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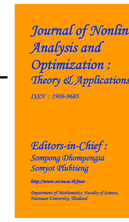

STRONG CONVERGENCE OF FINITE FAMILY OF PSEUDOCONTRACTIVE MAPPINGS BY A NEW IMPLICIT ITERATION

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ABSTRACT. In this paper, we propose Mann-Kirk type implicit iteration for a finite family of pseudocontractive mappings, and prove strong convergence of proposed iteration to a common fixed point in Banach spaces. The results in the paper extend and generalize well known corresponding results.

KEYWORDS : Mann iteration; Kirk iteration; implicit iteration; pseudocontractive mapping; common fixed point.
AMS Subject Classification: 47H09,47H10.

1. INTRODUCTION

Let E be a real Banach space, K be a closed convex subset of E and let J denote the normalized duality pairing from E into 2^{E^*} given by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|\,\|f\|\,, \|x\| = \|f\|\,\}\ ,\ \ \forall x \in E\,,$$

where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. We shall denote elements in Jx by $j(x)$ and define $Fix(T) = \{x \in E : Tx = x\}$ to be the fixed point set of a mapping T. When $\{x_n\}$ is a sequence in E, then $x_n \longrightarrow x$ ($x_n \rightharpoonup x$) will denote strong (weak) convergence of the sequence $\{x_n\}$ to x.

Let T be a mapping with domain $D(T)$ and range $R(T)$ in E. Then T is called

- Nonexpansive, if for any $x, y \in D(T)$

$$\|Tx - Ty\| \leq \|x - y\|\ .$$

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- Accretive, if for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that
$$\langle Tx - Ty, j(x-y)\rangle \geq 0\,.$$
- Pseudocontractive, if for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that
$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2\;.$$
- Hemicontractive, if for any $x \in D(T)$ and $x^* \in Fix(T)$, there exists $j(x-x^*) \in J(x-x^*)$ such that
$$\langle Tx - x^*, j(x-x^*)\rangle \leq \|x-x^*\|^2\;.$$
- Strongly pseudocontractive, if for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that
$$\langle Tx - Ty, j(x-y)\rangle \leq \alpha\,\|x-y\|^2\;,\quad \text{for some } 0 < \alpha < 1\,.$$
- Strictly pseudocontractive, if for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that
$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \lambda\,\|(x-y)-(Tx-Ty)\|^2\;,$$
for some $0 < \lambda < 1\,.$

The class of pseudocontractive mappings has close relations with the class of nonexpansive mappings and the class of accretive mappings. It is easy to see that if T is a pseudocontractive mapping, then $I-T$ is accretive.
If we define $A = (2I-T)^{-1}$, then $Fix(A) = Fix(T)$ and we have the following result :

Lemma 1.1 (Martin[18]). *A is a nonexpansive self mapping on* K.

Regarding iterative approximation of fixed points of nonexpansive mappings, it is well known that Picard (successive) iteration may fail to produce a norm convergence sequence $\{x_n\}$ for nonexpansive mappings. Thus when a fixed point of nonexpansive mappings exists, other approximation techniques are needed to approximate it. One such technique is to form a mapping
$$S_\lambda = \lambda I + (1-\lambda)T \quad (0 < \lambda < 1)\,,$$
and then show that under certain circumstances the Picard iterates of S_λ converges to a fixed point of T. The first such result was obtained by Krasnoselskii [14] in a uniformly convex Banach space for $\lambda = \frac{1}{2}$. Schaefer [22] noted that this result holds for arbitrary $\lambda \in (0,1)$. Edelstein [9] proved corresponding result in strictly convex Banach spaces.

Kirk [13] defined a more general mapping than those of S_λ. Let K be a closed convex subset of a Banach space, and T be a nonexpansive mapping of K into itself. Define the mapping $S : K \longrightarrow K$ by
$$S = \lambda_0 I + \lambda_1 T + \lambda_2 T^2 + \cdots + \lambda_k T^k\,,$$
where $\lambda_i \geq 0$, $\lambda_1 > 0$ and $\sum_{i=1}^{k} \lambda_i = 1$.
He proved that for arbitrary $x_0 \in K$, the sequence $\{S^n x_0\}$ converges weakly to a fixed point of T in K.

Maiti and Saha [16] extended this result of Kirk and proved the strong convergence of the sequence $\{S^n x_0\}$.

Liu et al. [15] extended Kirk's idea to finite family of nonexpansive mappings.

Let $T_i : K \longrightarrow K \ \ (i = 1, 2, \dots, k)$ be nonexpansive mappings and let

$$S = \lambda_0 I + \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k \,,$$

where $\lambda_i \geq 0$, $\lambda_0 > 0, \lambda_1 > 0$, and $\sum_{i=1}^{k} \lambda_i = 1$.
They proved that for arbitrary $x_0 \in K$, the sequence $\{S^n x_0\}$ converges to a common fixed point of T_i in K.

Iterative methods for nonexpansive mappings have been extensively investigated by many researchers, see ([2, 4, 5, 10, 11, 12, 19, 26]) and references there in.

The most popular iterative scheme to approximate a fixed point of a nonexpansive mapping is the following:

$$x_0 \in K \,;\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T x_n \,, \quad n \geq 0 \,, \tag{1.1}$$

where $\{\alpha_n\} \subset (0, 1)$ is a real sequence satisfying appropriate conditions.
Iteration process (1.1) is known as a Mann iteration [17].

Iterative method to approximate a fixed point of a pseudocontractive mapping was initiated by Browder and Petryshyn [3] in 1965, but iterative methods for pseudocontractive mappings are far less developed than those for nonexpansive mappings. In conncection with the iterative approximation of fixed points of pseudocontractive mappings, the following question is still open [6]:

> Does the Mann iteration process always converge for continuous pseudocontractive mapping? or for even Lipschitz pseudocontractive mappings?

Chidume and Mutangadura [7], negatively resolved the above problem by providing an example of a Lipschitz pseudocontractive mapping with a unique fixed point for which a Mann iteration process does not converge in a convex compact subset of a Hilbert space.

Rafiq [21], proposed a Mann type implicit iteration process for hemicontractive mapping T defined by:

$$x_0 \in K \,;\ x_n = (1 - \alpha_n) x_{n-1} + \alpha_n T x_n \,, \quad n \geq 0 \tag{1.2}$$

where $\{\alpha_n\}$ is a real sequence such that $\alpha_n \in [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$.

Song [24], discovered that the iteration (1.2) for hemicontractive mappings is not well defined. He observed that for an initial point $x_0 \in K$, x_1 is defined by the equation

$$x_1 = \alpha_1 x_0 + (1 - \alpha_n) T x_1 \,,$$

but the existence of x_1 is not established, because for hemicontractive mapping T, we do not know whether mapping $S_1 = \alpha_1 x_0 + (1 - \alpha_1) T$ has a fixed point $x_1 \in K$. Similarly the existence of $x_2, x_3, \dots, x_n$ is also doubtful, but the iteration (1.2) is well defined if we consider continuous pseudocontractive mappings.

Recently, Acedo and Xu [1] defined an iteration scheme in a Hilbert space for finite family of strict pseudocontractive mappings, where the sequence $\{x_n\}$ is generated by the algorithm :

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{i=1}^{N} \lambda_i^{(n)} T_i x_n ,$$

under appropriate assumptions on the sequence of weights $\left\{\lambda_i^{(n)}\right\}_{i=1}^{N}$.

It is important to note that, when we establish approximation results for pseudocontractive mappings, it looks more complicated than the results for strictly pseudocontractive mappings. Because T may increase distances which is not in the case of strictly pseudocontractive mappings due to the presence of a constant $\lambda \in (0,1)$. In order to overcome this difficulty caused by increasingness of T, one need to adjust the iteration or to make some additional assumptions.

Motivated by all above facts, in this paper we define a new implicit iteration for finite family of pseudocontractive mappings where the sequence $\{x_n\}$ is generated by the algorithm

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)\sum_{i=1}^{N} \lambda_i^{(n)} T_i x_n , \tag{1.3}$$

where the initial guess $x_0 \in K$ is arbitrary and $T_i \ (i = 1, 2, \ldots N)$ are pseudocontractive mappings, $\{\alpha_n\}$ is a real sequence in $(0, 1)$ and $\{\lambda_i^{(n)}\}_{i=1}^{N}$ is a sequence of weights satisfying appropriate assumptions. We shall prove strong convergence of iteration (1.3) to a point $x \in \bigcap_{i=1}^{N} Fix(T_i)$.

2. PRELIMINARIES

A Banach space E is said to satisfy Opial's condition [20] for any sequence $\{x_n\}$ in E converging weakly to a point $x \in E$, we have $\limsup_{n\longrightarrow\infty} \|x_n - x\| < \limsup_{n\longrightarrow\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$.

Now we establish the following result :

Proposition 2.1. *Given an integer $N \geq 1$, let $T_i : K \longrightarrow K$ be a pseudocontractive mapping for each $1 \leq i \leq N$. Define $S = \sum_{i=1}^{N} \lambda_i T_i$, where $\lambda_i > 0$ for all $1 \leq i \leq N$ such that $\sum_{i=1}^{N} \lambda_i = 1$. Then S is a pseudocontractive mapping.*

Proof. We have for $x, y \in K$,

$$\begin{aligned}
\langle Sx - Sy, j(x-y)\rangle &= \left\langle \left(\sum_{i=1}^{N} \lambda_i T_i\right) x - \left(\sum_{i=1}^{N} \lambda_i T_i\right) y, j(x-y) \right\rangle \\
&= \left\langle \sum_{i=1}^{N} \lambda_i \left(T_i x - T_i y\right), j(x-y) \right\rangle \\
&= \sum_{i=1}^{N} \lambda_i \left\langle T_i x - T_i y, j(x-y)\right\rangle \\
&\leq \sum_{i=1}^{N} \lambda_i \|x-y\|^2
\end{aligned}$$

$$= \|x-y\|^2 \sum_{i=1}^{N} \lambda_i$$
$$= \|x-y\|^2 ,$$

i.e.

$$\langle Sx - Sy, j(x-y)\rangle \leq \|x-y\|^2 .$$

Hence S is a pseudocontractive mapping. □

Next, we show that the iteration (1.3), is well defined, we need the following lemma to prove it :

Lemma 2.2 (Deimling[8]). *Let E be a Banach space, K be a nonempty closed convex subset of E and $T : K \longrightarrow K$ be a continuous and strong pseudocontractive mapping. Then T has a unique fixed point.*

Given an integer $N \geq 1$, let $T_i : K \longrightarrow K$ be a continuous pseudocontractive mapping for each $1 \leq i \leq N$ with $\bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$. For any fixed n, define $S_n = \sum_{i=1}^{N} \lambda_i^{(n)} T_i$, where $\left\{\lambda_i^{(n)}\right\}_{i=1}^{N}$ is a finite sequence of positive numbers such that $\sum_{i=1}^{N} \lambda_i^{(n)} = 1$ for all n and $\inf_{n\geq 1} \lambda_i^{(n)} > 0$ for all $1 \leq i \leq N$. Then for any fixed n, by Propostion 2.1, we observe that S_n is a pseudocontractive mapping which is continuous.
For any $u \in K$ and $t \in (0,1)$, define a mapping $G_t : K \longrightarrow K$ by

$$G_t x = tu + (1-t) S_n x .$$

Now,

$$\begin{aligned}\langle G_t x - G_t y, j(x-y)\rangle &= (1-t)\langle S_n x - S_n y, j(x-y)\rangle \\ &\leq (1-t)\|x-y\|^2 , \quad \forall x, y \in K .\end{aligned}$$

Hence G_t is continuous strongly pseudocontractive mapping and by Lemma 2.2, it has a unique fixed point, i.e. there exists a unique $x_t \in K$ satisfying the equation

$$x_t = tu + (1-t) S_n x_t .$$

This shows that the implicit iteration scheme (1.3) is well defined and can be employed to approximate a common fixed point of a finite family of pseudocontractive self mappings on K.

Now we prove the following result, its proof is motivated from [25] :

Lemma 2.3. *Let S be a self mapping on K and $\lim_{n\longrightarrow\infty} \|x_n - Sx_n\| = 0$. If $A = (2I - S)^{-1}$, then $\lim_{n\longrightarrow\infty} \|x_n - Ax_n\| = 0$.*

Proof. We have

$$x_n - Sx_n = (2I - S)x_n - x_n = A^{-1}x_n - x_n ,$$

also,

$$A^{-1}Ax_n = x_n = AA^{-1}x_n .$$

So,

$$\begin{aligned}\|x_n - Ax_n\| &= \left\|AA^{-1}x_n - Ax_n\right\| \\ &\leq \left\|A^{-1}x_n - x_n\right\| \\ &= \|x_n - Sx_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty .\end{aligned}$$

Hence,

$$\lim_{n\longrightarrow\infty} \|x_n - Ax_n\| = 0\,.$$

□

3. MAIN RESULTS

Lemma 3.1. *Assume for each n, there is a finite sequence $\left\{\lambda_i^{(n)}\right\}_{i=1}^{N}$ of positive numbers such that $\sum_{i=1}^{N}\lambda_i^{(n)} = 1$ for all n and $\inf_{n\geq 1}\lambda_i^{(n)} > 0$ for all $1 \leq i \leq N$. For any fixed n, set $S_n = \sum_{i=1}^{N}\lambda_i^{(n)}T_i$. Given $x_0 \in K$, let $\{x_n\}$ be generated by the algorithm (1.3). Assume that $\{\alpha_n\}$ is a real sequence satisfying $\alpha_n \in (0,b] \subset (0,1)$ for some constant $b \in (0,1)$, then*

(i) *for any $x^* \in \bigcap_{i=1}^{N} Fix(T_i)$, $\lim_{n\longrightarrow\infty}\|x_n - x^*\|$ exists,*
(ii) *$\{x_n\}$ and $\{S_nx_n\}$ are bounded.*

Proof. Since $x^* \in \bigcap_{i=1}^{N} Fix(T_i)$, we can see that $x^* \in Fix(S_n)$. Now

$$\begin{aligned}\|x_n - x^*\|^2 &= \langle \alpha_n (x_{n-1} - x^*) + (1-\alpha_n)(S_nx_n - x^*), j(x_n - x^*)\rangle \\ &= \alpha_n \langle x_{n-1} - x^*, j(x_n - x^*)\rangle + (1-\alpha_n)\langle S_nx_n - x^*, j(x_n - x^*)\rangle \\ &\leq \alpha_n \|x_{n-1} - x^*\|\,\|x_n - x^*\| + (1-\alpha_n)\|x_n - x^*\|^2,\end{aligned}$$

and

$$\|x_n - x^*\|^2 \leq \|x_{n-1} - x^*\|\,\|x_n - x^*\|\,. \tag{3.1}$$

Consequently, for each n,

$$\|x_n - x^*\| \leq \|x_{n-1} - x^*\|\,,$$

which implies that the sequence $\{\|x_n - x^*\|\}$ is monotone and nonincreasing. Hence $\lim_{n\longrightarrow\infty}\|x_n - x^*\|$ exists. Hence conclusion (i) is proved.
It follows from (i) that $\{x_n\}$ is bounded. Again from (1.3), we have

$$\begin{aligned}\|S_nx_n\| &= \left\|\frac{1}{1-\alpha_n}\,x_n - \frac{\alpha_n}{1-\alpha_n}\,x_{n-1}\right\| \\ &\leq \frac{1}{1-\alpha_n}\,\|x_n\| + \frac{\alpha_n}{1-\alpha_n}\,\|x_{n-1}\| \\ &\leq \frac{1}{1-b}\,\|x_n\| + \frac{b}{1-b}\,\|x_{n-1}\|\,.\end{aligned}$$

Hence $\{S_nx_n\}$ is bounded. This completes the proof of conclusion (ii). □

Theorem 3.2. *Let S_n and $\{x_n\}$ be as in Lemma 3.1. Assume that $\{\alpha_n\} \subset (0,1)$ is a sequence of real numbers satisfying $\lim_{n\longrightarrow\infty}\alpha_n = 0$, then*

$$\lim_{n\longrightarrow\infty}\|x_n - S_nx_n\| = 0\,.$$

If in addition E satisfies Opial's condition and K is weakly compact convex, then the sequence $\{x_n\}$ converges weakly to a fixed point of S, where S is as in Proposition 2.1.

Proof. Since $\lim_{n\longrightarrow\infty}\alpha_n = 0$ there exists a positive integer M such that $\alpha_n \in (0,b]$ for all $n \geq M$, $b \in (0,1)$. Hence Lemma 3.1 implies that, $\{x_n\}$ and $\{S_nx_n\}$ are bounded.
Using (1.3), we have

$$\|x_n - S_nx_n\| = \alpha_n\|x_{n-1} - S_nx_n\| \longrightarrow 0\,, \quad \text{as } n \longrightarrow \infty\,. \tag{3.2}$$

By weak compactness of K, there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \rightharpoonup x^* \in K$ as $l \longrightarrow \infty$.
With no loss of generality, we may assume that as $l \longrightarrow \infty$

$$\lambda_i^{(n_l)} \longrightarrow \lambda_i\,, \quad 1 \leq i \leq N\,. \tag{3.3}$$

Now for each $\lambda_i > 0$ and $\sum_{i=1}^{N} \lambda_i = 1$, for all $x \in K$ we have,

$$S_{n_l}x \longrightarrow Sx\,, \quad \text{as } \; l \longrightarrow \infty\,,$$

where $S = \sum_{i=1}^{N} \lambda_i T_i$, and S is pseudocontractive by proposition 2.1.
Since

$$\begin{aligned}\|x_{n_l} - Sx_{n_l}\| &= \|x_{n_l} - S_{n_l}x_{n_l}\| + \|S_{n_l}x_{n_l} - Sx_{n_l}\| \\ &\leq \|x_{n_l} - S_{n_l}x_{n_l}\| + \sum_{i=1}^{N} |\lambda_i^{n_l} - \lambda_i| \; \|T_i x_{n_l}\| \;,\end{aligned}$$

by (3.2), (3.3) and above inequality, we have

$$\|x_{n_l} - Sx_{n_l}\| \longrightarrow 0\,, \quad \text{as } l \longrightarrow \infty\,.$$

Also, by Lemma 2.3, we have

$$\lim_{l \longrightarrow \infty} \|x_{n_l} - Ax_{n_l}\| = 0\,,$$

where $A = (2I - S)^{-1}$.
Now we show that $x^* \in Fix(S)$. Suppose $x^* \neq Ax^*$. By nonexpansiveness of A and Opial's condition, we have

$$\begin{aligned}\limsup_{l \longrightarrow \infty} \|x_{n_l} - x^*\| &< \limsup_{l \longrightarrow \infty} \|x_{n_l} - Ax^*\| \\ &\leq \limsup_{l \longrightarrow \infty} \{\|x_{n_l} - Ax_{n_l}\| + \|Ax_{n_l} - Ax^*\|\} \\ &\leq \limsup_{l \longrightarrow \infty} \{\|x_{n_l} - Ax_{n_l}\| + \|x_{n_l} - x^*\|\} \\ &\leq \limsup_{l \longrightarrow \infty} \|x_{n_l} - x^*\| \;,\end{aligned}$$

i.e.

$$\limsup_{l \longrightarrow \infty} \|x_{n_l} - x^*\| < \limsup_{l \longrightarrow \infty} \|x_{n_l} - x^*\| \;,$$

which is a contradiction, so $x^* = Ax^*$. Since $Fix(A) = Fix(S)$, we have $x^* \in Fix(S)$.
Next, we prove that the sequence $\{x_n\}$ converges weakly to x^*. Suppose $\{x_n\}$ does not converges weakly to x^*. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $z \neq x^*$. Since $\lim_{n \longrightarrow \infty} \|x_n - p\|$ exists and E satisfies Opial's condition, we have

$$\begin{aligned}\lim_{n \longrightarrow \infty} \|x_n - x^*\| &= \lim_{l \longrightarrow \infty} \|x_{n_l} - x^*\| < \lim_{l \longrightarrow \infty} \|x_{n_l} - z\| \\ &= \lim_{k \longrightarrow \infty} \|x_{n_k} - z\| < \lim_{k \longrightarrow \infty} \|x_{n_k} - x^*\| \\ &= \lim_{n \longrightarrow \infty} \|x_n - x^*\|\end{aligned}$$

which is a contradiction, so we must have $z = x^*$. Thus $\{x_n\}$ converges weakly to $x^* \in Fix(S)$. This completes the proof.

□

Theorem 3.3. *Let S_n and $\{x_n\}$ be as in Lemma 3.1. Assume that K be compact convex subset of E, and $\{\alpha_n\} \subset (0,1)$ is a sequence of real numbers satisfying $\lim_{n\longrightarrow\infty} \alpha_n = 0$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of S, where S is as in Proposition 2.1.*

Proof. By Lemma 3.1, sequence $\{x_n\}$ is bounded, since K is compact, there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \longrightarrow x^* \in K$. By (3.2), we have

$$\lim_{l\longrightarrow\infty} \|x_{n_l} - S_{n_l}x_{n_l}\| = 0\,,$$

by repeating the arguments in the proof of Theorem 3.2, we get that

$$\lim_{l\longrightarrow\infty} \|x_{n_l} - Sx_{n_l}\| = 0\,. \tag{3.4}$$

By the continuity of the mapping S and the norm $\|\cdot\|$, together with (3.4), we have

$$\|x^* - Sx^*\| = \lim_{l\longrightarrow\infty} \|x_{n_l} - Sx_{n_l}\| = 0\,.$$

Therefore $x^* = Sx^*$. Since $\{\|x_n - x^*\|\}$ is nonincreasing by Lemma 3.1, so x^* is the strong limit of the sequence $\{x_n\}$ itself. This completes the proof. □

We recall the following definition:

Definition 3.4 (Senter and Dotson [23]). A mapping $T : K \longrightarrow K$ with $Fix(T) \neq \emptyset$ is said to satisfy *condition* (I) on K if there exists a nondecreasing function $f : [0,\infty) \longrightarrow [0,\infty)$ with $f(0) = 0$ and $f(r) > r$ for all $r \in (0,\infty)$ such that for all $x \in K$

$$\|x - Tx\| \geq f(d(x, Fix(T)))\,,$$

where $d(x, Fix(T)) = \inf\{\|x - p\| \,:\, p \in Fix(T)\}$.

Definition 3.5. We shall say that a finite family $\{T_1, T_2, ...T_N\}$ of N self mappings of K with $F = \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$ satisfies Condition (BS), if there exist f and d as in definition 3.4 such that

$$\|x - Sx\| \geq f(d(x,F))\,, \quad \forall x \in K\,,$$

where $S = \sum_{i=1}^{N} \lambda_i T_i$ and $\{\lambda_i\}_{i=1}^{N}$ is a sequence of positive number such that $\sum_{i=1}^{N} \lambda_i = 1$.

Definition 3.6. We shall say that a finite family $\{T_1, T_2, ...T_N\}$ of N self mappings of K with $F = \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$ satisfies Condition (BT), if there exist f and d as in definition 3.4 such that

$$\|x - S_n x\| \geq f(d(x,F))\,, \quad \forall x \in K\,,$$

where $S_n = \sum_{i=1}^{N} \lambda_i^{(n)} T_i$ and $\left\{\lambda_i^{(n)}\right\}_{i=1}^{N}$ is a sequence of positive number such that $\sum_{i=1}^{N} \lambda_i^{(n)} = 1$ for all n and $\inf_{n\geq 1} \lambda_i^{(n)} > 0$ for all $1 \leq i \leq N$.

Condition (BT) reduces to condition (BS) when sequence $\left\{\lambda_i^{(n)}\right\}_{i=1}^{N}$ is independent of iteration step n. Condition (BT) is identical to Condition (I) when $\lambda_i^{(n)} = 0$ for $i = 2, 3, \ldots, N$.

We now establish the main result of paper :

Theorem 3.7. *If K, S_n and $\{x_n\}$ be as in Lemma 3.1. Let T_i satisfy condition (BT), then $\{x_n\}$ converges strongly to a member of F.*

Proof. Since T_i, $i = 1, 2, \dots, N$ satisfies condition (BT), we have

$$\|x_n - S_n x_n\| \geq f\left(d(x_n, F)\right), \quad \text{for all } n \geq 0\,.$$

Let $x^* \in F$, then by Lemma 3.1, $\|x_n - x^*\| \leq \|x_{n-1} - x^*\|$ and $\lim_{n \longrightarrow \infty} \|x_n - x^*\|$ exists. This implies that $d(x_n, F) \leq d(x_{n-1}, F)$, so $\{d(x_n, F)\}$ is decreasing, it follows from Theorem 3.2, that

$$\lim_{n \longrightarrow \infty} f\left(d(x_n, F)\right) = 0\,.$$

By the nature of function f and the fact that $\lim_{n \longrightarrow \infty} d(x_n, F)$ exists, we have

$$\lim_{n \longrightarrow \infty} d(x_n, F) = 0\,.$$

We can thus choose a subsequence say $\{x_{n_l}\}$ of $\{x_n\}$ such that

$$\|x_{n_l} - x_l^*\| < 2^{-l}\,,$$

for all integer $l \geq 1$ and some sequence $\{x_l^*\}$ in $Fix(T)$. Again by Lemma 3.1, we have

$$\|x_{n_l} - x_l^*\| \leq \|x_{n_l - 1} - x_l^*\| < 2^{-l}\,,$$

and hence

$$\begin{aligned} \left\|x_l^* - x_{l-1}^*\right\| &\leq \|x_l^* - x_{n_l-1}\| + \left\|x_{n_l-1} - x_{l-1}^*\right\| \\ &\leq 2^{-(l+1)} + 2^{-l} \\ &< 2^{-l+1}\,. \end{aligned}$$

Which shows that $\{x_l^*\}$ is Cauchy and therefore converges strongly to a point $x^* \in K$, since F is closed, $x^* \in F$. Since $\lim_{n \longrightarrow \infty} \|x_n - x^*\|$ exists, $\{x_n\}$ converges strongly to x^*. This completes the proof.

□

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