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REMARKS ON THE BETTER ADMISSIBLE MULTIMAPS

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ABSTRACT. For a quite long period, we investigated the better admissible class $\mathfrak{B}$ of multimaps on abstract convex spaces. In a paper of Liu et al. [1] in 2010, an extended class $\mathfrak{B}^+$ is introduced and fixed point theorems for maps in such class are proved. As a consequence, they deduce fixed point theorems on abstract convex Φ-spaces. However, we note that $\mathfrak{B} = \mathfrak{B}^+$ and all results in [1] are known by the present author.

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1. Introduction

In 1929, Knaster, Kuratowski and Mazurkiewicz (simply, KKM) obtained the so-called KKM theorem from the weak Sperner lemma and applied it to a new proof of the Brouwer fixed point theorem. Later in 1961, Ky Fan extended the KKM theorem to any topological vector spaces and applied it to various results including the Schauder fixed point theorem.

Since then there have appeared a large number of works devoting applications of the KKM theorem. In 1992, such research field was called the KKM theory by ourselves, and since 2006 the theory has been extended to abstract convex spaces by the present author.

Note that, in the KKM theory, a large number of results were obtained on various classes of topological spaces having abstract convex structure called the (partial) KKM spaces and of multimap classes such as acyclic maps, admissible class $\mathfrak{A}_c^\kappa$, better admissible class $\mathfrak{B}$, and the KKM classes $\mathfrak{KC}$, $\mathfrak{KO}$. Such research is initiated by ourselves and followed by several hundreds of other authors.

In 2010, in a work of Liu, Zhang and Tan [1], a better admissible class $\mathfrak{B}^+$ is introduced and a new fixed point theorem for better admissible multimap is proved on abstract convex spaces. As a consequence, they claimed to deduce a new fixed

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point theorem on abstract convex Φ-spaces. They also claimed that their main results generalize some recent work due to Lassonde, Kakutani, Browder, and Park, without giving any justification.

In the present paper, we show that all of the results in [1] are already known by ourselves.

2. Abstract convex spaces

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. Multimaps are also called simply maps.

Definition 2.1. [6-8] Let E be a topological space, D a nonempty set, and $\Gamma : \langle D\rangle \multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Examples of abstract convex spaces are given, for example, in [6-8]

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \quad \text{or} \quad \Gamma_A \subset F^+G(A) \text{ for all } A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ to a set Z is called *a* $\mathfrak{K}$-*map* and we say that F belongs to *the KKM family* if, for a KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-*map* is defined for closed-valued maps G, and a $\mathfrak{KO}$-*map* for open-valued maps G. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp. $F \in \mathfrak{KO}(E, Z)$].

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, E)$, that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property.

The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$, that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

In our previous works, we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the (partial) KKM principle. See [6-8] and the references therein.

We obtained the following diagram for subclasses of abstract convex spaces $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space

$\Longrightarrow$ Horvath space $\Longrightarrow$ G-convex space $\Longleftrightarrow$ ϕ_A-space

$$\Longrightarrow \text{KKM space} \Longrightarrow \text{Partial KKM space}$$
$$\Longrightarrow \text{Abstract convex space.}$$

Recall that any simplex is a KKM space by the KKM theorem and its open-valued version, and that any convex subset of a t.v.s. is a KKM space by the proof of the 1961 KKM Lemma of Ky Fan; see [6]. For other subclasses of (partial) KKM spaces in the diagram, all proofs were well-established in the literature; see [6-8].

Recall also that, as subfamilies of the KKM classes $\mathfrak{KC}$ and $\mathfrak{KO}$, we investigated the better admissible class $\mathfrak{B}$ and the admissible class $\mathfrak{A}_c^\kappa$; see [6-8].

In fact, the authors of [1] formulated our concepts in [2-5] as follows:

Definition 2.4. Let $(X, D; \Gamma)$ be an abstract convex space and Y a topological space. A *better admissible class* $\mathfrak{B}$ of multimaps from X into Y is defined as follows. A multimap $F : X \multimap Y$ belongs to $\mathfrak{B}(X, Y)$ if for any $N \in \langle D \rangle$ with the cardinality $|N| = n + 1$, there exists a map $\varphi_N : \Delta_n \longrightarrow \Gamma_N$, and for any continuous function $f : F(\Gamma_N) \longrightarrow \Delta_n$, the composition

$$f \circ F|_{\Gamma_N} \circ \varphi_N : \Delta_n \longrightarrow \Delta_n$$

has a fixed point.

Motivated by the work of the present author, Liu et al. [1] defined *the better admissible class* $\mathfrak{B}^+$ of multimaps on abstract convex space as follows:

Definition 2.5. Let $(X, D; \Gamma)$ be an abstract convex space and Y a topological space. We define a class $\mathfrak{B}^+$ of multimaps from X into Y as follows. A multimap $F : X \multimap Y$ belongs to $\mathfrak{B}^+(X, Y)$ if for any $N \in \langle D \rangle$ with the cardinality $|N| = n+1$, there is a map $G \in \mathfrak{B}(\Gamma_N, Y)$ such that $G(x) \subset F(x)$ for each $x \in \Gamma_N$.

Note that Γ_N can be replaced by the compact set $\varphi_N(\Delta_n)$.

Here let us call F is *an extension* of G and we note the following:

Proposition 2.6. *Every extension of* $\mathfrak{B}$*-maps is also a* $\mathfrak{B}$*-map, that is,* $\mathfrak{B}^+ = \mathfrak{B}$.

Proof. As in Definition 2.5, since F is an extension of some $G \in \mathfrak{B}(\Gamma_N, Y)$, we have

$$f \circ G|_{\Gamma_N} \circ \varphi \subset f \circ F|_{\Gamma_N} \circ \varphi : \Delta_n \multimap \Delta_n.$$

Since $f \circ G|_{\Gamma_N} \circ \varphi$ has a fixed point, so does $f \circ F|_{\Gamma_N} \circ \varphi$. Hence $F \in \mathfrak{B}$. Q.E.D.

3. Fixed point theorems on abstract convex uniform space

In our previous work [5], we introduced the following concepts:

Definition 3.1. An *abstract convex uniform space* $(X, D; \Gamma; \mathcal{U})$ is an abstract convex space such that $(X, \mathcal{U})$ is a uniform space with a basis $\mathcal{U}$ of the uniformity consisting of symmetric entourages. For each $U \in \mathcal{U}$, let $U[x] = \{x \in X : (x, x) \in U\}$ be the U-ball around a given element $x \in X$. For $U \in \mathcal{U}$, a point $x \in X$ is called a *U-fixed point* of a map $F : X \multimap X$ if $F(x) \cap U[x] \neq \emptyset$. The map F is said to have *the almost fixed point property* if it has a U-fixed point for any $U \in \mathcal{U}$.

Definition 3.2.. For an abstract convex uniform space $(E, D; \Gamma; \mathcal{U})$, a subset X of E is said to be *admissible* (in the sense of Klee) if, for each nonempty compact subset K of X and for each entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \longrightarrow X$ satisfying

(1) $(x, h(x)) \in U$ for all $x \in K$;

(2) $h(K) \subset \Gamma_N$ for some $N \in \langle D \rangle$; and

(3) there exist continuous functions $p : K \longrightarrow \Delta_n$ and $\phi_N : \Delta_n \longrightarrow \Gamma_N$ with $|N| = n + 1$ such that $h = \phi_N \circ p$.

This definition was given in [5] in 2009, and as [1, Definition 3.2] in 2010.

Definition 3.3.. Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space. A subset K of E is said to be *Klee approximable* if, for each entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \longrightarrow E$ satisfying conditions (1)-(3) in the preceding definition. Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K) \subset \Gamma_N \subset X$ for some $N \in \langle D \rangle$ in condition (2).

This definition was given in [5] in 2009, and as [1, Definition 3.3] in 2010.

Theorem 3.4. *Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space, $X \subset Y$ subsets of E, and $F : Y \multimap Y$ a map such that $F|_X \in \mathfrak{B}(X, Y)$ and $F(X)$ is Klee approximable into X. Then F has the almost fixed point property.*

Further if $(E,\mathcal{U})$ is Hausdorff, F is closed, and $\overline{F(X)}$ is compact in Y, then F has a fixed point $x_0 \in Y$ (that is, $x_0 \in F(x_0)$).

This was given as [5, Theorem 8.3] in 2009, and, for $\mathfrak{B}^+$, as [1, Theorem 3.1] in 2010.

Theorem 3.5. *Let $(X, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space and $F \in \mathfrak{B}(X, X)$ a multimap such that $F(X)$ is Klee approximable. Then F has the almost fixed point property.*

Further if F is closed and compact, then F has a fixed point $x_0 \in X$.

This was given as [5, Theorem 8.4] in 2009, and, for $\mathfrak{B}^+$, as [1, Corollary 3.1] in 2010.

Theorem 3.6. *Let $(X, D; \Gamma; \mathcal{U})$ be an admissible abstract convex uniform space. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

This was given as [5, Theorem 8.5] in 2009, and, for $\mathfrak{B}^+$, as [1, Theorem 3.2] in 2010.

Corollary 3.7. *Let $(X, D; \Gamma; \mathcal{U})$ be a compact admissible abstract convex uniform space. Then any map $F \in \mathfrak{A}_c^\kappa(X, X)$ has a fixed point.*

This was given as [5, Corollary 8.6] in 2009, and, for $\mathfrak{B}^+$, as [1, Corollary 3.2] in 2010.

4. Fixed point theorems on abstract convex Φ-space

In this section, we begin with the following.

Definition 4.1. For a given abstract convex space $(E, D; \Gamma)$ and a topological space X, a map $H : X \multimap E$ is called a Φ-*map* (or a *Fan-Browder map*) if there exists a map $G : X \multimap D$ such that

(i) for each $x \in X$, $\mathrm{co}_\Gamma G(x) \subset H(x)$ [that is, $H(x)$ is Γ-convex relative to $G(x)$]; and

(ii) $X = \bigcup\{\mathrm{Int}\, G^-(y) \mid y \in D\}$.

Definition 4.2. In $(E, D; \Gamma; \mathcal{U})$, a subset Z of E is called a Φ-*set* if, for any entourage $U \in \mathcal{U}$, there exists a Φ-map $H : Z \multimap E$ such that $\mathrm{Gr}(H) \subset U$. If E itself is a Φ-set, then it is called a Φ-*space*.

These definitions were given in [5, Section 5] in 2009, and as Definitions 4.1 and 4.2 in [1] in 2010.

Proposition 4.3. *Every locally convex subset Y of a convex subset X of a t.v.s. E is a Φ-subset of X.*

This was given as [4, Proposition 7.1] in 2008. and as [1. Lemma 4.1] in 2010.

Now we have the following fixed point theorem:

Theorem 4.4. *Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space, and $F \in \mathfrak{KC}(E, E)$ be a compact map. If $\overline{F(E)}$ is a Φ-set, then F has the almost fixed point property. Further if $(E, \mathcal{U})$ is separated and if F is closed, then it has a fixed point.*

This is a combined form of [3, Theorem 12 and Corollary 12.1] in 2008 and reduces to [1, Theorem 4.1] for $\mathfrak{B}^+$.

Corollary 4.5. *Let X be a nonempty convex subset of a Hausdorff t.v.s. Then any compact closed map $F \in \mathfrak{B}(X, X)$ such that $\overline{F(X)}$ is locally convex has a fixed point.*

This was given as [4, Corollary 9.11] in 2008, and as [1, Corollary 4.1] in 2010 for $\mathfrak{B}^+$.

We have the following in [2]:

Theorem 4.6. *Let $(X, D; \Gamma; \mathcal{U})$ be a Hausdorff Φ-space. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

This was given as [2, Theorem 4.6] in 2000 and [4, Theorem 9.12] in 2008, and as [1, Corollary 4.2] in 2010 for $\mathfrak{B}^+$.

Remark 4.7. It should be noted that no references of ourselves in this paper appeared in [1]. Moreover, the authors of [1] claimed that their main results generalize some recent work due to Lassonde, Kakutani, Browder, and Park without giving any details or justifications.

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