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ON THE CONVERGENCE OF AN ITERATION PROCESS FOR TOTALLY ASYMPTOTICALLY I-NONEXPANSIVE MAPPINGS

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ABSTRACT. Suppose that K be a nonempty closed convex subset of a real Banach space X and $T, S : K \to K$ be two totally asymptotically I-nonexpansive mappings, where $I : K \to K$ is a totally asymptotically nonexpansive mapping. We define the iterative sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in K \\ x_{n+1} = (1-\alpha_n) x_n + \alpha_n S^n y_n \\ y_n = (1-\beta_n) x_n + \beta_n T^n z_n \\ z_n = (1-\gamma_n) x_n + \gamma_n I^n x_n \end{cases}, n \in \mathbb{N},$$

where $\{\alpha_n\}, \{\beta_n\}$ ve $\{\gamma_n\}$ are sequences in $[0,1]$ .Under some suitable conditions, the strong and weak convergence theorems of $\{x_n\}$ to a common fixed point of S,T and I are obtained.

KEYWORDS : Totally Asymptotically I-Nonexpansive Mapping; Total Asymptotically Nonexpansive Mapping; Opial Condition; (A') Condition; Strong and weak convergence; Common fixed point.
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1. Introduction

Let K be a nonempty subset of a real normed space X and $T : K \to K$ be a mapping. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$ and denote by $F := F(T) \cap F(S) \cap F(I) = \{x \in K : Tx = Sx = Ix = x\}$, the set of common fixed points of the mappings S, T and I. A mapping $T : K \to K$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in K$. A mapping $T : K \to K$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\| \tag{1.1}$$

for all $x, y \in K$ and $n \ge 1$. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5] as a generalization of the class of nonexpansive

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mappings. They proved that if K is a nonempty closed bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self-mapping has a fixed point.

A mapping $T : K \to K$ is called asymptotically nonexpansive in the intermediate sense if T is continuous and the following inequality holds:

$$\lim_{n\to\infty} \sup \sup_{x,y\in K} (\|T^n x - T^n y\| - \|x - y\|) \le 0.$$

Observe that if

$$a_n = \sup (\|T^n x - T^n y\| - \|x - y\|)$$

then last inequality reduces to the relation

$$\|T^n x - T^n y\| \le \|x - y\| + a_n. \tag{1.2}$$

The class of mappings which are asymptotically nonexpansive in the intermediate sense was introduced by Bruck et al. [2]. It is known [13] that if K is a nonempty closed convex bounded subset of a real uniformly convex Banach spaces E and T is a self-mapping of K which is asymptotically nonexpansive in the intermediate sense, then T has a fixed point.

The class of asymptotically nonexpansive mappings is an important generalization of the class of nonexpansive mappings. Note that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings.

In 2006, Alber et al. [1] introduced a more general class of asymptotically nonexpansive mappings called total asymptotically nonexpansive mappings. Their aim is to unify various definitions of classes of mappings associated with the class of asymptotically nonexpansive mappings and to prove a general convergence theorem applicable to all these classes of nonlinear mappings.

Definition 1.1. Let K be a nonempty closed subset of a real normed linear space X. A mapping $T : K \to K$ is called totally asymptotically nonexpansive if there exist nonnegative real sequences $\{\mu_n\}, \{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that

$$\|T^n x - T^n y\| \le \|x - y\| + \mu_n \phi(\|x - y\|) + \lambda_n, \text{for all } x, y \in K,\ n \ge 1. \tag{1.3}$$

Remark 1.2. If $\phi(\lambda) = \lambda$, then (1.3) reduces to $\|T^n x - T^n y\| \le (1 + \mu_n) \|x - y\| + \lambda_n, n \ge 1$. If $\phi(\lambda) = \lambda$ and $\lambda_n = 0$ for all $n \ge 1$, then total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings. In addition, if $\mu_n = 0$ and $\lambda_n = 0$ for all $n \ge 1$, then total asymptotically nonexpansive mappings becomes nonexpansive mappings. If $\mu_n = 0$ and $\lambda_n = \sigma_n = \max\{0, a_n\}$, where $a_n = \sup (\|T^n x - T^n y\| - \|x - y\|)$ for all $n \ge 1$, then (1.3) reduces to (1.2) which has been studied as mappings asymptotically nonexpansive in the intermediate sense.

The strong convergence theorems for iterative processes of finite family of total asymptotically nonexpansive mappings in Banach spaces have been studied by Chidume and Ofoedu [3, 4]. Also, Hu and Yang [11] obtained strong convergence theorems for three nonself total asymptotically nonexpansive mappings in real Banach spaces. In addition to these works, Hao [10] studied weak and strong convergence theorems in real Hilbert space.

On the other hand, in [17] an asymptotically I-nonexpansive mapping was introduced. Namely, let $T, I : K \to K$ be two mappings of a nonempty subset K of a real normed linear space X. Then T is said to be asymptotically

I-nonexpansive if there exists a sequence $\{\lambda_n\}$ with $\lim_{n\to\infty}\lambda_n = 1$ such that $\|T^n x - T^n y\| \leq \lambda_n \|I^n x - I^n y\|$ for all $x, y \in K$ and $n \geq 1$.

Recently, Mukhamedov and Saburov [14] introduced the folowing iteration process for common fixed points of totally asymptotically I-nonexpansive mappings in Banach spaces. For arbitrarily chosen $x_0 \in K$, $\{x_n\}$ is define as follows:

$$\begin{cases} x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T^n y_n \\ y_n = (1-\beta_n)\, x_n + \beta_n I^n x_n \end{cases}, \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. And they gave the following definition for totally asymptotically I-nonexpansive mappings.

Definition 1.3. Let $T, I : K \to K$ be two mappings of a nonempty subset K of a real normed linear space X. A mapping $T : K \to K$ is called totally asymptotically $I-$nonexpansive if there exist nonnegative real sequences $\{\mu_n\}$, $\{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\|T^n x - T^n y\| \leq \|I^n x - I^n y\| + \mu_n \phi\left(\|I^n x - I^n y\|\right) + \lambda_n, n \geq 1. \tag{1.5}$$

Note that (1.5) reduces to (1.3) when $I = Id$ (Id is the identity mapping). If $\phi(\lambda) = \lambda$, then one gets $\|T^n x - T^n y\| \leq (1+\mu_n)\|I^n x - I^n y\| + \lambda_n$, which is a generalization of the asymptotically I-nonexpansive mapping. If $\phi(\lambda) = \lambda$ and $\lambda_n = 0$ for all $n \geq 1$, then total asymptotically I-nonexpansive mappings coincide with asymptotically I-nonexpansive mappings.

In this paper, we construct an explicit iterative sequence for the approximation of common fixed points of two total asymptotically I-nonexpansive mappings and a total asymptotically nonexpansive mapping in Banach spaces.

2. Preliminary

For the sake of convenience, we restate the following concepts.

A Banach space X is said to satisfy Opial's condition if, for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$, where $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x. It is well know from that all Hilbert spaces and all l_p spaces for $1 < p < \infty$ have this property. However, the L_p spaces do not have this property unless $p = 2$.

Let K be a nonempty closed subset of a real Banach space X and $T : K \to K$ a mapping. The mapping T said to be demiclosed at zero if $Tx_0 = 0$ whenever $\{x_n\} \subset K, x_n \to x$ and $Tx_n \to 0$.

Recall that the mapping $T : K \to K$ with $F(T) \neq \emptyset$ is said to satisfy condition (A) [15] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in K$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

We can modify the condition (A) for our case as follows:

Let $S, T : K \to K$ be two totally asymptotically $I-$nonexpansive mappings and $I : K \to K$ be totally asymptotically nonexpansive mapping. Then S, T and I are said to satisfy condition (A') if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$ such that

$$\frac{1}{3}\left(\|x - Sx\| + \|x - Tx\| + \|x - Ix\|\right) \geq f\left(d\left(x, F\right)\right)$$

for all $x \in K$. This condition is a generalization of condition of Khan and Fukharuddin [12] from two mappings to three mappings.

Next, we state the following useful lemmas.

Lemma 2.1. [18] *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 + b_n)\, a_n + c_n, \; n \geq 1.$$

Suppose that $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then $\{a_n\}$ is bounded and $\lim\limits_{n\to\infty} a_n$ exists.

Lemma 2.2. [16] *Let X be a uniformly convex Banach space and let $\{t_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences of X such that*

$$\limsup_{n\to\infty} \|x_n\| \leq d, \;\; \limsup_{n\to\infty} \|y_n\| \leq d \text{ and } \lim_{n\to\infty} \|(1 - t_n)\, x_n + t_n y_n\| = d$$

hold for some $d \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

3. Main Results

Now, we introduce an iteration process which can be viewed as an extension for totally asymptotically I-nonexpansive mappings of iteration process of Mukhamedov and Saburov [14].

Let K be a nonempty closed convex subset of a real Banach space X. Let $T, S : K \to K$ be two totally asymptotically I-nonexpansive mappings, where $I : K \to K$ is a totally asymptotically nonexpansive mapping. We define the iterative sequence $\{x_n\}$ by

$$\left\{ \begin{array}{c} x_0 \in K \\ x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n S^n y_n \\ y_n = (1 - \beta_n)\, x_n + \beta_n T^n z_n \\ z_n = (1 - \gamma_n)\, x_n + \gamma_n I^n x_n \end{array} \right. , n \in \mathbb{N}, \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are *sequences in* $[0, 1]$.

Remark 3.1.

i. If $I = Id$ (Id is the identity mapping), then (3.1) reduces to the modified Ishikawa iterative scheme for two totally asymptotically nonexpansive mappings as follow.

$$\left\{ \begin{array}{c} x_0 \in K \\ x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n S^n y_n \\ y_n = (1 - \beta_n)\, x_n + \beta_n T^n x_n \end{array} \right. , n \in \mathbb{N}, \tag{3.2}$$

where $I : K \to K$ is a totally asymptotically nonexpansive mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

ii. If $I = Id$ (Id is the identity mapping) and $\beta_n = 0$ for all $n \geq 1$, then (3.1) reduces to the modified Mann iterative scheme for one total asymptotically nonexpansive mappings as follow.

$$\left\{ \begin{array}{c} x_0 \in K \\ x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n S^n x_n \end{array} \right. , n \in \mathbb{N}, \tag{3.3}$$

where $I : K \to K$ is a totally asymptotically nonexpansive mapping and $\{\alpha_n\}$ is *sequence in* $[0, 1]$.

iii. We remark that the iteration process (3.1) is more general than the iteration process (1.4), (3.2) and (3.3), and includes the process (3.2) and (3.3) as special cases.

Lemma 3.2. *Let K be a nonempty subset of a real Banach space X. Let $T, S : K \to K$ be two totally asymptotically I-nonexpansive mappings and let $I : K \to K$ be a totally asymptotically nonexpansive mapping. Then there exist nonnegative real sequences $\{\mu_n\}, \{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that for all $x, y \in K$,*

$$\begin{aligned} \|I^n x - I^n y\| &\le \|x - y\| + \mu_n \phi(\|x - y\|) + \lambda_n, \\ \|T^n x - T^n y\| &\le \|I^n x - I^n y\| + \mu_n \phi(\|I^n x - I^n y\|) + \lambda_n, \\ \|S^n x - S^n y\| &\le \|I^n x - I^n y\| + \mu_n \phi(\|I^n x - I^n y\|) + \lambda_n, \ n \ge 1. \end{aligned} \tag{3.4}$$

Proof. Since $T, S : K \to K$ are totally asymptotically I-nonexpansive mappings and $I : K \to K$ is a totally asymptotically nonexpansive mapping, then there exist nonnegative real sequences $\{\mu_n^{'}\}, \{\lambda_n^{'}\}, \{\mu_n^{''}\}, \{\lambda_n^{''}\}$ and $\{\tilde{\mu}_n\}, \{\tilde{\lambda}_n\}, n \ge 1$ with $\mu_n^{'}, \lambda_n^{'}, \mu_n^{''}, \lambda_n^{''}, \tilde{\mu}_n, \tilde{\lambda}_n \to 0$ $(n \to \infty)$ and there exist strictly increasing continuous functions $\phi_1, \phi_2, \phi_3 : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi_1(0) = \phi_2(0) = \phi_3(0) = 0$ such that for all $x, y \in K$,

$$\|I^n x - I^n y\| \le \|x - y\| + \tilde{\mu}_n \phi_1(\|x - y\|) + \tilde{\lambda}_n,$$

$$\|T^n x - T^n y\| \le \|I^n x - I^n y\| + \mu_n^{'} \phi_2(\|I^n x - I^n y\|) + \lambda_n^{'},$$

$$\|S^n x - S^n y\| \le \|I^n x - I^n y\| + \mu_n^{''} \phi_3(\|I^n x - I^n y\|) + \lambda_n^{''}, n \ge 1.$$

Setting

$$\begin{aligned} \mu_n &= \max\{\mu_n^{'}, \mu_n^{''}, \tilde{\mu}\}, \lambda_n = \max\{\lambda_n^{'}, \lambda_n^{''}, \tilde{\lambda}_n\}, \\ \phi(a) &= \max\{\phi_1(a), \phi_2(a), \phi_3(a)\} \text{ for } a \ge 0, \end{aligned}$$

then we obtain that there exist nonnegative real sequences $\{\mu_n\}$ and $\{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\|I^n x - I^n y\| \le \|x - y\| + \mu_n \phi(\|x - y\|) + \lambda_n,$$

$$\|T^n x - T^n y\| \le \|I^n x - I^n y\| + \mu_n \phi(\|I^n x - I^n y\|) + \lambda_n,$$

$$\|S^n x - S^n y\| \le \|I^n x - I^n y\| + \mu_n \phi(\|I^n x - I^n y\|) + \lambda_n, n \ge 1.$$

This completes the proof. □

Before proving our main results, throughout this paper we take mappings S, T and I as in (3.4).

Lemma 3.3. *Let K be a nonempty closed convex subset of a real Banach space X. Let $T, S : K \to K$ be two totally asymptotically I-nonexpansive mappings and $I : K \to K$ be a totally asymptotically nonexpansive mapping with sequences $\{\mu_n\}, \{\lambda_n\}$ defined by (3.4) such that $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \lambda_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \le M^* \lambda$ for all $\lambda \ge M$. Let $\{x_n\}$ be the sequence as defined (3.1). If $F = F(S) \cap F(T) \cap F(I) \ne \emptyset$, then $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F = F(S) \cap F(T) \cap F(I)$.*

Proof. Let $p \in F$. It follows from (3.1) and (3.4) that

$$\begin{aligned} \|z_n - p\| &= \|(1 - \gamma_n)(x_n - p) + \gamma_n (I^n x_n - p)\| \\ &\le (1 - \gamma_n) \|x_n - p\| + \gamma_n \|I^n x_n - p\| \\ &\le (1 - \gamma_n) \|x_n - p\| + \gamma_n [\|x_n - p\| + \mu_n \phi(\|x_n - p\|) + \lambda_n] \\ &\le \|x_n - p\| + \gamma_n \mu_n \phi(\|x_n - p\|) + \lambda_n. \end{aligned} \tag{3.5}$$

Note that ϕ is an increasing function, it follows that $\phi(\lambda) \leq \phi(M)$ whenever $\lambda \leq M$ and (by hypothesis) $\phi(\lambda) \leq M^*\lambda$ if $\lambda \geq M$. In either case, we have

$$\phi(\lambda) \leq \phi(M) + M^*\lambda \tag{3.6}$$

for some $M, M^* \geq 0$. Thus, from (3.5) and (3.6), we have

$$\begin{aligned} \|z_n - p\| &\leq \|x_n - p\| + \gamma_n\mu_n[\phi(M) + M^*\|x_n - p\|] + \lambda_n \\ &\leq (1 + M^*\mu_n)\|x_n - p\| + Q_1(\mu_n + \lambda_n) \end{aligned} \tag{3.7}$$

for some constant $Q_1 > 0$. Similarly, from (3.7) we have

$$\begin{aligned} \|y_n - p\| &= \|(1-\beta_n)(x_n - p) + \beta_n(T^n z_n - p)\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|T^n z_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n[\|I^n z_n - p\| + \mu_n\phi(\|I^n z_n - p\|) + \lambda_n] \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n[\|I^n z_n - p\| + \mu_n\phi(M) \\ &\quad + \mu_n M^*\|I^n z_n - p\| + \lambda_n] \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n[(1 + \mu_n M^*)\|I^n z_n - p\| + \mu_n\phi(M) + \lambda_n] \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n[(1 + \mu_n M^*)(\|z_n - p\| \\ &\quad + \mu_n\phi(\|z_n - p\|) + \lambda_n) + \mu_n\phi(M) + \lambda_n] \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n[(1 + \mu_n M^*)(\|z_n - p\| + \mu_n\phi(M) \\ &\quad + \mu_n M^*\|z_n - p\| + \lambda_n) + \mu_n\phi(M) + \lambda_n] \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n[(1 + \mu_n M^*)((1 + \mu_n M^*)\|z_n - p\| \\ &\quad + \mu_n\phi(M) + \lambda_n) + \mu_n\phi(M) + \lambda_n] \\ &\leq (1-\beta_n)\|x_n - p\| \\ &\quad + \beta_n[(1 + \mu_n M^*)((1 + \mu_n M^*)(1 + M^*\mu_n)\|x_n - p\| \\ &\quad + (1 + M^*\mu_n)Q_1(\mu_n + \lambda_n) + \mu_n\phi(M) + \lambda_n) + \mu_n\phi(M) + \lambda_n] \\ &\leq \|x_n - p\| + (3(\mu_n M^*)^2 + 3\mu_n M^* + (\mu_n M^*)^3)\|x_n - p\| \\ &\quad + (1 + M^*\mu_n)^2 Q_1(\mu_n + \lambda_n) + (1 + M^*\mu_n)(\mu_n\phi(M) + \lambda_n) \\ &\quad + \mu_n\phi(M) + \lambda_n \\ &\leq (1 + M_2\mu_n)\|x_n - p\| + Q_2(\mu_n + \lambda_n) \end{aligned} \tag{3.8}$$

for some constants $M_2, Q_2 > 0$. Hence, it follows from (3.6) and (3.8) that

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)(x_n - p) + \alpha_n(S^n y_n - p)\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|S^n y_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n[\|I^n y_n - p\| + \mu_n\phi(\|I^n y_n - p\|) + \lambda_n] \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n[\|I^n y_n - p\| + \mu_n\phi(M) \\ &\quad + \mu_n M^*\|I^n y_n - p\| + \lambda_n] \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n[(1 + \mu_n M^*)\|I^n y_n - p\| + \mu_n\phi(M) + \lambda_n] \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n[(1 + \mu_n M^*)(\|y_n - p\| \\ &\quad + \mu_n\phi(\|y_n - p\|) + \lambda_n) + \mu_n\phi(M) + \lambda_n] \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n[(1 + \mu_n M^*)(\|y_n - p\| + \mu_n\phi(M) \\ &\quad + \mu_n M^*\|y_n - p\| + \lambda_n) + \mu_n\phi(M) + \lambda_n] \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n[(1 + \mu_n M^*)((1 + \mu_n M^*)\|y_n - p\| \\ &\quad + \mu_n\phi(M) + \lambda_n) + \mu_n\phi(M) + \lambda_n] \\ &\leq (1-\alpha_n)\|x_n - p\| \end{aligned}$$

$$
\begin{aligned}
&+\alpha_n[(1+\mu_n M^*)\left((1+\mu_n M^*)\left(1+M_2\mu_n\right)\|x_n-p\|\right.\\
&\left.+\left(1+M^*\mu_n\right)Q_2\left(\mu_n+\lambda_n\right)+\mu_n\phi\left(M\right)+\lambda_n\right)+\mu_n\phi\left(M\right)+\lambda_n]\\
\leq\;&\left(1+\left(M_2+2M^*\right)\mu_n+\left(2M^*M_2+(M^*)^2\right)\mu_n^2\right.\\
&\left.+M_2(M^*)^2\mu_n^3\right)\|x_n-p\|+M_2(M^*)^2\mu_n^3\|x_n-p\|\\
&+\left(1+M^*\mu_n\right)^2Q_2\left(\mu_n+\lambda_n\right)\\
&+\left(1+M^*\mu_n\right)\left(\mu_n\phi\left(M\right)+\lambda_n\right)+\mu_n\phi\left(M\right)+\lambda_n\\
\leq\;&\left(1+M_3\mu_n\right)\|x_n-p\|+Q_3\left(\mu_n+\lambda_n\right) \qquad (3.9)
\end{aligned}
$$

for some constants $M_3, Q_3 > 0$. Since $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\lambda_n < \infty$, by Lemma 2.1, we get $\lim_{n\to\infty}\|x_n-p\|$ exists. This completes the proof. □

Theorem 3.1. *Let K be a nonempty closed subset of a real Banach space X. Let $T, S : K \to K$ be two continuous totally asymptotically I-nonexpansive mappings and $I : K \to K$ be a continuous totally asymptotically nonexpansive mapping with sequences $\{\mu_n\}, \{\lambda_n\}$ defined by (3.4) such that $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\lambda_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$. Let $\{x_n\}$ be the sequence as defined (3.1) and $F = F(S)\cap F(T)\cap F(I) \neq \emptyset$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of S, T and I if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x_n, F) = \inf_{p\in F}\|x_n-p\|, n \geq 1$.*

Proof. The necessity of the conditions is obvious. Thus, we need only prove the sufficiency. Since S, T and I are continuous mappings, $F(T), F(S)$ and $F(I)$ are closed. Hence $F = F(S)\cap F(T)\cap F(I)$ is a nonempty closed set.

For any given $p \in F$, we have from (3.9)

$$\|x_{n+1}-p\| \leq (1+M_3\mu_n)\|x_n-p\| + Q_3(\mu_n+\lambda_n)$$

which gives

$$d(x_{n+1}, F) \leq (1+M_3\mu_n)\, d(x_n, F) + Q_3(\mu_n+\lambda_n). \qquad (3.10)$$

Now applying Lemma 2.1 to (3.10), we get the existence of the limit $\lim_{n\to\infty} d(x_n, F)$.

But by hypothesis $\liminf_{n\to\infty} d(x_n, F) = 0$, thus we have $\lim_{n\to\infty} d(x_n, F) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence in X. As $1+t \leq \exp(t)$ for all $t > 0$, from (3.9), we obtain

$$\|x_{n+1}-p\| \leq \exp(M_3\mu_n)\left(\|x_n-p\| + Q_3(\mu_n+\lambda_n)\right). \qquad (3.11)$$

Thus, for any given m, n, iterating (3.11), we obtain

$$
\begin{aligned}
\|x_{n+m}-p\| \;&\leq\; \exp(M_3\mu_{n+m-1})\left(\|x_{n+m-1}-p\| + Q_3(\mu_{n+m-1}+\lambda_{n+m-1})\right)\\
&\;\;\vdots\\
&\leq\; \exp\Big(\sum_{i=n}^{n+m-1} M_3\mu_i\Big)\left(\|x_n-p\| + \sum_{i=n}^{n+m-1} Q_3(\mu_i+\lambda_i)\right)\\
&\leq\; \exp\Big(\sum_{i=n}^{\infty} M_3\mu_i\Big)\left(\|x_n-p\| + \sum_{i=n}^{\infty} Q_3(\mu_i+\lambda_i)\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|x_{n+m}-x_n\| \;&\leq\; \|x_{n+m}-p\| + \|x_n-p\|\\
&\leq\; \left[1+\left(\exp\Big(\sum_{i=n}^{\infty} M_3\mu_i\Big)\right)\right]\|x_n-p\|
\end{aligned}
$$

$$+\exp(\sum_{i=n}^{\infty} M_3\mu_i)\left(\sum_{i=n}^{\infty} Q_3\left(\mu_i+\lambda_i\right)\right).$$

This imply that

$$\|x_{n+m}-x_n\| \leq D\|x_n-p\|+D\left(\sum_{i=n}^{\infty}\left(\mu_i+\lambda_i\right)\right) \tag{3.12}$$

for same constant $D>0$. Taking infimum over $p\in F$ in (3.12) gives

$$\|x_{n+m}-x_n\| \leq Dd\left(x_n,F\right)+D\left(\sum_{i=n}^{\infty}\left(\mu_i+\lambda_i\right)\right).$$

Now, since $\lim_{n\to\infty} d\left(x_n,F\right)=0$ and $\sum_{i=n}^{\infty}\left(\mu_i+\lambda_i\right)<\infty$, given $\epsilon>0$, there exists an integer $N_1>0$ such that for all $n\geq N_1$, $d\left(x_n,F\right)<\epsilon/2D$ and $\sum_{i=n}^{\infty}\left(\mu_i+\lambda_i\right)<\epsilon/2D$. Consequently, from last inequality we have $\|x_{n+m}-x_n\|<\epsilon$, which means that $\{x_n\}$ is a Cauchy sequence in X, and completeness of X yield the existence of $x^*\in X$ such that $x_n\to x^*$. We now show that $x^*\in F$. Suppose that $x^*\notin F$. Since F closed subset of X, we have that $d\left(x^*,F\right)>0$. But, for all $p\in F$, we have

$$\|x^*-p\| \leq \|x^*-x_n\|+\|x_n-p\|.$$

This implies

$$d\left(x^*,F\right) \leq \|x^*-x_n\|+d\left(x_n,F\right),$$

so we obtain $d\left(x^*,F\right)=0$ as $n\to\infty$, which contradicts $d\left(x^*,F\right)>0$. Hence, x^* is a common fixed point of T,S and I. This completes the proof. □

Lemma 3.4. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space X. Let $T,S:K\to K$ be two continuous totally asymptotically I-nonexpansive mappings and $I:K\to K$ be a continuous totally asymptotically nonexpansive mapping with sequences $\{\mu_n\},\{\lambda_n\}$ defined by (3.4) such that $\sum_{n=1}^{\infty}\mu_n<\infty, \sum_{n=1}^{\infty}\lambda_n<\infty$. Assume that there exist $M,M^*>0$ such that $\phi(\lambda)\leq M^*\lambda$ for all $\lambda\geq M$ and $F=F(S)\cap F(T)\cap F(I)\neq\emptyset$. Let $\{x_n\}$ be the sequence as defined (3.1), where $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[a,1-a]$ for some $a\in(0,1)$. Then*

$$\lim_{n\to\infty}\|x_n-S^nx_n\|=\lim_{n\to\infty}\|x_n-T^nx_n\|=\lim_{n\to\infty}\|x_n-I^nx_n\|=0$$

and

$$\lim_{n\to\infty}\|x_n-Sx_n\|=\lim_{n\to\infty}\|x_n-Tx_n\|=\lim_{n\to\infty}\|x_n-Ix_n\|=0.$$

Proof. Let $p\in F$. Then by Lemma 3.3, we have

$$\lim_{n\to\infty}\|x_n-p\|=d. \tag{3.13}$$

It follows from (3.1) that

$$\|x_{n+1}-p\|=\|(1-\alpha_n)(x_n-p)+\alpha_n(S^ny_n-p)\|\to d,\ (n\to\infty). \tag{3.14}$$

From (3.4) and (3.6), we obtain

$$\begin{aligned}
\|S^ny_n-p\| &\leq \|I^ny_n-p\|+\mu_n\phi\left(\|I^ny_n-p\|\right)+\lambda_n\\
&\leq \|I^ny_n-p\|+\mu_n\phi\left(M\right)+\mu_nM^*\|I^ny_n-p\|+\lambda_n\\
&\leq \left(1+\mu_nM^*\right)\|I^ny_n-p\|+\mu_n\phi\left(M\right)+\lambda_n\\
&\leq \left(1+\mu_nM^*\right)\left(\|y_n-p\|+\mu_n\phi\left(\|y_n-p\|\right)+\lambda_n\right)+\mu_n\phi\left(M\right)+\lambda_n\\
&\leq \left(1+\mu_nM^*\right)\left(\|y_n-p\|+\mu_n\phi\left(M\right)+\mu_nM^*\|y_n-p\|+\lambda_n\right)\\
&\quad +\mu_n\phi\left(M\right)+\lambda_n
\end{aligned}$$

$$\leq \left(1+\mu_n M^*\right)\left(\left(1+\mu_n M^*\right)\|y_n-p\|+\mu_n\phi\left(M\right)+\lambda_n\right) \\ +\mu_n\phi\left(M\right)+\lambda_n. \tag{3.15}$$

Because of $\sum_{n=1}^{\infty}\mu_n<\infty, \sum_{n=1}^{\infty}\lambda_n<\infty$, we have

$$\limsup_{n\to\infty}\|S^n y_n-p\|\leq\limsup_{n\to\infty}\|y_n-p\|$$

and from (3.8)

$$\limsup_{n\to\infty}\|S^n y_n-p\|\leq\limsup_{n\to\infty}\left(1+M_2\mu_n\right)\|x_n-p\|+Q_2\left(\mu_n+\lambda_n\right)=d. \tag{3.16}$$

Using (3.13),(3.16) and applying Lemma 2.2 to (3.14), we get

$$\lim_{n\to\infty}\|x_n-S^n y_n\|=0. \tag{3.17}$$

Also from (3.15), we have

$$\begin{aligned}\|x_n-p\| &\leq \|x_n-S^n y_n\|+\|S^n y_n-p\| \\ &\leq \|x_n-S^n y_n\|+\left(1+\mu_n M^*\right)\left(\left(1+\mu_n M^*\right)\|y_n-p\| \right.\\ &\quad \left.+\mu_n\phi\left(M\right)+\lambda_n\right)+\mu_n\phi\left(M\right)+\lambda_n\end{aligned}$$

which implies $d\leq\lim_{n\to\infty}\|y_n-p\|$ and from (3.8), we have $\lim_{n\to\infty}\|y_n-p\|\leq d$. Thus we obtain

$$\lim_{n\to\infty}\|y_n-p\|=d. \tag{3.18}$$

On the other hand note that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\alpha_n\|x_n-S^n y_n\|=0. \tag{3.19}$$

It follows from (3.18) that

$$\|y_n-p\|=\|(1-\beta_n)(x_n-p)+\beta_n(T^n z_n-p)\|\to d,\ n\to\infty. \tag{3.20}$$

From (3.4) and (3.6), we obtain

$$\begin{aligned}\|T^n z_n-p\| &\leq \|I^n z_n-p\|+\mu_n\phi\left(\|I^n z_n-p\|\right)+\lambda_n \\ &\leq \|I^n z_n-p\|+\mu_n\phi\left(M\right)+\mu_n M^*\|I^n z_n-p\|+\lambda_n \\ &\leq \left(1+\mu_n M^*\right)\|I^n z_n-p\|+\mu_n\phi\left(M\right)+\lambda_n \\ &\leq \left(1+\mu_n M^*\right)\left(\|z_n-p\|+\mu_n\phi\left(\|z_n-p\|\right)+\lambda_n\right)+\mu_n\phi\left(M\right)+\lambda_n \\ &\leq \left(1+\mu_n M^*\right)\left(\|z_n-p\|+\mu_n\phi\left(M\right)+\mu_n M^*\|z_n-p\|+\lambda_n\right) \\ &\quad +\mu_n\phi\left(M\right)+\lambda_n \\ &\leq \left(1+\mu_n M^*\right)\left(\left(1+\mu_n M^*\right)\|z_n-p\|+\mu_n\phi\left(M\right)+\lambda_n\right) \\ &\quad +\mu_n\phi\left(M\right)+\lambda_n. \end{aligned} \tag{3.21}$$

Therefore,

$$\limsup_{n\to\infty}\|T^n z_n-p\|\leq\limsup_{n\to\infty}\|z_n-p\|$$

and from (3.7), we have

$$\limsup_{n\to\infty}\|T^n z_n-p\|\leq\limsup_{n\to\infty}\left(1+M^*\mu_n\right)\|x_n-p\|+Q_1\left(\mu_n+\lambda_n\right)=d. \tag{3.22}$$

Using (3.13), (3.22) and applying Lemma 2.2 to (3.20), we get

$$\lim_{n\to\infty}\|x_n-T^n z_n\|=0. \tag{3.23}$$

Also from (3.21), we have

$$\begin{aligned}\|x_n-p\| &\leq \|x_n-T^n z_n\|+\|T^n z_n-p\| \\ &\leq \|x_n-T^n z_n\|+\left(1+\mu_n M^*\right)\left(\left(1+\mu_n M^*\right)\|z_n-p\| \right.\\ &\quad \left.+\mu_n\phi\left(M\right)+\lambda_n\right)+\mu_n\phi\left(M\right)+\lambda_n,\end{aligned}$$

which implies $d \leq \lim_{n\to\infty} \|z_n - p\|$ and from (3.7) we have $\lim_{n\to\infty} \|z_n - p\| \leq d$. This gives

$$\lim_{n\to\infty} \|z_n - p\| = d. \tag{3.24}$$

As above, from (3.1) one can see that

$$\|z_n - p\| = \|(1-\gamma_n)(x_n - p) + \gamma_n(I^n x_n - p)\| \to d, \tag{3.25}$$

as $n \to \infty$. Besides this, we have

$$\begin{aligned} \|I^n x_n - p\| &\leq \|x_n - p\| + \mu_n \phi\left(\|x_n - p\|\right) + \lambda_n \\ &\leq \left(1 + M^* \mu_n\right) \|x_n - p\| + \mu_n \phi\left(M\right) + \lambda_n. \end{aligned}$$

Taking limit as $n \to \infty$ in last inequality, we obtain

$$\limsup_{n\to\infty} \|I^n x_n - p\| \leq d. \tag{3.26}$$

Hence from (3.13), (3.26) and (3.25), we get

$$\lim_{n\to\infty} \|x_n - I^n x_n\| = 0. \tag{3.27}$$

Meanwhile, note that from (3.27)

$$\begin{aligned} \|z_n - x_n\| &= \|(1-\gamma_n)(x_n - p) + \gamma_n(I^n x_n - p) - x_n\| \\ &= \gamma_n \|x_n - I^n x_n\| = 0. \end{aligned} \tag{3.28}$$

and from (3.23)

$$\begin{aligned} \|y_n - x_n\| &= \|(1-\beta_n)(x_n - p) + \beta_n(T^n z_n - p) - x_n\| \\ &= \beta_n \|x_n - T^n z_n\| = 0. \end{aligned} \tag{3.29}$$

Now we show that $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$. By way of (3.21), we have

$$\begin{aligned} \|T^n z_n - T^n x_n\| \leq\ & \left(1 + \mu_n M^*\right)\left(\left(1 + \mu_n M^*\right)\|z_n - x_n\|\right. \\ & \left.+ \mu_n \phi\left(M\right) + \lambda_n\right) + \mu_n \phi\left(M\right) + \lambda_n. \end{aligned}$$

Thus we get

$$\begin{aligned} \|x_n - T^n x_n\| &\leq \|x_n - T^n z_n\| + \|T^n z_n - T^n x_n\| \\ &\leq \|x_n - T^n z_n\| + \left(1 + \mu_n M^*\right)\left(\left(1 + \mu_n M^*\right)\|z_n - x_n\|\right. \\ &\quad \left.+ \mu_n \phi\left(M\right) + \lambda_n\right) + \mu_n \phi\left(M\right) + \lambda_n, \end{aligned}$$

which implies

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{3.30}$$

Now we show that $\lim_{n\to\infty} \|x_n - S^n x_n\| = 0$. This time, by way of (3.15), we have

$$\begin{aligned} \|S^n y_n - S^n x_n\| \leq\ & \left(1 + \mu_n M^*\right)\left(\left(1 + \mu_n M^*\right)\|y_n - x_n\|\right. \\ & \left.+ \mu_n \phi\left(M\right) + \lambda_n\right) + \mu_n \phi\left(M\right) + \lambda_n. \end{aligned}$$

Thus we get

$$\begin{aligned} \|x_n - S^n x_n\| &\leq \|x_n - S^n y_n\| + \|S^n y_n - S^n x_n\| \\ &\leq \|x_n - S^n y_n\| + \left(1 + \mu_n M^*\right)\left(\left(1 + \mu_n M^*\right)\|y_n - x_n\|\right. \\ &\quad \left.+ \mu_n \phi\left(M\right) + \lambda_n\right) + \mu_n \phi\left(M\right) + \lambda_n. \end{aligned}$$

Taking limit in last inequality, we obtain from (3.17) and (3.29)

$$\lim_{n\to\infty} \|x_n - S^n x_n\| = 0. \tag{3.31}$$

Finally, we prove

$$\lim_{n\to\infty} \|x_n - S x_n\| = \lim_{n\to\infty} \|x_n - T x_n\| = \lim_{n\to\infty} \|x_n - I x_n\| = 0.$$

Firstly, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$$

from (3.19) and

$$\lim_{n\to\infty} \|x_n - I^n x_n\| = 0$$

from (3.27). Hence we get that

$$\begin{aligned}
\left\|x_n - I^{n-1}x_n\right\| &\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - I^{n-1}x_{n-1}\right\| \\
&\quad + \left\|I^{n-1}x_{n-1} - I^{n-1}x_n\right\| \\
&\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - I^{n-1}x_{n-1}\right\| + \|x_n - x_{n-1}\| \\
&\quad + \mu_n \phi\left(\|x_n - x_{n-1}\|\right) + \lambda_n \\
&\leq (2 + \mu_n M^*)\|x_n - x_{n-1}\| + \left\|x_{n-1} - I^{n-1}x_{n-1}\right\| \\
&\quad + \mu_n \phi(M) + \lambda_n \\
&\to 0. \qquad (3.32)
\end{aligned}$$

Because I is continuous, therefore we have from (3.32)

$$\|I^n x_n - I x_n\| \leq \left\|I\left(I^{n-1}x_n\right) - I x_n\right\| \to 0.$$

Thus we obtain

$$\|x_n - I x_n\| \leq \|x_n - I^n x_n\| + \|I^n x_n + I x_n\| \to 0.$$

Also we have $\lim_{n\to\infty} \|x_n - S^n x_n\| = 0$ from (3.31). Thus from (3.19) and (3.27), we get that

$$\begin{aligned}
\left\|x_n - S^{n-1}x_n\right\| &\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - S^{n-1}x_{n-1}\right\| \\
&\quad + \left\|S^{n-1}x_{n-1} - S^{n-1}x_n\right\| \\
&\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - S^{n-1}x_{n-1}\right\| + \left\|I^{n-1}x_n - I^{n-1}x_{n-1}\right\| \\
&\quad + \mu_n \phi\left(\left\|I^{n-1}x_n - I^{n-1}x_{n-1}\right\|\right) + \lambda_n \\
&\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - S^{n-1}x_{n-1}\right\| + \left\|I^{n-1}x_n - I^{n-1}x_{n-1}\right\| \\
&\quad + \mu_n \phi(M) + \mu_n M^* \left\|I^{n-1}x_n - I^{n-1}x_{n-1}\right\| + \lambda_n \\
&\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - S^{n-1}x_{n-1}\right\| \\
&\quad + (1 + \mu_n M^*)\left\|I^{n-1}x_n - I^{n-1}x_{n-1}\right\| + \mu_n \phi(M) + \lambda_n \\
&\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - S^{n-1}x_{n-1}\right\| \\
&\quad + (1 + \mu_n M^*)[\|x_n - x_{n-1}\| + \mu_n \phi(\|x_n - x_{n-1}\|) + \lambda_n] \\
&\quad + \mu_n \phi(M) + \lambda_n \\
&\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - S^{n-1}x_{n-1}\right\| \\
&\quad + (1 + \mu_n M^*)[\|x_n - x_{n-1}\| + \mu_n \phi(M) \\
&\quad + \mu_n M^* \|x_n - x_{n-1}\| + \lambda_n] + \mu_n \phi(M) + \lambda_n \\
&\to 0. \qquad (3.33)
\end{aligned}$$

Because S is continuous, therefore we have from (3.33)

$$\|S^n x_n - S x_n\| \leq \left\|S\left(S^{n-1}x_n\right) - S x_n\right\| \to 0.$$

Thus we obtain

$$\|x_n - S x_n\| \leq \|x_n - S^n x_n\| + \|S^n x_n - S x_n\| \to 0.$$

Similarly, in the same way we get

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0.$$

This completes the proof. □

Using this lemma, we prove the following strong convergence theorems.

Theorem 3.2. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space X. Let $T, S : K \to K$ be two continuous totally asymptotically I-nonexpansive mappings and $I : K \to K$ be a continuous totally asymptotically nonexpansive mapping with sequences $\{\mu_n\}, \{\lambda_n\}$ defined by (3.4) such that $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \lambda_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$ and $F = F(S) \cap F(T) \cap F(I) \neq \emptyset$. Let $\{x_n\}$ be the sequence as defined (3.1), where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[a, 1-a]$ for some $a \in (0,1)$. If one of S, T and I is compact, then the sequence $\{x_n\}$ converges strongly to a common fixed point of S, T and I.*

Proof. Without loss of generality, let S be compact. Then there exists a subsequence $\{S^{n_k} x_{n_k}\}$ of $\{S^n x_n\}$ such that $\{S^{n_k} x_{n_k}\}$ converges strongly to $x^* \in K$. Thus, from (3.31), $\{x_{n_k}\}$ converges strongly to x^*. Using continuity of S we have $\{S^{n_k+1} x_{n_k+1}\}$ converges strongly to Sx^*. On the other hand, according to (3.30) and (3.27), we get that $\{T^{n_k} x_{n_k}\}$, $\{I^{n_k} x_{n_k}\}$ converge strongly to x^*. Since T and I are continuous, the squences $\{T^{n_k+1} x_{n_k}\}$ and $\{I^{n_k+1} x_{n_k}\}$ converge strongly to Tx^* and Ix^*, respectively. Besides these, since $\|x_{n+1} - x_n\|$ converges to 0; definition of S, T and I imply that $\left\|S^{n_k+1} x_{n+1} - S^{n_k+1} x_n\right\|, \left\|T^{n_k+1} x_{n+1} - T^{n_k+1} x_n\right\|$ and $\left\|I^{n_k+1} x_{n+1} - I^{n_k+1} x_n\right\|$ converge to 0. Observe that

$$\begin{aligned}
\|x^* - Sx^*\| &\leq \|x^* - x_{n_k+1}\| + \left\|x_{n_k+1} - S^{n_k+1} x_{n_k+1}\right\| \\
&\quad + \left\|S^{n_k+1} x_{n_k+1} - S^{n_k+1} x_{n_k}\right\| + \left\|S^{n_k+1} x_{n_k+1} - S^{n_k+1} x_{n_k}\right\| \\
&\quad + \left\|S^{n_k+1} x_{n_k} - Sx^*\right\|, \\
\|x^* - Tx^*\| &\leq \|x^* - x_{n_k+1}\| + \left\|x_{n_k+1} - T^{n_k+1} x_{n_k+1}\right\| \\
&\quad + \left\|T^{n_k+1} x_{n_k+1} - T^{n_k+1} x_{n_k}\right\| + \left\|T^{n_k+1} x_{n_k+1} - T^{n_k+1} x_{n_k}\right\| \\
&\quad + \left\|T^{n_k+1} x_{n_k} - Tx^*\right\|
\end{aligned}$$

and

$$\begin{aligned}
\|x^* - Ix^*\| &\leq \|x^* - x_{n_k+1}\| + \left\|x_{n_k+1} - I^{n_k+1} x_{n_k+1}\right\| \\
&\quad + \left\|I^{n_k+1} x_{n_k+1} - I^{n_k+1} x_{n_k}\right\| + \left\|I^{n_k+1} x_{n_k+1} - I^{n_k+1} x_{n_k}\right\| \\
&\quad + \left\|I^{n_k+1} x_{n_k} - Ix^*\right\|.
\end{aligned}$$

Taking limit as $k \to \infty$ in the last inequalities, we find $x^* = Sx^*, x^* = Tx^*$ and $x^* = Ix^*$. So $x^* \in F$. However, due to Lemma 2.1, the limit $\lim_{n\to\infty} \|x_n - p\|$ exists, $p \in F$. Hence, $\{x_n\}$ converges strongly to $x^* \in F$. This completes the proof. □

Theorem 3.3. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space X. Let $T, S : K \to K$ be two continuous totally asymptotically I-nonexpansive mappings, let $I : K \to K$ be a continuous totally asymptotically nonexpansive mapping with sequences $\{\mu_n\}, \{\lambda_n\}$ defined by (3.4) such that $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \lambda_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$ and $F = F(S) \cap F(T) \cap F(I) \neq \emptyset$. Let $\{x_n\}$ be the sequence as defined (3.1), where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[a, 1-a]$ for some $a \in (0,1)$. If S, T and I satisfy Condition $\left(A'\right)$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of S, T and I.*

Proof. By Lemma 3.4,

$$\lim_{n\to\infty}\|x_n - Sx_n\| = \lim_{n\to\infty}\|x_n - Tx_n\| = \lim_{n\to\infty}\|x_n - Ix_n\| = 0.$$

Using Condition $\left(A'\right)$, we get

$$\lim_{n\to\infty} f\left(d\left(x,F\right)\right) = \lim_{n\to\infty}\frac{1}{3}\left(\|x - Sx\| + \|x - Tx\| + \|x - Ix\|\right) = 0.$$

Since f is a nondecreasing function and $f(0) = 0$, so it follows that

$$\lim_{n\to\infty} d\left(x,F\right) = 0.$$

Now applying the Theorem 3.1, we obtain the result. □

Our weak convergence theorem is as follows:

Theorem 3.4. *Let X be a real uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of X. Let $T,S : K \to K$ be two continuous totally asymptotically I-nonexpansive mappings, let $I : K \to K$ be a continuous totally asymptotically nonexpansive mapping with sequences $\{\mu_n\},\{\lambda_n\}$ defined by (3.4) such that $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\lambda_n < \infty$. Let $E : X \to X$ be an identity mapping. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$ and $F = F(S)\cap F(T)\cap F(I) \neq \emptyset$. Let $\{x_n\}$ be the sequence as defined (3.1), where $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[a, 1-a]$ for some $a \in (0,1)$. If $E - S, E - T$ and $E - I$ are demiclosed at zero then the sequence $\{x_n\}$ converges weakly to a common fixed point of S, T and I.*

Proof. Let $p \in F$. Then by Lemma 3.3, $\lim_{n\to\infty}\|x_n - p\|$ exists. We prove that $\{x_n\}$ has a unique weak subsequential limit in F. Since $\{x_n\}$ is a bounded sequence in a uniformly convex Banach space X, there exist two weakly convergent subsequences $\{x_{n_i}\}$ and $\left\{x_{n_j}\right\}$ of $\{x_n\}$. Let $w_1 \in K$ and $w_2 \in K$ be weak limits of the subsequences $\{x_{n_i}\}$ and $\left\{x_{n_j}\right\}$, respectively. Since S is demiclosed with respect to zero (by hypothesis) then we obtain $Sw_1 = w_1$. Similarly, $Tw_1 = w_1$ and $Iw_1 = w_1$. That is, $w_1 \in F$. In the same way, we can prove that $w_2 \in F$.

Next, we prove the uniqueness. For this, suppose that $w_1 \neq w_2$. Then, by Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - w_1\| &= \lim_{i\to\infty}\|x_{n_i} - w_1\| \\
&< \lim_{i\to\infty}\|x_{n_i} - w_2\| \\
&= \lim_{n\to\infty}\|x_n - w_2\| \\
&= \lim_{j\to\infty}\left\|x_{n_j} - w_2\right\| \\
&< \lim_{j\to\infty}\left\|x_{n_j} - w_1\right\| \\
&= \lim_{n\to\infty}\|x_n - w_1\|,
\end{aligned}$$

which is a contradiction. Hence $\{x_n\}$ converges weakly to a point of F. □

Remark 3.5. Since the class of totally asymptotically I-nonexpansive mappings includes totally asymptotically nonexpansive mappings, our results improve and extend the corresponding ones announced by Ya.I. Alber et al. [1], Mukhamedov and Saburov [14], Chidume and Ofoedu [4, 3] and Gunduz [6]. Also our results genaralize corresponding results given in [7, 8, 9, 19, 20].

Remark 3.6. The iteration process (3.1) can be generalized for two finite families of totally asymptotically I_i-nonexpansive mappings $\{T_j : j \in J\}$ and $\{S_j : j \in J\}$,where $\{I_j : j \in J\}$ is a finite family of totally asymptotically nonexpansive mappings (where $J = \{1, 2, ..., N\}$).

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