Salahuddin, J. Nonlinear Anal. Optim. Vol. 11(2) (2020), 127-136

Journal of Nonlinear Analysis and Optimization
Volume 12(1) (2021)
http://www.math.sci.nu.ac.th
ISSN : 1906-9685

J. Nonlinear Anal. Optim.

GENERALIZED g-TYPE EXPONENTIAL VECTOR VARIATIONAL INEQUALITY PROBLEMS

SALAHUDDIN[1]

[1] Department of Mathematics, Jazan University, Jazan-45142, Kingdom of Saudi Arabia

ABSTRACT. In this works, we introduce a class of generalized g-type exponential vector variational inequality problems in Euclidean spaces and define α_g-relaxed exponentially (τ,μ)-monotone mapping. By utilizing KKM-mapping and Nadler's theorem with α_g-relaxed exponentially (τ,μ)-monotone mapping, we prove that the existence theorems of generalized g-type exponential vector variational inequality problems.

KEYWORDS:Generalized g-type exponential vector variational inequality problems, α_g-relaxed exponentially (τ,μ)-monotone mapping, KKM-mappings, Nadler's Theorem.
AMS Subject Classification: 49J40, 47H06, 47H09,47J20.

1. Introduction

The theory of variational inequalities was introduced by Stampacchia [18], provides a very powerful tools for studying problems arising in fluid mechanics, optimization, transportation, economics, contact problems in elasticity and other branches of physics, *for examples*, free boundary value problems can be studied effectively in the framework of variational inequalities, moving boundary value problems can be characterized by a class of variational inequalities, the traffic assignment problem is a variational inequality problem.

Many real-life problems associated with decision sciences involving multiple objectives or criteria within the treatment. Very often, these objectives and criteria square measure in conflict. Consequently, makes an attempt to model these phenomena by one objective or criterion function have provided only rough models, that are far away from reality. We believe that vector variational inequality problems could also be utilized in this respect, as innovative, powerful and unified modeling while not losing the vector nature of the problems, [1, 2, 3, 10, 13, 20].

Email address : drsalah12@hotmail.com, smohammad@jazanu.edu.sa.
Article history : Received 24 August 2019; Accepted 12 September 2020.

Inspired by recent research works [5, 9, 11, 12, 16, 17, 19, 21], in this article, we introduce a generalized g-type exponential vector variational inequality problems in $\mathbb{R}^n$-space and defined a class of α_g-relaxed exponential (τ, μ)-monotone mappings. We prove that the existence of generalized g-type exponential vector variational inequality problems by utilizing KKM-mapping and Nadler's Theorem.

The rest of this work is organized as follows. In section 2, we mathematically state the generalized g-type exponential vector variational inequality problems and discussed some concepts and remarks. In section 3, we present the main results of our paper and some corollaries to be discussed.

2. Preliminaries

Let $Y = \mathbb{R}^n$ be an Euclidean space and C be a nonempty subset of Y. C is called a cone if $\lambda C \subset C$, for any $\lambda \geq 0$. Further, the cone C is called convex cone if $C + C \subset C$ and C is pointed cone if C is cone and $C \cap \{-C\} = \{\mathbf{0}\}$, where $\mathbf{0}$ indicate a zero vector. C is said to be proper cone, if $C \neq Y$. Now, we consider $C \subseteq Y$ is a pointed closed convex cone with $intC \neq \emptyset$ with apex at origin, where $intC$ is a set of interior points of C. Then, C induces a vector ordering in Y as follows:

(i) $x \leq_C y \Leftrightarrow y - x \in C$;
(ii) $x \not\leq_C y \Leftrightarrow y - x \notin C$;
(iii) $x \leq_{intC} y \Leftrightarrow y - x \in intC$;
(iv) $x \not\leq_{intC} y \Leftrightarrow y - x \notin intC$.

Assume that (Y, C) is an ordered space with the ordering of Y defined by a set C and ordering relation " $\leq_C$ " is a partial order. Then, we have

(i) $x \not\leq_C y \Leftrightarrow x + z \not\leq_C y + z$, for any $x, y, z \in Y$;
(ii) $x \not\leq_C y \Leftrightarrow \lambda x \not\leq_C \lambda y$, for any $\lambda \geq 0$.

Let $K \subseteq X$ be a nonempty closed convex subset of an Euclidean space $X = \mathbb{R}^n$ and (Y, C) be an ordered space induces by the closed convex pointed cone C whose apex at origin with $intC \neq \emptyset$.

Lemma 2.1. [6, 7] *Let (Y, C) be an ordered space induced by the pointed closed convex cone C with $intC \neq \emptyset$. Then for any $x, y, z \in Y$, the following relation hold:*

$$z \not\leq_{intC} x \geq_C y \Rightarrow z \not\leq_{intC} y;$$
$$z \not\geq_{intC} x \leq_C y \Rightarrow z \not\geq_{intC} y.$$

Definition 2.2. A mapping $f : X \longrightarrow Y$ is said to be:

(i) C-convex on X if

$$f(tx + (1-t)y) \leq_C tf(x) + (1-t)f(y), \ \forall x, y \in X, t \in [0, 1];$$

(ii) affine if for any $x_i \in K$ and $\lambda_i \geq 0$, $(1 \leq i \leq n)$ with $\sum_{i=1}^n \lambda_i = 1$, we have

$$f(\sum_{i=1}^n \lambda_i x_i) = \sum_{i=1}^n \lambda_i f(x_i).$$

Definition 2.3. [4] A mapping $f : K \longrightarrow Y$ is said to be completely continuous if for any sequence $\{x_n\} \in K$, $x_n \rightharpoonup x_0 \in K$ weakly, then $f(x_n) \longrightarrow f(x_0)$.

Definition 2.4. Let $f : K \longrightarrow 2^X$ be a set valued mapping. Then, f is said to be KKM-mapping if for any $\{y_1, y_2, \cdots, y_n\}$ of K, we have

$$co\{y_1, y_2, \cdots, y_n\} \subset \bigcup_{i=1}^{n} f(y_i),$$

where $co\{y_1, y_2, \cdots, y_n\}$ denotes the convex hull of $y_1, y_2, \cdots, y_n$.

Lemma 2.5. [8] *Let M be a nonempty subset of Hausdorff topological vector space X and let $f : M \longrightarrow 2^X$ be KKM-mapping. If $f(y)$ is a closed in X for all $y \in M$ and compact for some $y \in M$, then*

$$\bigcap_{y \in M} f(y) \neq \emptyset.$$

Lemma 2.6. [15] *Let E be a normed vector space and H be the Hausdorff metric on the collection $CB(E)$ of all closed bounded subsets of E, induced by a metric d in terms of $d(x, y) = \|x - y\|$ which is defined by*

$$\mathrm{H}(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\|\},$$

for $A, B \in CB(E)$. If A and B are compact subset in E, then for each $x \in A$, there exists $y \in B$ such that

$$\|x - y\| \leq \mathrm{H}(A, B).$$

Definition 2.7. Let $\mu : X \times X \longrightarrow X$ be a mapping and $Q : K \longrightarrow L(X, Y)$ be a single valued mapping, where $L(X, Y)$ is a space of all continuous linear mappings from X to Y. Let $T : K \longrightarrow 2^{L(X,Y)}$ be a nonempty compact set valued mapping, then

(i) Q is said to be μ-hemicontinuous if

$$\lim_{t \longrightarrow 0^+} \langle Q(x + t(y - x)), \mu(y, x)\rangle = \langle Qx, \mu(y, x)\rangle, \text{ for each } x, y \in K.$$

(ii) T is said to be H-hemicontinuous, if for any given $x, y \in K$, the mapping $t \longrightarrow \mathrm{H}(T(x + t(y - x)), Tx)$ is continuous at 0^+, where H is the Hausdorff metric defined on $CB(L(X, Y))$.

Definition 2.8. A function $f : X \longrightarrow R$ is said to be

(i) lower semicontinuous at $x_0 \in X$ if

$$f(x_0) \leq \liminf_{n} f(x_n)$$

for any sequence $\{x_n\} \subset X$ such that $\{x_n\}$ converges to x_0;

(ii) weakly upper semicontinuous at $x_0 \in X$ if

$$f(x_0) \geq \limsup_{n} f(x_n)$$

for any sequence $\{x_n\} \subset X$ such that $\{x_n\}$ converges to x_0 weakly.

Let $K \subseteq X$ be a nonempty closed convex subset of an Euclidean space $\mathbb{R}^n$ and (Y, C) be an ordered Euclidean space induced by a closed convex pointed cone C whose apex at origin with $intC \neq \emptyset$. Let $\tau \in R$ be a nonzero real number, $\mu : K \times K \longrightarrow X$, $g : K \longrightarrow K$, $f : K \times K \longrightarrow Y$ and $Q : L(X, Y) \longrightarrow L(X, Y)$ be the mappings, where $L(X, Y)$ be the space of all continuous linear mappings from X to Y. Let $T : K \longrightarrow 2^{L(X,Y)}$ be a vector set valued mapping. Then, generalized *g-type exponential vector variational inequality problems* is to find $x \in K$, $u \in T(x)$ such that

$$\langle Qu, \frac{1}{\tau}(e^{(\tau\mu(y,g(x)))} - \mathbf{1})\rangle + f(g(x), y) \nleq_{intC} 0, \ \forall y \in K, \tag{2.1}$$

here $\mathbf{1}$ is not a real number, because we deal with a vector in $\mathbb{R}^n$.

Definition 2.9. Let $Q : L(X,Y) \longrightarrow L(X,Y)$ be single-valued mapping. A multivalued mapping $T : K \longrightarrow 2^{L(X,Y)}$ with compact valued is said to be α_g-relaxed exponentially (τ, μ)-monotone with respect to Q and g, if for each pair of points $x, y \in K$, we have

$$\langle Qu - Qv, \frac{1}{\tau}(e^{(\tau\mu(x,g(y)))} - \mathbf{1})\rangle \geq_C \alpha_g(x - y), \ \forall u \in T(x), v \in T(y) \tag{2.2}$$

where $\alpha_g : X \longrightarrow Y$ with $\alpha_g(tx) = t^q \alpha_g(x)$ for all $t > 0$ and $x \in X$, where $q > 1$ is a real number.

Remark 2.10.

(i) Assume that Q is an identity mapping and $T : K \longrightarrow L(X,Y)$ is single-valued mapping in (2.2), then T is said to be α_g-relaxed exponentially (τ, μ)-monotone for every pair of points $x, y \in K$, such that

$$\langle Tx - Ty, \frac{1}{\tau}(e^{(\tau\mu(x,g(y)))} - \mathbf{1})\rangle \geq_C \alpha_g(x - y) \tag{2.3}$$

where $\alpha_g : X \longrightarrow Y$ with $\alpha_g(tx) = t^q \alpha_g(x)$ for all $t > 0$ and $x \in X$, where $q > 1$ is a real number.

(ii) If $g \equiv I$ is an identity mapping in (2.3), then T is said to be α-relaxed exponentially (τ, μ)-monotone, for each pair of points $x, y \in K$, such that

$$\langle Tx - Ty, \frac{1}{\tau}(e^{(\tau\mu(x,y))} - \mathbf{1})\rangle \geq_C \alpha(x - y), \tag{2.4}$$

studied in [14].

(iii) If $\alpha \equiv 0$, then (2.4) is said to be exponentially (τ, μ)-monotone, for each pair of points $x, y \in K$, such that

$$\langle Tx - Ty, \frac{1}{\tau}(e^{(\tau\mu(x,y))} - \mathbf{1})\rangle \geq_C 0. \tag{2.5}$$

(iv) If $g \equiv I$ is an identity mapping, then Definition 2.9 becomes an α-relaxed exponentially (τ, μ)-monotone with respect to Q, for each pair of points $x, y \in K$, such that

$$\langle Qu - Qv, \frac{1}{\tau}(e^{(\tau\mu(x,y))} - \mathbf{1})\rangle \geq_C \alpha(x - y), \ \forall u \in T(x), v \in T(y). \tag{2.6}$$

(v) If $\alpha \equiv 0$, then (2.6) is said to be exponentially (τ, μ)-monotone with respect to Q, for each pair of points $x, y \in K$, such that

$$\langle Qu - Qv, \frac{1}{\tau}(e^{(\tau\mu(x,y))} - \mathbf{1})\rangle \geq_C 0, \ \forall u \in T(x), v \in T(y). \tag{2.7}$$

3. Main Results

Theorem 3.1. *Let $g : K \longrightarrow K$ be a single-valued mapping and $\mu : K \times K \longrightarrow X$ be affine in the first variable with $\mu(x, g(x)) = 0$. Let $f : K \times K \longrightarrow Y$ be affine in second variable with condition $f(g(x), x) = 0$ for all $x \in K$. Let $Q : L(X,Y) \longrightarrow L(X,Y)$ be a continuous mapping and $T : K \longrightarrow 2^{L(X,Y)}$ be a nonempty compact valued mapping, which is* H*-hemicontinuous and α_g-relaxed exponentially (τ, μ)-monotone with respect to Q and g. Then, the following two statements (a) and (b) are equivalent:*

(a) *there exists* $\bar{x} \in K$, $\bar{u} \in T(\bar{x})$ *such that*

$$\langle Q\bar{u}\,,\, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}), y) \nleq_{intC} 0,\ \forall y \in K,$$

(b) *there exists* $\bar{x} \in K$ *such that*

$$\langle Qv\,,\, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}), y) \nleq_{intC} \alpha_g(y-\bar{x})\ \forall y \in K, v \in T(y).$$

Proof. Assume that the statement (a) is true, then there exists $\bar{x} \in K$, $\bar{u} \in T(\bar{x})$ such that

$$\langle Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}), y) \nleq_{intC} 0, \forall y \in K. \tag{3.1}$$

Since T is α_g-relaxed exponentially (τ,μ)-monotone with respect to Q and g, we have

$$\langle Qv - Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}), y) \geq_C \alpha_g(y-\bar{x}) + f(g(\bar{x}), y), \forall y \in K, v \in T(y),$$

$$\begin{aligned}\Rightarrow &\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}), y) \geq_C \langle Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle \\ &\quad + \alpha_g(y-\bar{x}) + f(g(\bar{x}), y), \forall y \in K, v \in T(y), \\ \Rightarrow &\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}), y) - \alpha_g(y-\bar{x}) \geq_C \langle Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle \\ &\quad + f(g(\bar{x}), y), \forall y \in K, v \in T(y).\end{aligned} \tag{3.2}$$

Utilizing (3.1), (3.2) and Lemma 2.1, we obtain

$$\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}), y) \nleq_{intC} \alpha_g(y-\bar{x}), \forall y \in K, v \in T(y).$$

Conversely, assume that the statement (b) is true, then there exists $\bar{x} \in K$ such that

$$\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}), y) \nleq_{intC} \alpha_g(y-\bar{x}), \forall y \in K, v \in T(y). \tag{3.3}$$

For any $y \in K$, let $y_t = ty + (1-t)\bar{x},\ t \in (0,1]$, $y_t \in K$ and K is convex. Let for all $v_t \in T(y_t)$, we have from (3.3),

$$\langle Qv_t, \frac{1}{\tau}(e^{(\tau\mu(y_t,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}), y_t) \nleq_{intC} \alpha_g(y_t-\bar{x}) = t^q\alpha_g(y-\bar{x}). \tag{3.4}$$

Now

$$\begin{aligned}&\langle Qv_t, \frac{1}{\tau}(e^{(\tau\mu(y_t,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}), y_t) \\ &= \langle Qv_t, \frac{1}{\tau}(e^{(\tau\mu(ty+(1-t)\bar{x},g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}), ty+(1-t)\bar{x}) \\ &= \langle Qv_t, \frac{1}{\tau}(e^{(\tau t\mu(y,g(\bar{x}))+(1-t)\tau\mu(\bar{x},g(\bar{x})))}-\mathbf{1})\rangle + tf(g(\bar{x}), y) + (1-t)f(g(\bar{x}), \bar{x}) \\ &\leq_C \langle Qv_t, \frac{1}{\tau}(t(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1}) + (1-t)(e^{(\tau\mu(\bar{x},g(\bar{x})))}-\mathbf{1}))\rangle + tf(g(\bar{x}), y) \\ &= t\{\langle Qv_t, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}), y)\}.\end{aligned} \tag{3.5}$$

Utilizing (3.4), (3.5) and Lemma 2.1, we obtain

$$\langle Qv_t, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}), y) \nleq_{intC} t^{q-1}\alpha_g(y-\bar{x}). \tag{3.6}$$

Seeing as $T(y_t)$ and $T(\bar{x})$ are compact, from Lemma 2.6, for each $v_t \in T(y_t)$, there exists $u_t \in T(\bar{x})$ such that

$$\|v_t - u_t\| \leq \mathrm{H}(T(y_t), T(\bar{x})). \tag{3.7}$$

Since $T(\bar{x})$ is compact, without loss of generality, we may possibly assume that

$$u_t \longrightarrow \bar{u} \in T(\bar{x}) \text{ as } t \longrightarrow 0^+.$$

Furthermore, T is H-hemicontinuous, thus it follows that

$$\mathrm{H}(T(y_t), T(\bar{x})) \longrightarrow 0 \text{ as } t \longrightarrow 0^+.$$

Now from (3.7), we have

$$\begin{aligned}\|v_t - \bar{u}\| &\leq \|v_t - u_t\| + \|u_t - \bar{u}\| \\ &\leq \mathrm{H}(T(y_t), T(x_t)) + \|u_t - \bar{u}\| \longrightarrow 0 \text{ as } t \longrightarrow 0^+. \end{aligned} \tag{3.8}$$

As Q is continuous, let $t \longrightarrow 0^+$, we have

$$\begin{aligned}&\|\langle Qv_t, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle - t^{q-1}\alpha_g(y-\bar{x}) - \langle Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle\| \\ &\leq \|\langle Qv_t - Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle\| + \|t^{q-1}\alpha_g(y-\bar{x})\| \\ &\leq \frac{1}{\tau}\|Qv_t - Q\bar{u}\|\|e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1}\| + t^{q-1}\|\alpha_g(y-\bar{x})\| \longrightarrow 0 \text{ as } t \longrightarrow 0^+. \end{aligned} \tag{3.9}$$

From (3.4), we get

$$\langle Qv_t, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle + f(g(\bar{x}), y) - t^{q-1}\alpha_g(y-\bar{x}) \in Y \setminus (-intC).$$

Since $Y \setminus (-intC)$ is closed, therefore from (3.9) we have

$$\begin{aligned}&\langle Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle + f(g(\bar{x}), y) \in Y \setminus (-intC) \\ \Longrightarrow &\langle Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle + f(g(\bar{x}), y) \not\leq_{intC} 0, \ \forall y \in K,\end{aligned}$$

the proof is completed. □

Theorem 3.2. *Let $g : K \longrightarrow K$ be a single-valued mapping, $\mu : K \times K \longrightarrow X$ be affine in the first variable with $\mu(x, g(x)) = 0$ for $x \in K$ and continuous in both variable. Let $f : K \times K \longrightarrow Y$ be affine in second variable with the condition $f(g(x), x) = 0$ for $x \in K$. Let $\alpha_g : X \longrightarrow Y$ be weakly lower semicontinuous with respect to g. Let $Q : L(X,Y) \longrightarrow L(X,Y)$ be a continuous mapping and $T : K \longrightarrow 2^{L(X,Y)}$ be nonempty compact valued mapping, which is* H*-hemicontinuous and α_g-relaxed exponentially (τ, μ)-monotone with respect to Q and g. Then (2.1) is solvable, that is, there exist $x \in K$, $u \in T(x)$ such that*

$$\langle Qu, \frac{1}{\tau}(e^{(\tau\mu(y,g(x)))} - \mathbf{1})\rangle + f(g(x), y) \not\leq_{intC} 0, \ \forall y \in K.$$

Proof. Consider the set valued mapping $F : K \longrightarrow 2^X$ such that

$$F(y) = \{x \in K : \langle Qu, \frac{1}{\tau}(e^{(\tau\mu(y,g(x)))} - \mathbf{1})\rangle + f(g(x), y) \not\leq_{intC} 0, \forall u \in T(x)\}, y \in K.$$

First, we claim that F is a KKM mapping. If F is not a KKM-mapping, then there exists $(x_1, x_2, \cdots, x_m) \subset K$ such that

$$co\{x_1, x_2, \cdots, x_m\} \not\subseteq \bigcup_{i=1}^{m} F(x_i),$$

there exists at least $x \in co\{x_1, x_2, \cdots, x_m\}, x = \sum_{i=1}^m t_i x_i$, where $t_i \geq 0, i = 1, 2 \cdots, m$, $\sum_{i=1}^m t_i = 1$, but $x \notin \bigcup_{i=1}^m F(x_i)$. From the construction of F, for any $u \in T(x)$, we have

$$\langle Qu, \frac{1}{\tau}(e^{(\tau\mu(x_i,g(x)))} - \mathbf{1})\rangle + f(g(x), x_i) \leq_{intC} 0, \text{ for } i = 1, 2, \cdots, m. \tag{3.10}$$

From (3.10), since μ is affine in first variable and f is affine with respect to second variable, it follows that

$$\begin{aligned}
0 &= \langle Qu, \frac{1}{\tau}(e^{(\tau\mu(x,g(x)))} - \mathbf{1})\rangle + f(g(x), x) \\
&= \langle Qu, \frac{1}{\tau}(e^{(\tau\mu(\sum_{i=1}^m t_i x_i, g(x)))} - \mathbf{1})\rangle + f(g(x), \sum_{i=1}^m t_i x_i) \\
&= \langle Qu, \frac{1}{\tau}(e^{(\sum_{i=1}^m t_i \tau\mu(x_i, g(x)))} - \mathbf{1})\rangle + \sum_{i=1}^m t_i f(g(x), x_i) \\
&\leq_C \langle Qu, \frac{1}{\tau}\sum_{i=1}^m t_i (e^{(\tau\mu(x_i,g(x)))} - \mathbf{1})\rangle + \sum_{i=1}^m t_i f(g(x), x_i) \\
&= \sum_{i=1}^m t_i \{\langle Qu, \frac{1}{\tau}(e^{(\tau\mu(x_i,g(x)))} - \mathbf{1})\rangle + f(g(x), x_i)\} \\
&\leq_{intC} 0,
\end{aligned}$$

this show that $0 \in intC$, which is a contradiction that the fact C is proper. Hence F is KKM-mapping. Define another set valued mapping $G : K \longrightarrow 2^X$ such that

$$G(y) = \{x \in K : \langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y,g(x)))} - \mathbf{1})\rangle + f(g(x), y) \not\leq_{intC} \alpha_g(y - x), \forall v \in T(y)\}, y \in K.$$

Since by Theorem 3.1, we have $F(y) \equiv G(y)$ for all $y \in K$. This implies that G is also KKM-mapping.

We claim that for each $y \in K, G(y) \subset K$ is closed in the weak topology of X. Let us suppose that $\bar{x} \in \overline{G(y)}^w$, the weak closure of $G(y)$. Since X is reflexive, there is a sequence $\{x_n\}$ in $G(y)$ such that $\{x_n\}$ converges weakly to $\bar{x} \in K$. Then for each $v \in T(y)$, we have

$$\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y,g(x_n)))} - \mathbf{1})\rangle + f(g(x_n), y) \not\leq_{intC} \alpha_g(y - x_n)$$

$$\Rightarrow \langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y,g(x_n)))} - \mathbf{1})\rangle + f(g(x_n), y) - \alpha_g(y - x_n) \in Y \setminus (-intC).$$

Since Qv, f and g are continuous, $Y \setminus (-intC)$ is closed, α_g is weakly lower semi-continuous, therefore the sequence

$$\{\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y,g(x_n)))} - \mathbf{1})\rangle + f(g(x_n), y) - \alpha_g(y - x_n)\}$$

converges to

$$\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle + f(g(\bar{x}), y) - \alpha_g(y - \bar{x})$$

and

$$\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle + f(g(\bar{x}), y) - \alpha_g(y - \bar{x}) \in Y \setminus (-intC).$$

Therefore

$$\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle + f(g(\bar{x}), y) \not\leq_{intC} \alpha_g(y - \bar{x}).$$

Hence $\bar{x} \in G(y)$. This confirm $G(y)$ is weakly closed for all $y \in K$. Furthermore, X is reflexive and $K \subset X$ is a nonempty closed convex and bounded. Therefore, K is weakly compact subset of X and so $G(y)$ is also weakly compact. Therefore, from Lemma 2.5, it follows

$$\bigcap_{y \in K} G(y) \neq \emptyset.$$

There exists $\bar{x} \in K$ such that

$$\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle + f(g(\bar{x}), y) \nleq_{intC} \alpha_g(y - \bar{x}), \forall y \in K, v \in T(y).$$

Hence, we conclude that from Theorem 3.1, there exists $\bar{x} \in K, \bar{u} \in T(\bar{x})$ such that

$$\langle Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle + f(g(\bar{x}), y) \nleq_{intC} 0, \ \forall y \in K$$

the proof is completed. □

Theorem 3.3. *Let $g : K \longrightarrow K$ be a single-valued mapping and $\mu : K \times K \longrightarrow X$ be affine in the first variable with $\mu(x, g(x)) = 0$ for all $x \in K$. Let $f : K \times K \longrightarrow Y$ be a continuous mapping and affine in the second variable with the condition $f(g(x), x) = 0$ for all $x \in K$. Let $\alpha_g : X \longrightarrow Y$ be weakly lower semicontinuous. Let $Q : L(X,Y) \longrightarrow L(X,Y)$ be a mapping and $T : K \longrightarrow 2^{L(X,Y)}$ be a nonempty compact valued mapping, which is* H*-hemicontinuous and α_g-relaxed exponentially (τ, μ)-monotone with respect to Q and g. There exists $r > 0$ such that*

$$\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(0,g(y)))} - \mathbf{1})\rangle + f(g(y), 0) \leq_{intC} 0, \ \forall y \in K, v \in T(y) \text{ with } \|y\| = r. \tag{3.11}$$

Then (2.1) is solvable.

Proof. For $r > 0$, assume that $K_r = \{x \in X, \|x\| \leq r\}$. From Theorem 3.2, we know that (2.1) is solvable over K_r, *i.e.*, there exists $x_r \in K \bigcap K_r$ and $u_r \in T(x_r)$ such that

$$\langle Qu_r, \frac{1}{\tau}(e^{(\tau\mu(y,g(x_r)))} - \mathbf{1})\rangle + f(g(x_r), y) \nleq_{intC} 0, \ \forall y \in K \bigcap K_r. \tag{3.12}$$

Putting $y = 0$ in (3.12) we have

$$\langle Qu_r, \frac{1}{\tau}(e^{\tau\mu(0,g(x_r))} - \mathbf{1})\rangle + f(g(x_r), 0) \nleq_{intC} 0. \tag{3.13}$$

If $\|x_r\| = r$ for all r, then it is contradicts to (3.11). Hence $r > \|x_r\|$. For any $z \in K$, let us choose $t \in (0, 1)$ small enough such that $(1-t)x_r + tz \in K \bigcap K_r$. Putting $y = (1-t)x_r + tz$ in (3.12), we get

$$\langle Qu_r, \frac{1}{\tau}(e^{(\tau\mu((1-t)x_r+tz,g(x_r)))} - \mathbf{1})\rangle + f(g(x_r), (1-t)x_r + tz) \nleq_{intC} 0. \tag{3.14}$$

Since μ is affine in the first variable and f is affine with respect to the second variable, we have

$$\begin{aligned}
&\langle Qu_r, \frac{1}{\tau}(e^{(\tau\mu((1-t)x_r+tz,g(x_r)))} - \mathbf{1})\rangle + f(g(x_r), (1-t)x_r + tz) \\
&= \langle Qu_r, \frac{1}{\tau}(e^{((1-t)\tau\mu(x_r,g(x_r))+t\tau\mu(z,g(x_r)))} - \mathbf{1})\rangle + tf(g(x_r), z) \\
&\leq_C \langle Qu_r, \frac{1}{\tau}(1-t)(e^{(\tau\mu(x_r,g(x_r)))} - \mathbf{1}) + \frac{1}{\tau}t(e^{(\tau\mu(z,g(x_r)))} - \mathbf{1})\rangle + tf(g(x_r), z) \\
&= t\{\langle Qu_r, \frac{1}{\tau}(e^{(\tau\mu(z,g(x_r)))} - \mathbf{1})\rangle + f(g(x_r), z)\}.
\end{aligned} \tag{3.15}$$

Hence from (3.14), (3.15) and Lemma 2.1, we get

$$\langle Qu_r, \frac{1}{\tau}(e^{(\tau\mu(z,g(x_r)))}-\mathbf{1})\rangle + f(g(x_r),z) \nleq_{intC} 0,\ \forall z \in K. \tag{3.16}$$

Therefore, (2.1) is solvable and proof is completed. □

We note that, if $g = I$ is an identity mapping, then we have following corollary:

Corollary 3.1. *Let $\mu : K \times K \longrightarrow X$ be an affine in the first variable with $\mu(x,x) = 0$ for all $x \in K$ and $f : K \times K \longrightarrow Y$ be C-convex in the second variable with the condition $f(x,x) = 0$ for all $x \in K$. Let $Q : L(X,Y) \longrightarrow L(X,Y)$ be a continuous mapping and $T : K \longrightarrow 2^{L(X,Y)}$ be a nonempty compact valued mapping, which is* H*-hemicontinuous and α-relaxed exponentially (τ,μ)-monotone with respect to Q. Then the following two statements (a) and (b) are equivalent:*

(a) *there exists $\bar{x} \in K$, $\bar{u} \in T(\bar{x})$ such that*

$$\langle Q\bar{u}\,, \frac{1}{\tau}(e^{(\tau\mu(y,\bar{x}))}-\mathbf{1})\rangle + f(\bar{x},y) \nleq_{intC} 0,\ \forall y \in K,$$

(b) *there exists $\bar{x} \in K$ such that*

$$\langle Qv\,, \frac{1}{\tau}(e^{(\tau\mu(y,\bar{x}))}-\mathbf{1})\rangle + f(\bar{x},y) \nleq_{intC} \alpha(y-\bar{x})\ \forall y \in K, v \in T(y).$$

We note that, if $g = Q = I$ are identity mapping, then we have following corollary:

Corollary 3.2. *Let $\mu : K \times K \longrightarrow X$ be an affine in the first variable with $\mu(x,x) = 0$ for all $x \in K$ and $f : K \times K \longrightarrow Y$ be C-convex in the second variable with the condition $f(x,x) = 0$ for all $x \in K$. Let $T : K \longrightarrow 2^{L(X,Y)}$ be a nonempty compact valued mapping, which is* H*-hemicontinuous and α-relaxed exponentially (τ,μ)-monotone. Then the following two statements (a) and (b) are equivalent:*

(a) *there exists $\bar{x} \in K$, $\bar{u} \in T(\bar{x})$ such that*

$$\langle \bar{u}\,, \frac{1}{\tau}(e^{(\tau\mu(y,\bar{x}))}-\mathbf{1})\rangle + f(\bar{x},y) \nleq_{intC} 0,\ \forall y \in K,$$

(b) *there exists $\bar{x} \in K$ such that*

$$\langle v\,, \frac{1}{\tau}(e^{(\tau\mu(y,\bar{x}))}-\mathbf{1})\rangle + f(\bar{x},y) \nleq_{intC} \alpha(y-\bar{x})\ \forall y \in K, v \in T(y).$$

We note that, if T is a single valued mapping and $f(x,y) \equiv 0$, a zero mapping, then Corollary 3.2 reduces to the following:

Corollary 3.3. *Let $\mu : K \times K \longrightarrow X$ be an affine in the first variable with $\mu(x,x) = 0$ for all $x \in K$ and $T : X \longrightarrow L(X,Y)$ be α-relaxed exponentially (τ,μ)-monotone. Then the following two statements (a) and (b) are equivalent:*

(a) *there exists $\bar{x} \in K$ such that*

$$\langle \bar{x}\,, \frac{1}{\tau}(e^{(\tau\eta(y,\bar{x}))}-\mathbf{1})\rangle \nleq_{intC} 0,\ \forall y \in K,$$

(b) *there exists $\bar{x} \in K$ such that*

$$\langle Tz\,, \frac{1}{\tau}(e^{(\tau\mu(y,\bar{x}))}-\mathbf{1})\rangle \nleq_{intC} \alpha(y-\bar{x})\ \forall y,z \in K.$$

4. Acknowledgements

The author wishes to express his gratitude to the Editor Professor Narin Petrot, the two anonymous referees for their detailed comments and useful suggestions that allowed to improve significantly the presentation of this paper.

References

1. R. P. Agarwal, M. K. Ahmad and Salahuddin, Hybrid type generalized multivalued vector complementarity problems, Ukrainian Mathematics Journal, **65**(1), 6-20, 2013, ISSN: 1027-3190.
2. G. A. Anastassiou and Salahuddin, Weakly set valued generalized vector variational inequalities, Journal Computational Analysis and Applications, **15**(4), 2013, 622–632.
3. L. Brouwer, Zur invarianz des n-dimensional gebietes. Mathematical Annllen, **71**(3), 1912, 305–313.
4. S. S. Chang, Salahuddin, C. F. Wen and X. R. Wang, On the existence problem of solutions to a class of fuzzy mixed exponential vector variational inequalities, Journal Nonlinear Science and Applications, **11**, 2018, 916–926.
5. S. S. Chang, Salahuddin, X. R. Wang, L. C. Zhao and J. F. Tang, Generalized (η, g, ϕ)-mixedvector equilibrium problems in fuzzy events, Communications on Applied Nonlinear Analysis, **27** (4), 2020, 49 – 64.
6. L. C. Ceng and J. C. Yao, On generalized variational-like inequalities with generalized monotone multivalued mappings, Applied Mathematics Letters, **22**(3), 2009, 428–434.
7. G. Y. Chen, Existence of solutions for a vector variational inequalities: An extension of Hartman-Stampacchia theorems, Journal of Optimization Theory and Applications, **74**(3), 1992, 445–456.
8. Ky Fan, A generalization of tychonoff's fixed point theorem, Mathematica Annllen **142**, 1961, 305–310.
9. Y. P. Fang and N. J. Huang, Variational-like inequalities with generalized monotone mappings in Banach spaces, Journal of Optimization Theory and Applications, **118**(2), 2003, 327–338.
10. F. Giannessi, Vector variational inequalities and vector equilibria, Mathematical Theories, Nonconvex Optimization and its Applications, **38** Kluwer Academic, Dordrecht, 2000.
11. P. Q. Khanh and N. H. Quan, Generic stability and essential components of generalized KKM points and applications, Journal of Optimization Theory and Applications, **148**, 2011, 488-504.
12. B. S. Lee, M. F. Khan and Salahuddin, Hybrid-type set-valued variational-like inequalities in Reflexive Banach spaces, Journal of Appled Mathematics and Informatics, **27**(5-6), 2009, 1371–1379.
13. B. S. Lee and Salahuddin, Minty lemma for inverted vector variational inequalities, Optimization, **66**(3), 2017, 351–359.
14. N. K. Mahto and R. N. Mohapatra, Existence results of a generalized mixed exponential type vector variational like inequalities, Mathematical Computing, Springer Nature, 2017, 209–220.
15. S. B. Nadler, Multi-valued contraction mappings, Pacific Journal of Mathematics, **30**(2), 1969, 475–488.
16. Salahuddin, General set valued vector variational inequality problems, Communication of Optimization Theory, **2017** , Article ID 13, 2017, 1–16.
17. Salahuddin, Iterative method for non-stationary mixed variational inequalities, Discontinuity, Nonlinearity and Complexity, **9** (4), 2020, 647–655.
18. G. Stampacchia, Formes bilineaires coercitives sur les ensembles convexes, C. R. Academic Science Paris Ser I Math. **258**, 1964, 4413–4416.
19. R. U. Verma, A. Jayswal and S. Choudhury, Exponential type vector variational-like inequalities and vector optimization problems with exponential type invexities, Journal of Applied Mathematic and Computation, **45**(1-2), 2014, 87–97.
20. G. Wang, S. S. Chang and Salahuddin, On the existence theorems of solutions for generalized vector variational inequalities, Journal of Inequality and Application, **2015**, 2015, DOI 10.1186/s13660-015-0856-4.
21. K. Q. Wu and N. J. Huang, Vector variational-like inequalities with relaxed $\eta - \alpha$-Pseudomonotone mappings in Banach spaces, Journal of Mathematical Inequality, **1**, 2007, 281–290.