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DUALITY FOR A NONDIFFERENTIABLE MULTIOBJECTIVE HIGHER-ORDER SYMMETRIC FRACTIONAL PROGRAMMING PROBLEMS WITH CONE CONSTRAINTS

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ABSTRACT. In this paper, a pair of nondifferentiable multiobjective higher-order symmetric fractional dual problem with cone constraints is formulated. For a differentiable function, we introduce the definition of higher-order (C, α, ρ, d)-convexity. Next, we prove appropriate duality relations under aforesaid assumptions.

KEYWORDS : Higher-order; symmetric duality; multiobjective fractional programming; (C, α, ρ, d)-convexity; cone constraints.
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1. INTRODUCTION

Higher-order duality is significant due to its computational importance as it provides more higher bounds whenever approximation is used. By introducing two different functions, $h : R^n \times R^n \to R$ and $k : R^n \times R^n \to R^m$, Mangasarian [8] formulated higher-order dual for a single objective nonlinear problems, $\{\min f(x),\ \text{subject to}\ \ g(x) \leqq 0\}$. Inspired by this concept, many researchers have worked in this direction. Chen [1] has formulated higher-order multiobjective symmetric dual programs and established duality relations under higher-order $F-$ convexity assumptions. A higher-order vector optimization problem and its dual has been studied by Kassem [9].

In last several years, various optimality and duality results have been obtained for multiobjective fractional programming problems. In Chen [1], multiobjective fractional problem and its duality theorems have been considered under higher-order (F, α, ρ, d)- convexity. Later on, Suneja et al. [10] discussed higher-order Mond-Weir and Schaible type nondifferentiable dual programs and their duality theorems under higher-order (F, ρ, σ) -type I- assumptions. Recently, Ying [12]

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has studied higher-order multiobjective symmetric fractional problem and formulated its Mond- Weir type dual. Further, duality results are obtained under higher-order (F, α, ρ, d)-convexity.

In this paper, we introduce a pair of nondifferentiable multiobjective Mond-Weir type higher-order symmetric fractional programming problems over cones. For a differentiable function $h : R^n \times R^n \longrightarrow R$, we introduce the definition of higher-order (C, α, ρ, d)-convexity, which extends some kinds of generalized convexity. Under the higher-order (C, α, ρ, d)-convexity assumptions, we prove the higher-order weak, strong and strict converse duality theorems.

2. PRELIMINARIES AND NOTATIONS

Let R^n be the n-dimensional Euclidean space and R^n_+ be its non-negative orthant. The following conventions for vectors in R^n will be used:

$$
\begin{array}{lll}
x < y & \text{if and only if} & x_i < y_i,\ i = 1, 2, ..., n, \\
x \leqq y & \text{if and only if} & x_i \leqq y_i,\ i = 1, 2, ..., n, \\
x \leq y & \text{if and only if} & x_i \leq y_i,\ i = 1, 2, ..., n \text{ but } x \neq y, \\
x \nleq y & \text{is the negation of} & x \leq y.
\end{array}
$$

For a real-valued twice differentiable function $h(x, y)$ defined on an open set in $R^n \times R^m$, denote by $\nabla_x h(\bar{x}, \bar{y})-$ the gradient vector of h with respect to x at $(\bar{x}, \bar{y})$, $\nabla_{xx} h(\bar{x}, \bar{y})-$ the Hessian matrix with respect to x at $(\bar{x}, \bar{y})$. Similarly, $\nabla_y h(\bar{x}, \bar{y})$, $\nabla_{xy} h(\bar{x}, \bar{y})$ and $\nabla_{yy} h(\bar{x}, \bar{y})$ are also defined.

Definition 2.1 [4]. Let C be a compact convex set in R^n. The support function of C is defined by

$$s(x|C) = \max\{x^T y : y \in C\}.$$

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists a $z \in R^n$ such that

$$s(y|C) \geqq s(x|C) + z^T(y - x), \forall x \in C.$$

The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) = \{z \in C : z^T x = s(x|C)\}.$$

For a convex set $D \subset R^n$, the normal cone to D at a point $x \in D$ is defined by

$$N_D(x) = \{y \in R^n : y^T(z - x) \leqq 0, \forall z \in D\}.$$

When C is a compact convex set, $y \in N_C(x)$ if and only if $s(y|C) = x^T y$, or equivalently, $x \in \partial s(y|C)$.

Definition 2.2 [4]. The positive polar cone S^* of a cone $S \subseteq R^n$ is defined by

$$S^* = \{y \in R^n : x^T y \geqq 0, \forall x \in S\}.$$

A general multiobjective programming problem can be expressed in the following form :

(P) Minimize $f(x) = (f_1(x), f_2(x), ..., f_k(x))^T$

subject to $x \in X^0 = \{x \in X : g(x) \leqq 0\}$,

where $X \subset R^n$ is open, $f : X \to R^k$, $g : X \to R^m$, are differentiable on X.

Definition 2.3 [3]. A feasible solution $\bar{x} \in X^0$ is said to be a weakly efficient solution of (P) if there exists no $x \in X^0$ such that $f(x) < f(\bar{x})$.

Definition 2.4 [3]. A feasible solution $\bar{x} \in X^0$ is said to be an efficient (or Pareto optimal) solution of (P) if there exists no $x \in X^0$ such that $f(x) \leq f(\bar{x})$.

Definition 2.5 [7]. Let $C : X \times X \times R^n \to R$ $(X \subseteq R^n)$ be a function which satisfies $C_{x,u}(0) = 0, \forall (x, u) \in X \times X$. Then, the function C is said to be convex on R^n with respect to third argument iff for any fixed $(x, u) \in X \times X$,

$$C_{x,u}(\lambda x_1 + (1 - \lambda)x_2) \leqq \lambda C_{x,u}(x_1) + (1 - \lambda)C_{x,u}(x_2),\ \forall \lambda \in (0, 1),\ \forall x_1, x_2 \in R^n.$$

Now, we introduce the definition of higher-order (C, α, ρ, d)-convex function:

Definition 2.6 A differentiable function $f : X \to R$ is said to be higher order (C, α, ρ, d) - convex at $u \in X$ with respect to $h : X \times R^n \to R$ if for all $x \in X$ and $p \in R^n$, $\exists\, \rho \in R$, a real valued function $\alpha : X \times X \to R_+ \setminus \{0\}$ and $d : X \times X \to R$ (satisfying $d(x, z) = 0 \Leftrightarrow x = z$) such that

$$\frac{1}{\alpha(x, u)}\big[f(x) - f(u) - h(u, p) + p^T \nabla_p h(u, p) - \rho d^2(x, u)\big] \geqq C_{x,u}\big[\nabla_x f(u) + \nabla_p h(u, p)\big].$$

The function f is higher-order (C, α, ρ, d)- convex over X if, $\forall\ u \in X$, it is higher (C, α, ρ, d)- convex.

3. Higher-order symmetric duality

Consider the following multiobjective fractional symmetric dual programs:

(MFP) Minimize $L(x, y, p) = (L_1(x, y, p_1), L_2(x, y, p_2), ..., L_k(x, y, p_k))^T$
subject to

$$-\sum_{i=1}^{k} \lambda_i \big[(\nabla_y f_i(x, y) - z_i + \nabla_{p_i} H_i(x, y, p_i)) - L_i(x, y, p_i)(\nabla_y g_i(x, y) + r_i + \nabla_{p_i} G_i(x, y, p_i))\big] \in C_2^*,$$

$$y^T\left[\sum_{i=1}^{k} \lambda_i \big[(\nabla_y f_i(x, y) - z_i + \nabla_{p_i} H_i(x, y, p_i)) - L_i(x, y, p_i)(\nabla_y g_i(x, y) + r_i + \nabla_{p_i} G_i(x, y, p_i))\big]\right] \geqq 0,$$

$$\lambda > 0,\ \lambda^T e = 1,\ x \in C_1,\ z_i \in D_i,\ r_i \in F_i,\ i = 1, 2, ..., k.$$

(MFD) Maximize $M(u, v, q) = (M_1(u, v, q_1), M_2(u, v, q_2), ..., M_k(u, v, q_k))^T$
subject to

$$\sum_{i=1}^{k} \lambda_i \big[(\nabla_x f_i(u, v) + w_i + \nabla_{q_i} \Phi_i(u, v, q_i)) - M_i(u, v, q_i)(\nabla_x g_i(u, v) - t_i + \nabla_{q_i} \Psi_i(u, v, q_i))\big] \in C_1^*,$$

$$u^T\left[\sum_{i=1}^{k} \lambda_i \big[(\nabla_x f_i(u, v) + w_i + \nabla_{q_i} \Phi_i(u, v, q_i)) - M_i(u, v, q_i)(\nabla_x g_i(u, v) - t_i + \nabla_{q_i} \Psi_i(u, v, q_i))\big]\right] \leqq 0,$$

$$\lambda > 0,\ \lambda^T e = 1,\ v \in C_2,\ w_i \in Q_i,\ t_i \in E_i,\ i = 1, 2, ..., k,$$

where

$$L_i(x,y,p_i) = \frac{f_i(x,y) + s(x|Q_i) - y^T z_i + H_i(x,y,p_i) - p_i^T \nabla_{p_i} H_i(x,y,p_i)}{g_i(x,y) - s(x|E_i) + y^T r_i + G_i(x,y,p_i) - p_i^T \nabla_{p_i} G_i(x,y,p_i)},$$

$$M_i(u,v,q_i) = \frac{f_i(u,v) - s(v|D_i) + u^T w_i + \Phi_i(u,v,q_i) - q_i^T \nabla_{q_i} \Phi_i(u,v,q_i)}{g_i(u,v) + s(v|F_i) - u^T t_i + \Psi_i(u,v,q_i) - q_i^T \nabla_{q_i} \Psi_i(u,v,q_i)},$$

where $f_i : S_1 \times S_2 \to R$; $g_i : S_1 \times S_2 \to R$; $H_i, G_i : S_1 \times S_2 \times R^m \to R$ and $\Phi_i, \Psi_i : S_1 \times S_2 \times R^n \to R$ are differentiable functions for all $i = 1, 2, ..., k$. $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are such that $C_1 \times C_2 \subset S_1 \times S_2$, where C_1 and C_2 are the closed convex cones in R^n and R^m, respectively. $Q_i,\ E_i$ are compact convex sets in R^n and $D_i,\ F_i$ are compact convex sets in R^m, $e = (1, 1, ..., 1)^T \in R^k$, $p_i \in R^n,\ q_i \in R^m,\ i = 1, 2, ..., k,\ p = (p_1, p_2, ..., p_k),\ q = (q_1, q_2, ..., q_k)$. C_1^* and C_2^* are positive polar cones of C_1 and C_2, respectively. It is assumed that in the feasible regions, the numerators are nonnegative and denominators are positive.

Let $U = (U_1, U_2, ..., U_k)^T$ and $V = (V_1, V_2, ..., V_k)^T$. Then, we can express the programs (MFP) and (MFD) equivalently as:

(MFP)$_U$ Minimize U subject to

$$\begin{aligned}(f_i(x,y) + s(x|Q_i) - y^T z_i + H_i(x,y,p_i)) - p_i^T \nabla_{p_i} H_i(x,y,p_i)) - U_i(g_i(x,y)\\ - s(x|E_i) + y^T r_i + G_i(x,y,p_i) - p_i^T \nabla_{p_i} G_i(x,y,p_i)) = 0,\ i = 1, 2, ..., k.\end{aligned} \tag{3.1}$$

$$\begin{aligned}-\sum_{i=1}^{k} \lambda_i \big[\nabla_y f_i(x,y) - z_i + \nabla_{p_i} H_i(x,y,p_i)\\ -U_i(\nabla_y g_i(x,y) + r_i + \nabla_{p_i} G_i(x,y,p_i))\big] \in C_2^*,\end{aligned} \tag{3.2}$$

$$\begin{aligned}y^T \Bigg[\sum_{i=1}^{k} \lambda_i \big[\nabla_y f_i(x,y) - z_i + \nabla_{p_i} H_i(x,y,p_i)\\ -U_i(\nabla_y g_i(x,y) + r_i + \nabla_{p_i} G_i(x,y,p_i))\big]\Bigg] \geqq 0,\end{aligned} \tag{3.3}$$

$$\lambda > 0,\ \lambda^T e = 1,\ x \in C_1,\ z_i \in D_i,\ r_i \in F_i,\ i = 1, 2, ..., k.$$

(MFD)$_V$ Maximize V subject to

$$\begin{aligned}(f_i(u,v) - s(v|D_i) + u^T w_i + \Phi_i(u,v,q_i) - q_i^T \nabla_{q_i} \Phi_i(u,v,q_i)) - V_i(g_i(u,v)\\ + s(v|F_i) - u^T t_i + \Psi_i(u,v,q_i) - q_i^T \nabla_{q_i} \Psi_i(u,v,q_i)) = 0,\ i = 1, 2, ..., k.\end{aligned} \tag{3.4}$$

$$\begin{aligned}\sum_{i=1}^{k} \lambda_i \big[\nabla_x f_i(u,v) + w_i + \nabla_{q_i} \Phi_i(u,v,q_i)\\ -V_i(\nabla_x g_i(u,v) - t_i + \nabla_{q_i} \Psi_i(u,v,q_i))\big] \in C_1^*,\end{aligned} \tag{3.5}$$

$$u^T\Big[\sum_{i=1}^{k}\lambda_i\big[(\nabla_x f_i(u,v)+w_i+\nabla_{q_i}\Phi_i(u,v,q_i))$$

$$-V_i(\nabla_x g_i(u,v)-t_i+\nabla_{q_i}\Psi_i(u,v,q_i))\big]\Big]\leqq 0, \tag{3.6}$$

$$\lambda>0,\ \lambda^T e=1,\ v\in C_2,\ w_i\in Q_i,\ t_i\in E_i,\ i=1,2,...,k.$$

Next, we prove weak, strong and converse duality theorems for (MFP)$_U$ and (MFD)$_V$, which therefore are equally applicable for (MFP) and (MFD). Let $z=(z_1,z_2,...,z_k)$, $r=(r_1,r_2,...,r_k)$, $w=(w_1,w_2,...,w_k)$ and $t=(t_1,t_2,...,t_k)$.

Theorem 3.1. *(Weak duality). Let (x,y,U,z,r,λ,p) be feasible for (MFP)$_U$ and let (u,v,V,w,t,λ,q) be feasible for (MFD)$_V$. Let $\forall i\in\{1,2,...,k\}$, $f_i(.,v)+(.)^T w_i$ be higher order (C,α,ρ_i,d_i)- convex at u with respect to $\Phi_i(u,v,q_i)$, $-(g_i(.,v)-(.)^T t_i)$ be higher-order (C,α,ρ_i,d_i)- convex at u with respect to $-\Psi_i(u,v,q_i)$, $-(f_i(x,.)-(.)^T z_i)$ be higher -order $(\bar{C},\bar{\alpha},\bar{\rho_i},\bar{d_i})$ -convex at y with respect to $-H_i(x,y,p_i)$ and $(g_i(x,.)+(.)^T r_i)$ be higher -order $(\bar{C},\bar{\alpha},\bar{\rho_i},\bar{d_i})$- convex at y with respect to $G_i(x,y,p)$ where $C:R^n\times R^n\times R^n\to R$ and $\bar{C}:R^m\times R^m\times R^m\to R$. If the following conditions hold:*

$$g_i(x,v)+v^T r_i-x^T t_i>0,\ i=1,2,...,k, \tag{3.7}$$

$$\text{either}\sum_{i=1}^{k}\lambda_i[(1+V_i)\rho_i d_i^2(x,u)+(1+U_i)\bar{\rho_i}\bar{d_i}^2(v,y)]\geqq 0 \text{ or } \rho_i\geqq 0$$

$$\text{and}\quad \bar{\rho_i}\geqq 0, i=1,2,...,k, \tag{3.8}$$

$$C_{x,u}(a)+a^T u\geqq 0,\ \forall a\in C_1^*,\ \ \bar{C}_{v,y}(b)+b^T y\geqq 0,\ \forall b\in C_2^*. \tag{3.9}$$

Then, $U\nleq V$.

Proof. Since $\forall i\in\{1,2,...,k\}$, $f_i(.,v)+(.)^T w_i$ and $-(g_i(.,v)-(.)^T t_i)$ is higher-order (C,α,ρ_i,d_i) - convex in the first variable at u for fixed v, we have

$$\frac{1}{\alpha(x,u)}\big[f_i(x,v)+x^T w_i-f_i(u,v)-u^T w_i-\Phi_i(u,v,q_i)+q_i^T\nabla_{q_i}\Phi_i(u,v,q_i)$$

$$-\rho_i d_i^2(x,u)\big]\geqq C_{x,u}\big(\nabla_x f_i(u,v)+w_i+\nabla_{q_i}\Phi_i(u,v,q_i)\big) \tag{3.10}$$

and

$$\frac{1}{\alpha(x,u)}\big[(-g_i(x,v)+x^T t_i+g_i(u,v)-u^T t_i)+(\Psi_i(u,v,q_i)-q_i^T\nabla_{q_i}\Psi_i(u,v,q_i))$$

$$-\rho_i d_i^2(x,u)\big]\geqq C_{x,u}\big((-\nabla_x g_i(u,v)+t_i)-\nabla_{q_i}\Psi_i(u,v,q_i)\big). \tag{3.11}$$

Multiplying $\frac{\lambda_i}{\tau}>0$ and $\frac{\lambda_i V_i}{\tau}\geqq 0,\ i=1,2,...,k$, where $\tau=1+\sum_{i=1}^{k}\lambda_i V_i$ together with (3.10) and (3.11), respectively, we obtain

$$\frac{\lambda_i}{\alpha(x,u)\tau}\big(f_i(x,v)+x^T w_i-f_i(u,v)-u^T w_i-\Phi_i(u,v,q_i)+q_i^T\nabla_{q_i}\Phi_i(u,v,q_i)\big)$$

$$-\frac{\lambda_i}{\alpha(x,u)\tau}\rho_i d_i^2(x,u) \geqq \frac{\lambda_i}{\tau} C_{x,u}\big(\nabla_x f_i(u,v) + w_i + \nabla_{q_i}\Phi_i(u,v,q_i)\big)$$

and

$$\frac{\lambda_i V_i}{\alpha(x,u)\tau}\big[-g_i(x,v) + x^T t_i + g_i(u,v) - u^T t_i + \Psi_i(u,v,q_i) - q_i^T \nabla_{q_i}\Psi_i(u,v,q_i)\big]$$

$$-\frac{\lambda_i V_i}{\alpha(x,u)\tau}\rho_i d_i^2(x,u) \geqq \frac{\lambda_i V_i}{\tau} C_{x,u}\big(-\nabla_x g_i(u,v) + t_i - \nabla_{q_i}\Psi_i(u,v,q_i)\big).$$

Now, summing over i and adding the above two inequalities, using convexity of C, we have

$$\sum_{i=1}^{k}\frac{\lambda_i}{\alpha(x,u)\tau}\big[f_i(x,v) + x^T w_i - f_i(u,v) - u^T w_i - \Phi_i(u,v,q_i) + q_i^T\nabla_{q_i}\Phi_i(u,v,q_i)\big]$$
$$+\sum_{i=1}^{k}\frac{\lambda_i V_i}{\alpha(x,u)\tau}\big[-g_i(x,v)+x^T t_i+g_i(u,v)-u^T t_i+\Psi_i(u,v,q_i)-q_i^T\nabla_{q_i}\Psi_i(u,v,q_i)\big]-$$
$$\sum_{i=1}^{k}\frac{\lambda_i}{\alpha(x,u)\tau}(1+V_i)\rho_i d_i^2(x,u) \geqq C_{x,u}\Bigg[\sum_{i=1}^{k}\frac{\lambda_i}{\tau}\Bigg(\big(\nabla_x f_i(u,v) + w_i + \nabla_{q_i}\Phi_i(u,v,q_i)\big)$$
$$-V_i(\nabla_x g_i(u,v) - t_i + \nabla_{q_i}\Psi_i(u,v,q_i))\Bigg)\Bigg]. \tag{3.12}$$

Now, from (3.5), as $\tau > 0$, we have

$$a = \sum_{i=1}^{k}\frac{\lambda_i}{\tau}[(\nabla_x f_i(u,v)+w_i+\nabla_{q_i}\Phi_i(u,v,q_i))-V_i(\nabla_x g_i(u,v)-t_i+\nabla_{q_i}\Psi_i(u,v,q_i))] \in C_1^*.$$

Hence, for this a, $C_{x,u}(a) \geqq -u^T a \geqq 0$ $\big($from (3.9)$\big)$. Using this, in (3.12), we obtain

$$\sum_{i=1}^{k}\frac{\lambda_i}{\alpha(x,u)\tau}\big(f_i(x,v) + x^T w_i - f_i(u,v) - u^T w_i - \Phi_i(u,v,q_i) + q_i^T\nabla_{q_i}\Phi_i(u,v,q_i)\big) +$$
$$\sum_{i=1}^{k}\frac{\lambda_i V_i}{\alpha(x,u)\tau}\big[-g_i(x,v) + x^T t_i + g_i(u,v) - u^T t_i + \Psi_i(u,v,q_i) - q_i^T\nabla_{q_i}\Psi_i(u,v,q_i)\big]$$

$$\geqq \sum_{i=1}^{k}\frac{\lambda_i}{\alpha(x,u)\tau}(1+V_i)\rho_i d_i^2(x,u).$$

Since $v^T r_i \leqq s(v|F_i)$ and using (3.4) in above inequality, we get

$$\sum_{i=1}^{k}\lambda_i[f_i(x,v) + x^T w_i - s(v|D_i) + V_i(x^T t_i - v^T r_i - g_i(x,v))]$$

$$\geqq \sum_{i=1}^{k}\lambda_i(1+V_i)\rho_i d_i^2(x,u). \tag{3.13}$$

Similarly, by the higher-order $(\bar{C}, \bar{\alpha}, \bar{\rho}_i, \bar{d}_i)$ - convexity of $-f_i(x,.) + (.)^T z_i$ and $(g_i(x,.) + (.)^T r_i)$, $\forall i \in \{1, 2, ..., k\}$, in the second variable at y, for fixed x and from the condition (3.9), for

$$b = -\sum_{i=1}^{k} \frac{\lambda_i}{\tau}[(\nabla_y f_i(x,y) - z_i + \nabla_{p_i} H_i(x,y,p_i) - U_i(\nabla_y g_i(x,y) + r_i + \nabla_{p_i} G_i(x,y,p_i))] \in C_2^*,$$

we get

$$\sum_{i=1}^{k} \lambda_i[-f_i(x,v) + v^T z_i - s(x|Q_i) + U_i(v^T r_i - x^T t_i + g_i(x,v))]$$
$$\geqq \sum_{i=1}^{k} \lambda_i (1+U_i)\bar{\rho}_i \bar{d}_i^{\,2}(v,y). \tag{3.14}$$

Adding the inequalities (3.13), (3.14) and applying (3.8), we get

$$\sum_{i=1}^{k} \lambda_i (v^T z_i - s(v|D_i) + x^T w_i - s(x|Q_i))$$
$$+\sum_{i=1}^{k} \lambda_i (U_i - V_i)(g_i(x,v) + v^T r_i - x^T t_i) \geqq 0. \tag{3.15}$$

Since $\lambda > 0$, $v^T z_i \leqq s(v|D_i)$ and $x^T w_i \leqq s(x|Q_i)$, the above inequality gives

$$\sum_{i=1}^{k} \lambda_i (U_i - V_i)(g_i(x,v) + v^T r_i - x^T t_i) \geqq 0. \tag{3.16}$$

From the fact that $\lambda > 0$ and using (3.7), it follows that $U \nleq V$. This completes the proof. $\square$

Theorem 3.2. *(Weak duality). Let $(x, y, U, z, r, \lambda, p)$ and $(u, v, V, w, t, \lambda, q)$ be feasible solutions of (MFP)$_U$ and (MFD)$_V$, respectively. Suppose that*

(i) $(f_i(.,v) + (.)^T w_i) - V_i(g_i(.,v) - (.)^T t_i)$ is higher-order (C, α, ρ_i, d_i)- convex at u with respect to $(\Phi_i(u,v,q) - V_i \Psi_i(u,v,q))$,

(ii) $(-f_i(x,.) + (.)^T z_i) + U_i(g_i(x,.) + (.)^T r_i)$ is higher-order $(\bar{C}, \bar{\alpha}, \bar{\rho}_i, \bar{d}_i)$ -convex at y with respect to $-H_i(x,y,p) + U_i G_i(x,y,p)$,

(iii) either $\sum_{i=1}^{k} \lambda_i \left[\rho_i d_i^2(x,u) + \bar{\rho}_i \bar{d}_i^{\,2}(v,y)\right] \geqq 0$ or $\rho_i \geqq 0$ and $\bar{\rho}_i \geqq 0$, $i = 1, 2, ..., k$,

(iv) $C_{x,u}(a) + a^T u \geqq 0, \forall a \in C_1^*$, $\bar{C}_{v,y}(b) + b^T y \geqq 0, \forall b \in C_2^*$,

(v) $g_i(x,v) + v^T r_i - x^T t_i > 0$, $i = 1, 2, ..., k$.

Then, $U \nleq V$.

Proof. By hypothesis (i), we have

$$\frac{1}{\alpha(x,u)}\big[f_i(x,v) + x^T w_i - V_i(g_i(x,v) - x^T t_i) - (f_i(u,v) + u^T w_i) - V_i(g_i(u,v) - u^T t_i) -$$
$$(\Phi_i(u,v,q_i) - V_i(\Psi_i(u,v,q_i)) + q_i^T(\nabla_{q_i}\Phi_i(u,v,q_i) - V_i \nabla_{q_i}\Psi_i(u,v,q_i)) - \rho_i d_i^2(x,u)\big]$$
$$\geqq C_{x,u}\big[\nabla_x f_i(u,v) + w_i - V_i(\nabla_x g_i(u,v) - t_i) + (\nabla_{q_i}\Phi_i(u,v,q_i) - V_i \nabla_{q_i}\Psi_i(u,v,q_i))\big].$$

Since $\lambda > 0$, we obtain

$$\sum_{i=1}^{k} \frac{\lambda_i}{\alpha(x,u)}[f_i(x,v)+x^T w_i - f_i(u,v) - u^T w_i - \Phi_i(u,v,q_i) + q_i^T \nabla_{q_i}\Phi_i(u,v,q_i) + V_i(-g_i(x,v)+ x^T t_i + g_i(u,v) - u^T t_i + \Psi_i(u,v,q_i) - q_i^T \nabla_{q_i}\Psi_i(u,v,q_i)) - \rho_i d_i^2(x,u)$$

$$\geqq \sum_{i=1}^{k} \lambda_i C_{x,u}\big[(\nabla_x f_i(u,v) + w_i) - V_i(\nabla_x g_i(u,v) - t_i\big) + (\nabla_{q_i}\Phi_i(u,v,q_i) - V_i \nabla_{q_i}\Psi_i(u,v,q_i))\big].$$

Using convexity of C, we have

$$\frac{1}{\alpha(x,u)}\Bigg[\sum_{i=1}^{k}\lambda_i\Big((f_i(x,v)+x^T w_i - f_i(u,v) - u^T w_i) - \Phi_i(u,v,q_i) + q_i^T \nabla_{q_i}\Phi_i(u,v,q_i)\Big)$$
$$+\sum_{i=1}^{k}\lambda_i V_i\big(-g_i(x,v)+x^T t_i - v^T r_i\big) + \sum_{i=1}^{k}\lambda_i V_i\big(g_i(u,v) + v^T r_i - u^T t_i + \Psi_i(u,v,q_i) -$$
$$q_i^T \nabla_{q_i}\Psi_i(u,v,q_i))\Bigg] - \frac{1}{\alpha(x,u)}\sum_{i=1}^{k}\lambda_i \rho_i d_i^2(x,u)$$

$$\geqq C_{x,u}\Bigg[\sum_{i=1}^{k}\lambda_i\Big(\big(\nabla_x f_i(u,v) + w_i + \nabla_{q_i}\Phi_i(u,v,q_i)\big) - V_i(\nabla_x g_i(u,v) - t_i - \nabla_{q_i}\Psi_i(u,v,q_i))\Big)\Bigg].$$

From (3.5),

$$a = \sum_{i=1}^{k}\lambda_i[\nabla_x f_i(u,v) + w_i + \nabla_{q_i}\Phi_i(u,v,q_i) - V_i(\nabla_x g_i(u,v) - t_i + \nabla_{q_i}\Psi_i(u,v,q_i))] \in C_1^*$$

and using hypothesis (iv), we get

$$\frac{1}{\alpha(x,u)}\Bigg[\sum_{i=1}^{k}\lambda_i\big(f_i(x,v) + x^T w_i - f_i(u,v) - u^T w_i) - \Phi_i(u,v,q_i) + q_i^T \nabla_{q_i}\Phi_i(u,v,q_i))$$
$$+\sum_{i=1}^{k}\lambda_i V_i\big(-g_i(x,v) + x^T t_i - v^T r_i\big) + \sum_{i=1}^{k}\lambda_i V_i(g_i(u,v) - u^T t_i$$
$$+\Psi_i(u,v,q_i) - q_i^T \nabla_{q_i}\Psi_i(u,v,q_i)\Bigg] \geqq \sum_{i=1}^{k}\frac{\lambda_i}{\alpha(x,u)}\rho_i d_i^2(x,u). \tag{3.17}$$

Since $v^T r_i \leqq s(v|F_i)$, using (3.4) in (3.17), we get

$$\sum_{i=1}^{k}\lambda_i[(f_i(x,v) + x^T w_i - s(v|D_i)) + V_i(x^T t_i - v^T r_i - g_i(x,v))]$$
$$\geqq \sum_{i=1}^{k}\lambda_i \rho_i d_i^2(x,u). \tag{3.18}$$

Similarly, by the higher-order $(\bar{C}, \bar{\alpha}, \bar{\rho}_i, \bar{d}_i)-$ convexity of $-f_i(x,.)+(.)^T z_i + U_i(g_i(x,.)+ (.)^T r_i)$ in the second variable at y, for fixed x, we get

$$\sum_{i=1}^{k} \lambda_i[-f_i(x,v) + v^T z_i - s(x|Q_i) + U_i(v^T r_i - x^T t_i + g_i(x,v))]$$

$$\geqq \sum_{i=1}^{k} \lambda_i \bar{\rho}_i \bar{d}_i^{\,2}(v,y). \tag{3.19}$$

Using $\lambda > 0,\ v^T z_i \leqq s(v|D_i)$ and $x^T w_i \leqq s(x|Q_i)$, it follows from (3.18) and (3.19) that

$$\sum_{i=1}^{k} \lambda_i (U_i - V_i)(g_i(x,v) + v^T r_i - x^T t_i) \geqq 0. \tag{3.20}$$

Since $\lambda > 0$ and using hypothesis (v), it follows that $U \nleq V$. Hence, the result. $\square$

Theorem 3.3. *(Strong duality). Let $(\bar{x}, \bar{y}, \bar{U}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p})$ be an efficient solution of (MFP)$_U$ and fix $\lambda = \bar{\lambda}$ in (MFD)$_V$. If the following conditions hold*

- (i) $\nabla_x H_i(\bar{x}, \bar{y}, 0) = \nabla_x G_i(\bar{x}, \bar{y}, 0) = 0, \nabla_{q_i} \Phi_i(\bar{x}, \bar{y}, 0) = \nabla_{q_i} \Psi_i(\bar{x}, \bar{y}, 0) = 0, H_i(\bar{x}, \bar{y}, 0) = G_i(\bar{x}, \bar{y}, 0) = 0, \Phi_i(\bar{x}, \bar{y}, 0) = \Psi_i(\bar{x}, \bar{y}, 0) = 0, \nabla_y H_i(\bar{x}, \bar{y}, 0) = \nabla_y G_i(\bar{x}, \bar{y}, 0) = 0, \nabla_{p_i} H_i(\bar{x}, \bar{y}, 0) = \nabla_{p_i} G_i(\bar{x}, \bar{y}, 0) = 0,\ i = 1, 2, ..., k,$
- (ii) *for all* $i \in \{1, 2, ..., k\}$, *the Hessian matrix* $\nabla_{p_i p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_{p_i p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)$ *is positive or negative definite,*
- (iii) *the set of vectors* $\{\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))\}_{i=1}^{k}$ *is linearly independent,*
- (iv) *for* $\bar{p}_i \in R^n,\ \bar{p}_i \neq 0\ (i = 1, 2, ..., k)$ *implies that* $\sum_{i=1}^{k} \bar{p}_i^T [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))] \neq 0,$
- (v) $\bar{U}_i > 0,\ \forall i \in \{1, 2, ..., k\}$.

Then

- (a) $\bar{p}_i = 0,\ i = 1, 2, ..., k,$
- (b) *there exists* $\bar{w}_i \in Q_i$ *and* $\bar{t}_i \in E_i,\ i = 1, 2, ..., k$ *such that* $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ *is a feasible solution of* $(MFD)_V$.

Furthermore, if the hypotheses in Theorem 3.1 *or* 3.2 *are satisfied, then* $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ *is an efficient solution of* $(MFD)_V$*, and the two objective values are equal.*

Proof. Since $(\bar{x}, \bar{y}, \bar{U}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p})$ is an efficient solution of (MFP)$_U$, therefore, by the Fritz John necessary optimality conditions [2], there exists $\alpha \in R^k\ , \beta \in R^k,\ \gamma \in C_2,\ \delta \in R_+,\ \xi \in R^k\ , \eta \in R,\ \bar{w}_i \in R^n$ and $\bar{t}_i \in R^n,\ i = 1, 2, ..., k$ such that

$$\begin{aligned}
&(x - \bar{x})^T \Bigg[\sum_{i=1}^{k} \beta_i (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i + \nabla_x H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i + \nabla_x G_i(\bar{x}, \bar{y}, \bar{p}_i))) + \\
&(\gamma - \delta \bar{y})^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_{yx} f_i(\bar{x}, \bar{y}) - \bar{U}_i \nabla_{yx} g_i(\bar{x}, \bar{y})) + \sum_{i=1}^{k} (\nabla_{p_i x} H_i(\bar{x}, \bar{y}, \bar{p}_i) \\
&\qquad - \bar{U}_i(\nabla_{p_i x} G_i(\bar{x}, \bar{y}, \bar{p}_i))^T ((\gamma - \delta \bar{y}) \bar{\lambda}_i - \beta_i \bar{p}_i) \Bigg] \geqq 0, \forall x \in C_1,
\end{aligned} \tag{3.21}$$

$$\sum_{i=1}^{k} \beta_i (\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i))$$

$$+\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yy}f_i(\bar{x},\bar{y})-\bar{U}_i\nabla_{yy}g_i(\bar{x},\bar{y}))^T(\gamma-\delta\bar{y})+\sum_{i=1}^{k}(\nabla_{p_iy}H_i(\bar{x},\bar{y},\bar{p}_i)$$
$$-\bar{U}_i\nabla_{p_iy}G_i(\bar{x},\bar{y},\bar{p}_i))^T(-\beta_i\bar{p}_i+(\gamma-\delta\bar{y})\bar{\lambda}_i)-\delta\sum_{i=1}^{k}\bar{\lambda}_i[\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i$$
$$+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{U}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))]=0, \quad (3.22)$$

$$\alpha_i-\beta_i(g_i(\bar{x},\bar{y})-s(\bar{x}|E_i)+\bar{y}^T\bar{r}_i+G_i(\bar{x},\bar{y},\bar{p}_i)-\bar{p}_i^T\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))$$
$$-(\gamma-\delta\bar{y})^T\nabla_y(\bar{\lambda}_i(g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))=0,\ i=1,2,...,k, \quad (3.23)$$

$$(\gamma-\delta\bar{y})^T(\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i)$$
$$-\bar{U}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))-\xi_i+\eta=0,\ i=1,2,...,k, \quad (3.24)$$

$$\bar{\lambda}_i(\gamma-\delta\bar{y})-\beta_i\bar{p}_i)^T(\nabla_{p_ip_i}H_i(\bar{x},\bar{y},\bar{p}_i)$$
$$-\bar{U}_i\nabla_{p_ip_i}G_i(\bar{x},\bar{y},\bar{p}_i))=0,\ i=1,2,...,k, \quad (3.25)$$

$$\beta_i\bar{y}+(\gamma-\delta\bar{y})\bar{\lambda}_i\in N_{D_i}(\bar{z}_i),\ i=1,2,...,k, \quad (3.26)$$

$$\beta_i\bar{U}_i\bar{y}+\bar{\lambda}_i\bar{U}_i(\gamma-\delta\bar{y})\in N_{F_i}(\bar{r}_i),\ i=1,2,...,k, \quad (3.27)$$

$$\gamma^T\sum_{i=1}^{k}\bar{\lambda}_i((\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i))$$
$$-\bar{U}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i)))=0, \quad (3.28)$$

$$\delta\bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i)$$
$$-\bar{U}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))=0, \quad (3.29)$$

$$\bar{\lambda}^T\xi=0, \quad (3.30)$$

$$\eta(\bar{\lambda}^Te-1)=0, \quad (3.31)$$

$$\bar{w}_i\in Q_i,\bar{t}_i\in E_i,\bar{x}^Tt_i=s(\bar{x}|E_i),\bar{x}^T\bar{w}_i=s(\bar{x}|Q_i),\ i=1,2,...,k, \quad (3.32)$$

$$(\alpha,\delta,\xi)\geqq 0,\ (\alpha,\beta,\gamma,\delta,\xi,\eta)\neq 0. \quad (3.33)$$

Since $\bar{\lambda}>0$, and $\xi\geqq 0,\ (30)$ implies that $\xi=0$.

Equation (3.22) can be re-written as

$$\sum_{i=1}^{k}(\beta_i-\delta\bar{\lambda}_i)((\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i))-\bar{U}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i)))$$
$$+\sum_{i=1}^{k}\beta_i((\nabla_y H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{U}_i\nabla_y G_i(\bar{x},\bar{y},\bar{p}_i))-(\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{U}_i\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))$$
$$+\sum_{i=1}^{k}\bar{\lambda}_i\ ((\nabla_{yy}f_i(\bar{x},\bar{y})-\bar{U}_i\nabla_{yy}g_i(\bar{x},\bar{y}))^T(\gamma-\delta\bar{y})$$
$$+\sum_{i=1}^{k}((\nabla_{p_iy}H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{U}_i\nabla_{p_iy}G_i(\bar{x},\bar{y},\bar{p}_i))^T(-\beta_i\bar{p}_i+(\gamma-\delta\bar{y})\bar{\lambda}_i)=0. \quad (3.34)$$

By hypothesis (ii) and (3.25), we have

$$\bar{\lambda}_i(\gamma - \delta\bar{y}) = \beta_i\bar{p}_i,\ i = 1, 2, ..., k. \tag{3.35}$$

Now, we claim that $\beta_i \neq 0,\ \forall i$. If possible, let $\beta_{t_0} = 0$ for some t_0, $1 \leq t_0 \leq k$, then from $\bar{\lambda}_{t_0} > 0$ and equation (3.35), we have

$$\gamma = \delta\bar{y}. \tag{3.36}$$

Using (3.35) and (3.36), we obtain $\beta_i\bar{p}_i = 0,\ i = 1, 2, ..., k$. Hence, by hypothesis (i), we get

$$\sum_{i=1}^{k} \beta_i((\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) \\ -\bar{U}_i\nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i\nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0. \tag{3.37}$$

Using (3.35)-(3.37) in (3.34), we obtain

$$\sum_{i=1}^{k}(\beta_i - \delta\bar{\lambda}_i)(\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) \\ -\bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0, \tag{3.38}$$

which by hypothesis (iii), follows that

$$\bar{\beta}_i - \delta\bar{\lambda}_i = 0,\ i = 1, 2, ..., k. \tag{3.39}$$

Now, for $i = t_0$, we have $\delta\bar{\lambda}_{t_0} = 0$. This implies $\delta = 0$ as $\bar{\lambda} > 0$. Hence, from (3.39), $\beta_i = 0,\ \forall\ i$. Thus, from relation (3.23) and (3.36), we get $\alpha_i = 0,\ i = 1, 2, ..., k$. Also, from relations (3.24) and (3.36), we get $\eta = 0$ and $\gamma = 0$, respectively, which contradicts the fact that $(\alpha, \beta, \gamma, \delta, \xi, \eta) \neq 0$. Hence $\beta_i \neq 0,\ i = 1, 2, ..., k$.
Now, equation (3.24) reduces to

$$(\gamma - \delta\bar{y})^T(\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) \\ - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)) + \eta = 0,\ i = 1, 2, ..., k, \tag{3.40}$$

Multiplying by $\bar{\lambda}_i$ and summing over i, we get

$$(\gamma - \delta\bar{y})^T \sum_{i}^{k} \bar{\lambda}_i(\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) \\ - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)) + \eta\ \bar{\lambda}^T e_k = 0. \tag{3.41}$$

Subtracting (3.28) and (3.29), we get

$$(\gamma - \delta\bar{y})^T \sum_{i}^{k} \bar{\lambda}_i(\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) \\ - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)) = 0. \tag{3.42}$$

Using (3.42) in (3.41), we get, $\eta = 0$.

Now, equation (3.40), yield

$$(\gamma - \delta\bar{y})^T(\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) \\ - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)) = 0,\ i = 1, 2, ..., k, \tag{3.43}$$

Since $\bar{\lambda} > 0$, using (3.35) in (3.43), we get

$$\beta_i \bar{p}_i^T [\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x},\bar{y},\bar{p}_i) \\ -\bar{U}_i(\nabla_y g_i(\bar{x},\bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p}_i))] = 0,\ i = 1,2,...,k, \tag{3.44}$$

Since $\beta_i \neq 0,\ i = 1,2,...,k$, we obtain

$$\bar{p}_i^T [\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x},\bar{y},\bar{p}_i) \\ -\bar{U}_i(\nabla_y g_i(\bar{x},\bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p}_i))] = 0,\ i = 1,2,...,k, \tag{3.45}$$

or

$$\sum_{i=1}^{k} \bar{p}_i^T [\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x},\bar{y},\bar{p}_i) \\ -\bar{U}_i(\nabla_y g_i(\bar{x},\bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p}_i))] = 0. \tag{3.46}$$

By the hypothesis (iv), we have $\bar{p}_i = 0,\ i = 1,2,...,k$. Further using, hypothesis (i), (3.35)-(3.36) in (3.21) and (3.34), respectively, we get

$$(x-\bar{x})^T \left[\sum_{i=1}^{k} \beta_i(\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i - \bar{U}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i)) \right] \geqq 0, \forall x \in C_1. \tag{3.47}$$

$$\sum_{i=1}^{k} (\beta_i - \delta\bar{\lambda}_i)[\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i - \bar{U}_i(\nabla_y g_i(\bar{x},\bar{y}) + \bar{r}_i)] = 0. \tag{3.48}$$

Using hypothesis (iii) in (3.48), we have

$$\beta_i = \delta\bar{\lambda}_i,\ i = 1,2,...,k. \tag{3.49}$$

Since $\beta_i \neq 0,\ \bar{\lambda}_i > 0,\ i = 1,2,...,k$ and $\delta \geqq 0$, this implies that $\beta_i > 0,\ \forall i$. Now, using (3.49) in (3.47), we obtain

$$(x-\bar{x})^T \left[\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i - \bar{U}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i)) \right] \geqq 0, \forall x \in C_1. \tag{3.50}$$

Let $x \in C_1$. Then $x + \bar{x} \in C_1$, as C_1 is a closed convex cone. On substituting $x + \bar{x}$ in place of x in (3.50), we get

$$x^T \sum_{i=1}^{k} \bar{\lambda}_i \big[(\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i) - \bar{U}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i)\big] \geqq 0, \tag{3.51}$$

which in turn implies that for all $x \in C_1$, we have

$$\sum_{i=1}^{k} \bar{\lambda}_i[(\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i) - \bar{U}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i))] \in C_1^*. \tag{3.52}$$

Also, by letting $x = 0$ and $x = 2\bar{x}$, simultaneously in (3.50), yields

$$\bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i) - \bar{U}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i)) = 0, \tag{3.53}$$

Using $\bar{p}_i = 0$ in (3.35), we get, $\gamma = \delta\bar{y}$ and $\delta > 0$, we have

$$\bar{y} = \frac{\gamma}{\delta} \in C_2.$$

Since $\beta > 0$ by (3.26) and the fact that $\gamma = \delta\bar{y}$, we get $\bar{y} \in N_{D_i}(\bar{z}_i), i = 1, 2, ..., k$. This implies

$$\bar{y}^T \bar{z}_i = s(\bar{y}|D_i), i = 1, 2, ..., k. \tag{3.54}$$

By (3.27) and hypothesis (v), we have $\bar{y} \in N_{F_i}(\bar{r}_i), i = 1, 2, ..., k$. Hence,

$$\bar{y}^T \bar{r}_i = s(\bar{y}|F_i), i = 1, 2, ..., k. \tag{3.55}$$

Combining (3.32), (3.54)-(3.55) and given equation (3.1), reduce to

$$\begin{aligned}&(f_i(\bar{x}, \bar{y}) + \bar{x}^T \bar{w}_i - s(\bar{y}|D_i))\\ &\qquad -\bar{U}_i(g_i(\bar{x}, \bar{y}) - \bar{x}^T \bar{t}_i - s(\bar{y}|F_i)) = 0, i = 1, 2, ..., k.\end{aligned} \tag{3.56}$$

Therefore, (3.52)-(3.53) and (3.56) shows that $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ is a feasible solution of $(MFD)_V$.

Under the assumptions Theorems 3.1 or 3.2, if $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ is not an efficient solution of $(MFD)_V$, then there exists other feasible solution $(u, v, V, w, t, \lambda, q)$, of $(MFD)_V$, such that $\bar{U} \leq V$.
Since $(\bar{x}, \bar{y}, \bar{U}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p})$ is a feasible solution of $(MFP)_U$, by Weak duality theorem, we have $\bar{U} \nleq V$, hence the contradiction implies that $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ is an efficient solution of $(MFD)_V$. Hence, the result. □

Theorem 3.4. *(Strong duality). Let $(\bar{x}, \bar{y}, \bar{U}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p})$ be efficient solution of (MFP)$_U$ and fix $\lambda = \bar{\lambda}$ in (MFD)$_V$. Suppose that*

(i) $\nabla_x H_i(\bar{x}, \bar{y}, 0) = \nabla_x G_i(\bar{x}, \bar{y}, 0) = 0, \nabla_{q_i} \Phi_i(\bar{x}, \bar{y}, 0) = \nabla_{q_i} \Psi_i(\bar{x}, \bar{y}, 0) = 0,\ H_i(\bar{x}, \bar{y}, 0) = G_i(\bar{x}, \bar{y}, 0) = 0, \Phi_i(\bar{x}, \bar{y}, 0) = \Psi_i(\bar{x}, \bar{y}, 0) = 0, \nabla_y H_i(\bar{x}, \bar{y}, 0) = \nabla_y G_i(\bar{x}, \bar{y}, 0) = 0, \nabla_{p_i} H_i(\bar{x}, \bar{y}, 0) = \nabla_{p_i} G_i(\bar{x}, \bar{y}, 0) = 0,\ i = 1, 2, ..., k,$

(ii) $\bar{U}_i > 0, \forall i \in 1, 2, ..., k,$

(iii) $\nabla_{p_i p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_{p_i p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)$ *is nonsingular* $\forall i = 1, 2, ..., k,$

(iv) $\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_{yy} f_i(\bar{x}, \bar{y}) - \bar{U}_i \nabla_{yy} g_i(\bar{x}, \bar{y}))$ *is positive definite and* $\bar{p}_i^T \big((\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))\big) \geqq 0, \quad \forall i = 1, 2, ..., k,$
or $\sum_{i=1}^{k} \bar{\lambda}_i((\nabla_{yy} f_i(\bar{x}, \bar{y}) - \bar{U}_i \nabla_{yy} g_i(\bar{x}, \bar{y}))$ *is negative definite and* $\bar{p}_i^T(\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)) \leqq 0, \forall i = 1, 2, ..., k,$

(v) *the set of vectors* $\{\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i) : i = 1, 2, ..., k\}$ *is linearly independent.*

Then $\bar{p} = 0$, and there exists $\bar{w}_i \in Q_i$ and $\bar{t}_i \in E_i$, $i = 1, 2, ..., k$ such that $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ is a feasible solution of $(MFD)_V$. Furthermore, if the hypotheses in theorem (3.1) or (3.2) are satisfied, then $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ is efficient solution of $(MFD)_V$, and the two objective values are equal.

Proof. It follows on the lines of Theorem 3.3. □

Theorem 3.5. *(Strict converse duality). Let $(\bar{u}, \bar{v}, \bar{V}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q})$ be efficient solution of (MFP)$_V$ and fix $\lambda = \bar{\lambda}$ in (MFD)$_U$. If the following conditions hold*

(i) $\nabla_x \Phi_i(\bar{u}, \bar{v}, 0) = \nabla_x \Psi_i(\bar{u}, \bar{v}, 0) = 0, \nabla_{q_i} \Phi_i(\bar{u}, \bar{v}, 0) = \nabla_{q_i} \Psi_i(\bar{u}, \bar{v}, 0) = 0, H_i(\bar{u}, \bar{v}, 0) = G_i(\bar{u}, \bar{v}, 0) = 0, \Phi_i(\bar{u}, \bar{v}, 0) = \Psi_i(\bar{u}, \bar{v}, 0) = 0, \nabla_y \Phi_i(\bar{u}, \bar{v}, 0) = \nabla_y \Psi_i(\bar{u}, \bar{v}, 0) = 0, \nabla_{p_i} H_i(\bar{u}, \bar{v}, 0) = \nabla_{p_i} G_i(\bar{u}, \bar{v}, 0) = 0,\ i = 1, 2, ..., k,$

(ii) *for all* $i \in \{1, 2, ..., k\}$, *the Hessian matrix* $\nabla_{p_ip_i}\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i\nabla_{p_ip_i}\Psi_i(\bar{u}, \bar{v}, \bar{q}_i)$ *is positive or negative definite,*

(iii) *the set of vectors* $\{\nabla_x f_i(\bar{u}, \bar{v}) + \bar{w}_i + \nabla_{q_i}\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i(\nabla_x g_i(\bar{u}, \bar{v}) - \bar{t}_i + \nabla_{q_i}\Psi_i(\bar{u}, \bar{v}, \bar{q}_i)))\}_{i=1}^k$ *is linearly independent,*

(iv) *for* $\bar{q}_i \in R^n,\ \bar{q}_i \neq 0, (i = 1, 2, ..., k)$ *implies that*
$\sum_{i=1}^k \xi_i\bar{q}_i^T[\nabla_x f_i(\bar{u}, \bar{v}) + \bar{w}_i + \nabla_{q_i}\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i(\nabla_x g_i(\bar{u}, \bar{v}) - \bar{t}_i + \nabla_{q_i}\Psi_i(\bar{u}, \bar{v}, \bar{q}_i))] \neq 0,$

(v) $\bar{V}_i > 0,\ \forall i \in 1, 2, ..., k.$

Then

(a) $\bar{q}_i = 0,\ i = 1, 2, ..., k,$

(b) *there exists* $\bar{z}_i \in D_i$ *and* $\bar{r}_i \in F_i$, $i = 1, 2, ..., k$ *such that* $(\bar{u}, \bar{v}, \bar{V}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p} = 0)$ *is a feasible solution of* $(MFD)_U$.

Furthermore, if the hypotheses in Theorem 3.1 *or* 3.2 *are satisfied, then* $(\bar{u}, \bar{v}, \bar{V}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p} = 0)$ *is an efficient solution of* $(MFD)_U$, *and the two objective values are equal.*

Proof. It follows on the lines of Theorem 3.3. □

Theorem 3.6. *(Strict converse duality). Let* $(\bar{u}, \bar{v}, \bar{V}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q})$ *be efficient solution of (MFP)*$_V$ *and fix* $\lambda = \bar{\lambda}$ *in (MFD)*$_U$. *Suppose that*

(i) $\nabla_x\Phi_i(\bar{u}, \bar{v}, 0) = \nabla_x\Psi_i(\bar{u}, \bar{v}, 0) = 0, \nabla_{q_i}\Phi_i(\bar{u}, \bar{v}, 0) = \nabla_{q_i}\Psi_i(\bar{u}, \bar{v}, 0) = 0, H_i(\bar{u}, \bar{v}, 0) = G_i(\bar{u}, \bar{v}, 0) = 0, \Phi_i(\bar{u}, \bar{v}, 0) = \Psi_i(\bar{u}, \bar{v}, 0) = 0, \nabla_y\Phi_i(\bar{u}, \bar{v}, 0) = \nabla_y\Psi_i(\bar{u}, \bar{v}, 0) = 0, \nabla_{p_i}H_i(\bar{u}, \bar{v}, 0) = \nabla_{p_i}G_i(\bar{u}, \bar{v}, 0) = 0,\ i = 1, 2, ..., k,$

(ii) $\bar{V}_i > 0, \forall i \in 1, 2, ..., k,$

(iii) $\nabla_{q_iq_i}\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i\nabla_{q_iq_i}\Psi_i(\bar{u}, \bar{v}, \bar{q}_i)$ *is nonsingular* $\forall i = 1, 2, ..., k,$

(iv) $\sum_{i=1}^k \bar{\lambda}_i(\nabla_{xx}f_i(\bar{u}, \bar{v}) - \bar{V}_i\nabla_{xx}g_i(\bar{u}, \bar{v}))$ *is positive definite and* $\bar{q}_i^T((\nabla_x\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i\nabla_x\Psi_i(\bar{u}, \bar{v}, \bar{q}_i)) - ((\nabla_{q_i}\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i\nabla_{q_i}\Psi_i(\bar{u}, \bar{v}, \bar{q}_i)) \geqq 0, \forall\ i = 1, 2, ..., k,$
or $\sum_{i=1}^k \bar{\lambda}_i(\nabla_{xx}f_i(\bar{u}, \bar{v}) - \bar{V}\nabla_{xx}g_i(\bar{u}, \bar{v}))$ *is negative definite and* $\bar{q}_i^T((\nabla_x\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i\nabla_x\Psi_i(\bar{u}, \bar{v}, \bar{q}_i)) - ((\nabla_{q_i}\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i\nabla_{q_i}\Phi_i\Psi_i(\bar{u}, \bar{v}, \bar{q}_i)) \leqq 0, \forall i = 1, 2, ..., k,$

(v) *the set of vectors* $\{\nabla_x f_i(\bar{u}, \bar{v}) + \bar{w}_i + \nabla_{q_i}\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i + \nabla_{q_i}\Psi_i(\bar{u}, \bar{v}, \bar{q}_i)) :\ i = 1, 2, ..., k\}$ *is linearly independent.*

Then, $\bar{q} = 0$ *and there exists* $\bar{z}_i \in D_i$ *and* $\bar{r}_i \in F_i$, $i = 1, 2, ..., k$ *such that* $(\bar{u}, \bar{v}, \bar{V}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p} = 0)$ *is a feasible solution of (MFD)*$_U$. *Furthermore, if the hypothesis in Theorems* 3.1 *or* 3.2 *are satisfied, then* $(\bar{u}, \bar{v}, \bar{V}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p} = 0)$ *is an efficient solution of (MFD)*$_U$, *and the two objective values are equal.*

Proof. It follows on the lines of Theorem 3.3. □

4. Special cases

We consider some of the special cases of the problems studied in section 3. In all the cases, $C_1 = R^n_+$ and $C_2 = R^m_+$,

(i) then, our problems $(MFP)_U$ and $(MFD)_V$ reduce to the programs studied in Ying [12].

(ii) If $k = 1,\ g_1(x, y) = 1,\ H_1(x, y, p_1) = \frac{1}{2}p_1^T\nabla_{yy}f_1(x, y)p_1,\ \Phi_1(u, v, q_1) = \frac{1}{2}q_1^T\nabla_{xx}f_1(u, v)q_1,\ g_1(u, v) = 1, F_1 = \{0\},\ E_1 = \{0\}$, then, $(MFP)_U$ and $(MFD)_V$ reduce to the problems considered by Hou and Yang [5].

(iii) If $g_i(x,y)=1,\ E_i=\{0\},\ g_i(u,v)=1,\ F_i=\{0\}$ for all $i\in\{1,2,...,k\}$, then, $(MFP)_U$ and $(MFD)_V$ becomes the problems considered by Chen [1].

(iv) If $g_i(x,y)\ =\ 1,\ E_i\ =\ \{0\},\ F_i\ =\ \{0\},\ H_i(x,y,p_i)\ =\ \frac{1}{2}p_i^T\nabla_{yy}f_i(x,y)p_i,$ $\Phi_i(u,v,q_i)\ =\ \frac{1}{2}q_i^T\nabla_{xx}f_i(u,v)q_i,$ for all $i\in\{1,2,...,k\}$ in $(MFP)_U$ and $(MFD)_V$, then, the problems reduce to the problems considered by Yang et al.[11].

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