{q^t}(x_i, s_i))$ is nonempty and q-hyperconvex.

Proof. We first observe that since X is q-hyperconvex, then $D \neq \emptyset$. For each $\alpha \in S$, let $x_\alpha \in D$ and let r_α, s_α be nonnegative vectors such that $q(x_\alpha, x_\beta) \preceq r_\alpha + s_\beta$ whenever $\alpha, \beta \in S$.

We show that the family satisfies the hypothesis of q-hyperconvexity. Indeed, in particular, for each $\alpha \in S$ and $i \in I$, we have that $q(x_\alpha, x_i) \preceq s_i \preceq r_\alpha + s_i$ and $q(x_i, x_\alpha) \preceq r_i \preceq r_i + s_\alpha$. Hence by q-hyperconvexity of X,

$$\begin{aligned}\emptyset &\neq \left[\bigcap_{i\in I} (C_q(x_i, r_i) \cap C_{q^t}(x_i, s_i))\right] \bigcap \left[\bigcap_{\alpha\in S} (C_q(x_\alpha, r_\alpha) \cap C_{q^t}(x_\alpha, s_\alpha))\right]\\ &= D \bigcap \left[\bigcap_{\alpha\in S} (C_q(x_\alpha, r_\alpha) \cap C_{q^t}(x_\alpha, s_\alpha))\right].\end{aligned}$$

Hence, D is q-hyperconvex.

□

References

1. M. Abbas, M. Alikhan and S. Radenovic; *Common coupled fixed point theorems in come metric spaces for W-compatible mappings*, Applied Mathematics and Computation, 217(1) (2010), 195 - 202, doi: 10.1016/j.amc2010.05.042 1, 1.3, 1.4 .
2. M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in come metric spaces*, J. Math. Anal. App. 341 (2008) 416-420.
3. C. A. Agyingi and Y. U. Gaba; *A fixed point like theorem in a T_0-ultra-quasi-metric space*, Advances in Inequalities and Applications. Vol 2014 (2014), Article ID 16.
4. K. Deimling; *Nonlinear Functional Analysis*, Springer, Berlin, Germany, 1985.
5. Y. U. Gaba; *Weakly C-Contractive mappings in Cone Metric Spaces.*, Eng. Math. Lett., Vol 2014 (2014), Article ID 7.
6. Y. U. Gaba; Unique fixed point theorems for contractive maps type in T_0-quasi-metric spaces, Advances in Fixed Point Theory. Vol 4, No 1 (2014),117-124.
7. L.-G. Huang and X. Zhang; *Cone Metric Spaces and Fixed Point Theorems of Contractive Mappings*, Journal of Mathematical Analysis and Applications, vol 332, no. 2, pp 1468-1476, 2007.
8. E. F. Kazeem, C. A. Agyingi and Y. U. Gaba; *On Quasi-Pseudometric Type Spaces*, Chinese Journal of Mathematics, vol. 2014, Article ID 198685, 7 pages, 2014. doi:10.1155/2014/198685.
9. E. F. Kazeem and Y. U. Gaba; *Weakly contractive maps in T_0-quasi-metric spaces*. Advances in Fixed Point Theory, Vol 4, No 3 (2014), 355-364.
10. E. F. Kazeem and Y. U. Gaba; *Some results on quasi-cone pseudometric spaces*, Communications in Optimization Theory, Vol 2014 (2014), Article ID 9.
11. E. Kemajou, H.-P.A. Künzi, and O.O. Otafudu, *q-Hyperconvexity in Quasipseudometric Spaces and Fixed Point Theorems.*, Topology Appl. 159, pp. 2463-2475, 2012.
12. Sh. Rezapour and R. Hamlbarani; *Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings"* Journal of Mathematical Analysis and Applications, volume 345, pp. 719-724, 2008.
13. $\dot{I}$. Şahin and M. Telci; *Fixed Points of Contractive Mappings on Complete Cone Metric Spaces*, Hacettepe Journal of Mathematics and Statistics, vol 38(1) (2009), 59-67.
14. F. Shadda and M. S. Md Noorani; *Fixed Point Results in Quasi-Cone Metric Spaces*, Abstract and Applied Analysis, vol 2013, Article ID 303626. 7 pages, http://dx.doi.org/10.1155/2013/303626.