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SOME FIXED POINT THEOREMS OF HARDY-ROGER CONTRACTION IN COMPLEX VALUED B-METRIC SPACES

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ABSTRACT. The aim of this paper is to prove the existence and uniqueness of a fixed point of a mapping satisfying the Hardy-Rogers contraction in complex valued b-metric space, we have obtained some fixed point theorems in complex-valued b-metric spaces. This work is generalized and improved some results of Hasanah [5], and well known results in the literature.

KEYWORDS: b-metric space, complex valued b-metric space, Hardy-Rogers contraction, fixed point.
AMS Subject Classification: :46C05, 47D03, 47H09, 47H10, 47H20.

1. Introduction

The axiomatic development of a metric space was essentially carried out by French mathematician Frechet in the year 1906 [4]. After the Banach contraction principle, because of various applications. Many mathematics used the Banach contractive principle to study an existence and uniqueness of fixed points. Banach fixed point theorem in a complete metric space introduced by Banach [2], because it was generalized in many spaces. In 1973, Hardy and Rogers [6], define the generalzed Kannan contraction and prove some fixed point theorem in metric space. In 2011 Azam et.al [1], introduced the notion of complex valued metric space and established sufficient conditions for the existence of common fixed point of a pair of mappings satisfying a contractive condition. In 2012, Sintunavarat and Kumam [10] introduced new spaces called the complex valued metric spaces and established the existence of fixed point theorems under the contraction condition. One year later, Sintunavarat, Cho and Kumam, [11] established the existence of fixed point theorems under the contractive condition in complex valued metric spaces, they introduce the concepts of a C-Cauchy sequence and C-complete in

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complex-valued metric spaces and establish the existence of common fixed point theorems in C-complete complex-valued metric spaces. In 2015, Jleli and Samet [7] introduced a very interesting concept of a generalized metric space, which covers different well-known metric structures including classical metric spaces, b-metric spaces, dislocated metric spaces, modular spaces, and so on.

In 2017, Hasanah [5], study the existence and proved the uniqueness of fixed point of some contractive condition in complete complex valued b-metric spaces.

Motivate by Hasanah [5] and Hardy and Rogers [6], we introduce the Hardy-Rogers contraction it has generalized than the contractive condition of [5], and then we proved the existence and uniqueness of fixed point in complete complex valued b-metric space.

2. Preliminaries

In this section, we suppose some definitions and define the definition of b-metric space in the complex plane, and suppose some lemmas for study in this work.

Definition 2.1. Let X be a nonempty set. A function $d : X \times X \to [0, \infty]$ is called a metric space if for $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.
The pair (X, d) is called a metric space, and d is called a matric on X.

Next, we provide the definition of b-metric space, this space is generalized than metric space.

Definition 2.2. [3] Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$.
The pair (X, d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (X, d).

We give some example for b-metric space.

Example 2.3. Let (X, d) be a metric space. The funcion $\rho(x, y)$ is defined by $\rho(x, y) = (d(x, y))^2$. Then (X, ρ) is a b-metric space with coefficient $s = 2$. This can be seen from the nonnegativity property and triangle inequality of metric to prove the property (iii).

Since in real numbers which has completeness property, order is not welll-defined in complex numbers. Before giving the definition of complex valued metric spaces and complex valued b-metric spaces, we define partial order in complex numbers (see [8]). Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define partial order $\precsim$ on $\mathbb{C}$ as follows;

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

This means that we would have $z_1 \precsim z_2$ if and only if one of the following conditions holds:
(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(iii) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.
If one of the conditions (ii), (iii), and (iv) holds, then we write $z_1 \precnsim z_2$. Particulaly, we have $z_1 \prec z_2$ if the condition (iv) is satisfied.

Remark 2.4. We can easily check the following:
(i) If $a, b \in \mathbb{R}, 0 \leq a \leq b$ and $z_1 \preccurlyeq z_2$ then $az_1 \preccurlyeq bz_2, \forall z_1, z_2 \in \mathbb{C}$.
(ii) $0 \preccurlyeq z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$.
(iii) $z_1 \preccurlyeq z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.
(iv) If $z \in \mathbb{C}, a, b \in \mathbb{R}$ and $a \leq b$, then $az \preccurlyeq bz$.

A b-metric on a b-metric sapce is a funcion having real value. Based on the definition of partial order on complex number, real valued b-metric can be generalized into complex valued b-metric as folllows.

Definition 2.5. [1] Let X be a nonmpty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in \mathbb{C}$, the following conditions are satisfied:
(i) $0 \preccurlyeq d(x, y)$
(ii) $d(x, y) = 0$ if and only if $x = y$,
(iii) $d(x, y) = d(y, x)$,
(iv) $d(x, z) \preccurlyeq d(x, y) + d(y, z)$.
Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Next, we give the definition of complex valued b-metric space.

Definition 2.6. [9] Let X be a nonmpty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if, for all $x, y, z \in \mathbb{C}$, the following conditions are satisfied:
(i) $0 \preccurlyeq d(x, y)$
(ii) $d(x, y) = 0$ if and only if $x = y$,
(iii) $d(x, y) = d(y, x)$,
(iv) $d(x, y) \preccurlyeq s[d(x, z) + d(z, y)]$.
The pair (X, d) is called a complex valued b-metric space. We see that if $s = 1$ then (X, d) is complex valued metric space is defined in Definition 2.5.

For Definition 2.6, we can suppose some example of complex valued b-metric space.

Example 2.7. Let $X = \mathbb{C}$. Define the mapping $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$ for all $x, y \in X$. Then $(\mathbb{C}, d)$ is complex valued b-metriic space with $s = 2$.

Definition 2.8. [9] Let (X, d) be a complex valued b-metric space.
(i) A point $x \in X$ is called interior point of set $A \subseteq X$ if there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = y \in Y : d(x, y) \prec r \subseteq A$.
(ii) A point $x \in X$ is called limit point of a set A if for every $0 \prec r \in \mathbb{C}, B(x, r) \cap (A - x) \neq \emptyset$
(iii) A subset $A \subseteq X$ is open if each element of A is an interior point of A.
(iv) A subset $A \subseteq X$ is closed if each limit point of A is contained in A.

For study this work we suppose the definition of convergent sequence, Cauchy sequence and complete complex space.

Definition 2.9. [9] Let (X, d) be complex valued b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(i) $\{x_n\}$ is convergent to $x \in X$ if for every $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, x) \prec r$. Thus x is the limit of $\{x_n\}$ and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) $\{x_n\}$ is said to be Cauchy sequence if for ever $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, x_{n+m}) \prec r$, where $m \in \mathbb{N}$.

(iii) If for every Cauchy sequence in$X$ is convergent, then (X, d) is said to be a complete complex valued b-metric space.

Finally, we give some lemmas for proof the main theorems.

Lemma 2.10. [9] *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.11. [9] *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

3. Main Results

In this section we give some comditions and prove the existence theorem and unique fixed point of Hardy-Rogers contraction in complete complex valued b-metric space.

Theorem 3.1. *Let (X, d) be a complete complex valued b-metric space with constant $s \geq 1$, and let $T : X \to X$ be a mapping with satisfying Hardy-Rogers contraction, that is*

$$d(Tx, Ty) \precsim \lambda_1 d(x, y) + \lambda_2 d(x, Tx) + \lambda_3 d(y, Ty) + \lambda_4 d(y, Tx) + \lambda_5 d(x, Ty)$$

for all $x, y \in X$ and λ_i are nonegative real number with $\Sigma_{i=1}^{5} \lambda_i \in [0, \frac{1}{s})$ and $\lambda_4 \leq \frac{\lambda_5}{2s-1}$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ from $T : X \to X$, we have there exists $x_1 \in X$ such that $x_1 = Tx_0$. From $x_1 \in X$, there exists $x_2 \in X$ such that $x_2 = Tx_1$. By induction of this process, we have the sequence $\{x_n\} \subseteq X$ such that,

$$x_n = Tx_{n-1} = T^n x_0, \forall n \in \mathbb{N}.$$

Note that for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
d(x_{n+2}, x_{n+1}) &= d(Tx_{n+2-1}, Tx_{n+1-1}) \\
&= d(Tx_{n+1}, Tx_n) \\
&\precsim \lambda_1 d(x_{n+1}, x_n) + \lambda_2 d(x_{n+1}, Tx_{n+1}) + \lambda_3 d(x_n, Tx_n) \\
&\quad +\lambda_4 d(x_n, Tx_{n+1}) + \lambda_5 d(x_{n+1}, Tx_n) \\
&\precsim \lambda_1 d(x_{n+1}, x_n) + \lambda_2 d(x_{n+1}, x_{n+2}) + \lambda_3 d(x_n, x_{n+1}) \\
&\quad +\lambda_4 d(x_n, x_{n+2}) + \lambda_5 d(x_{n+1}, x_{n+1}) \\
&\precsim \lambda_1 d(x_{n+1}, x_n) + \lambda_2 d(x_{n+1}, x_{n+2}) + \lambda_3 d(x_n, x_{n+1}) \\
&\quad +\lambda_4 s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + 0.
\end{aligned}$$

$$\begin{aligned}
(1 - (\lambda_2 + \lambda_4 s))d(x_{n+2}, x_{n+1}) &\precsim \lambda_1 d(x_{n+1}, x_n) + \lambda_3 d(x_{n+1}, x_n) + \lambda_4 s d(x_n, x_{n+1}) \\
d(x_{n+2}, x_{n+1}) &\precsim \frac{\lambda_1 + \lambda_3 + \lambda_4 s}{1 - (\lambda_2 + \lambda_4 s)} d(x_{n+1}, x_n).
\end{aligned}$$

If we take $\gamma = \frac{\lambda_1+\lambda_3+\lambda_4 s}{1-(\lambda_2+\lambda_4 s)}$ and continuing this process, then we have

$$d(x_{n+2}, x_{n+1}) \precsim \gamma d(x_{n+1}, x_n).$$

It follows that,

$$d(x_{n+1}, x_n) \precsim \gamma d(x_n, x_{n-1})$$

and

$$d(x_n, x_{n-1}) \precsim \gamma d(x_{n-1}, x_{n-2})$$
$$\vdots$$
$$d(x_{n+1}, x_n) \precsim \gamma^n d(x_1, x_0),$$

for all $n \in \mathbb{N}$. Hence, $d(x_{n+2}, x_{n+1}) \precsim \gamma^{n+1} d(x_1, x_0)$. For $m \in \mathbb{N}$, we have

$$\begin{aligned}
d(x_n, x_{n+m}) &\precsim s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+m})] \\
&\precsim sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+m}) \\
&\precsim sd(x_n, x_{n+1}) + s(s[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+m})]) \\
&\precsim sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2(s[d(x_{n+2}, x_{n+3}) \\
&\quad +d(x_{n+3}, x_{n+m})]) \\
&\precsim sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3} \\
&\quad + \cdots + s^m d(x_{n+m-1}, x_{n+m}) \\
&\precsim s\gamma^n d(x_0, x_1) + s^2\gamma^{n+1} d(x_0, x_1) + s^3\gamma^{n+2} d(x_0, x_1) \\
&\quad + \cdots + s^m \gamma^{n+m-1} d(x_0, x_1) \\
&\precsim s\gamma^n d(x_0, x_1)[1 + s\gamma + (s\gamma)^2 + \cdots + (s\gamma)^{m-1}].
\end{aligned}$$

It follows that

$$d(x_n, x_{n+m}) \precsim s\gamma^n d(x_0, x_1)[1 + s\gamma + (s\gamma)^2 + \cdots + (s\gamma)^{m-1}].$$

By Remark 2.4, taking absolute value on both sides, we have

$$\begin{aligned}
|d(x_n, x_{n+m})| &\leq |s\gamma^n d(x_0, x_1)[1 + s\gamma + (s\gamma)^2 + \cdots + (s\gamma)^{m-1}]| \\
&\leq |s\gamma^n||d(x_0, x_1)[1 + s\gamma + (s\gamma)^2 + \cdots + (s\gamma)^{m-1}]| \\
&= s\gamma^n |d(x_0, x_1)|[1 + s\gamma + (s\gamma)^2 + \cdots + (s\gamma)^{m-1}].
\end{aligned}$$

Since, $\Sigma_{i=1}^5 \lambda_i \in [0, \frac{1}{s})$ for $s \geq 1$ and $\lambda_4 \leq \frac{\lambda_5}{2s-1}$ then $\gamma < 1$ and $s\gamma < 1$. Since $d(x_0, x_1) \in \mathbb{C}$ and $[1 + s\gamma + (s\gamma)^2 + \cdots + (s\gamma)^{m-1}]$ exists, taking limit $n \to \infty$ we have $\gamma^n \to 0$. This implies $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$. By Lemma 2.11, the sequence $\{x_n\}$ is a Cauchy sequence in X. Since, X is a complete complex valued b-metric space then $\{x_n\}$ is a convergent sequence. It follows that $\{x_n\}$ converges to u for some $u \in X$. Next, we can show that u is a fixed point of T. Consider,

$$\begin{aligned}
d(u, Tu) &\precsim s[d(u, x_n) + d(x_n, Tu)] \\
&= s[d(u, x_n) + d(Tx_{n-1}, Tu)] \\
&\precsim s[d(u, x_n) + \lambda_1 d(x_{n-1}, u) + \lambda_2 d(x_{n-1}, Tx_{n-1}) + \lambda_3 d(u, Tu) \\
&\quad +\lambda_4 d(u, Tx_{n-1}) + \lambda_5 d(x_{n-1}, Tu)] \\
(1 - s\lambda_3) d(u, Tu) &\precsim s[d(u, x_n) + \lambda_1 d(x_{n-1}, u) + \lambda_2 d(x_{n-1}, x_n) \\
&\quad +\lambda_4 d(u, x_n) + \lambda_5 d(x_{n-1}, Tu)].
\end{aligned}$$

From Remark 2.4, taking absolute value on both sides, we have

$$\begin{aligned}
|(1 - s\lambda_3) d(u, Tu)| &\leq |s[d(u, x_n) + \lambda_1 d(x_{n-1}, u) + \lambda_2 d(x_{n-1}, x_n) + \lambda_4 d(u, x_n) \\
&\quad +\lambda_5 d(x_{n-1}, Tu)]|
\end{aligned}$$

$$\begin{aligned}
&\leq |s||[d(u,x_n)+\lambda_1 d(x_{n-1},u)+\lambda_2 d(x_{n-1},x_n)+\lambda_4 d(u,x_n)\\
&\quad +\lambda_5 d(x_{n-1},Tu)]|\\
&\leq s[|d(u,x_n)|+|\lambda_1 d(x_{n-1},u)|+|\lambda_2 d(x_{n-1},x_n)|+|\lambda_4 d(u,x_n)|\\
&\quad +|\lambda_5 d(x_{n-1},Tu)|]\\
(1-s\lambda_3)|d(u,Tu)| &\leq s[|d(u,x_n)|+\lambda_1|d(x_{n-1},u)|+\lambda_2|d(x_{n-1},x_n)|+\lambda_4|d(u,x_n)|\\
&\quad +\lambda_5|d(x_{n-1},Tu)|].
\end{aligned}$$

Taking $n\to\infty$ implies $|d(x_n,u)|\to 0$, $|d(x_{n-1},u)|\to 0$. From $\{x_n\}$ is Cauchy sequence in X we have $|d(x_{n-1},x_n)|\to 0$ as $n\to\infty$, then

$$(1-s\lambda_3)|d(u,Tu)|\leq s\lambda_5|d(u,Tu)|.$$

It follows that $(1-s\lambda_3-\lambda_5)|d(u,Tu)|\leq 0$. From $\Sigma_{i=1}^5\lambda_i\in[0,\frac{1}{s})$. Thus $(1-s\lambda_3-\lambda_5)>0$ and then $|d(u,Tu)|=0$. Hence $u=Tu$. Therefore u is a fixed point of T.

Finally, we show the uniqueness of the fixed point of T. We assume that there are two fixed points of T which are $x=Tx$ and $y=Ty$. Thus,

$$\begin{aligned}
d(x,y) &= d(Tx,Ty)\\
&\precsim \lambda_1 d(x,y)+\lambda_2 d(x,Tx)+\lambda_3 d(y,Ty)+\lambda_4 d(y,Tx)+\lambda_5 d(x,Ty)\\
&\precsim \lambda_1 d(x,y)+\lambda_2 d(x,x)+\lambda_3 d(y,y)+\lambda_4 d(y,Tx)+\lambda_5 d(x,Ty)\\
&\precsim \lambda_1 d(x,y)+\lambda_4 d(y,x)+\lambda_5 d(x,y)\\
&\precsim (\lambda_1+\lambda_4+\lambda_5)d(x,y).
\end{aligned}$$

By Remark 2.4, taking the absolute value on both sides, we have

$$\begin{aligned}
|d(x,y)| &\leq |(\lambda_1+\lambda_4+\lambda_5)d(x,y)|\\
&\leq (\lambda_1+\lambda_4+\lambda_5)|d(x,y)|.
\end{aligned}$$

From, $\Sigma_{i=1}^5\lambda_i\in[0,\frac{1}{s})$. Then $\lambda_1+\lambda_4+\lambda_5<1$, this implies that $|d(x,y)|=0$. Hence $x=y$. This completes the proof. □

From Theorem 3.1, we have some corollary, as follows:

Corollary 3.2. *Let (X,d) be a complete complex valued b-metric space with constant $s\geq 1$ and let $T:X\to X$ be a function with the following*

$$d(Tx,Ty)\precsim ad(x,Tx)+bd(y,Ty)+cd(x,y),\forall x,y\in X$$

where $a,b,$ and c are nonnegative real numbers and satisfies $s(a+b+c)<1$. Then T has a uniqe fixed point.

Proof. We put $\lambda_4=\lambda_5=0$, $a=\lambda_2$, $b=\lambda_3$ and $c=\lambda_1$. By theorem 3.1, T has a unique fixed point. This complete the proof. □

Corollary 3.3. *Let (X,d) be a complete complex valued b-metric space with constant $s\geq 1$ and let $T:X\to X$ be a mapping such that*

$$d(Tx,Ty)\precsim \alpha d(x,Ty)+\beta d(y,Tx)$$

for every $x,y\in X$, where α,β are nonnegative real numbers with $\alpha+\beta<\frac{1}{s}$ and $\beta<\frac{\alpha}{2s-1}$. Then T has a fixed point in X.

Proof. We put $\lambda_1=\lambda_2=\lambda_3=0$, $\alpha=\lambda_5$ and $\beta=\lambda_4$. By theorem 3.1, T has a unique fixed point. This complete the proof. □

Next, we can applied Theorem 3.1 to prove the following theorem.

Theorem 3.4. *Let (X, d) be a complete complex valued b-metric space, with the constant $s \geq 1$. Let $T : X \to X$ be a mapping with satisfying*

$$d(T^n x, T^n y) \precsim \lambda_1 d(x, y) + \lambda_2 d(x, T^n x) + \lambda_3 d(y, T^n y) + \lambda_4 d(y, T^n x) + \lambda_5 d(x, T^n y)$$

for all $x, y \in X$ and λ_i are nonegative real number $\Sigma_{i=1}^{5} \lambda_i \in [0, \frac{1}{s})$ and $\lambda_4 \leq \frac{\lambda_5}{2s-1}$. Then T has a unique fixed point.

Proof. Suppose $S = T^n$, by Theorem 3.1, there exists a fixed point u of S, such that

$$Su = u.$$

Thus $T^n u = u$. We obtain that

$$\begin{aligned}
d(Tu, u) &= d(T(T^n u), T^n u) \\
&= d(T^n(Tu), T^n u) \\
&\precsim \lambda_1 d(Tu, u) + \lambda_2 d(Tu, T^n(Tu)) + \lambda_3 d(u, T^n u) \\
&\quad + \lambda_4 d(u, T^n(Tu)) + \lambda_5 d(Tu, T^n u) \\
&= \lambda_1 d(Tu, u) + \lambda_2 d(Tu, T(T^n u)) + \lambda_3 d(u, u) \\
&\quad + \lambda_4 d(u, T(T^n u)) + \lambda_5 d(Tu, u) \\
&= \lambda_1 d(Tu, u) + \lambda_2 d(Tu, Tu) + \lambda_3 d(u, u) + \lambda_4 d(u, Tu) \\
&\quad + \lambda_5 d(Tu, u)
\end{aligned}$$

$\therefore (1 - \lambda_1 - \lambda_4 - \lambda_5) d(Tu, u) \precsim 0$.
By Remark 2.4, taking absolute value on both side, we have

$$(1 - \lambda_1 - \lambda_4 - \lambda_5)|d(Tu, u)| \leq 0.$$

From $\Sigma_{i=1}^{5} \lambda_i < 1$, $(1 - \lambda_1 - \lambda_4 - \lambda_5) > 0$, then $|d(Tu, u)| = 0$. It follows that, $Tu = u$, hence u is a fixed point of T, and then $Tu = u = T^n u$.
Finally, we show that u is a unique fixed point of T. Let v be a fixed point of T, we must show that $u = v$. We see that,

$$\begin{aligned}
d(u, v) &= d(T^n u, T^n v) \\
&\precsim \lambda_1 d(u, v) + \lambda_2 d(u, T^n u) + \lambda_3 d(v, T^n u) + \lambda_4 d(v, T^n u) \\
&\quad + \lambda_5 d(u, T^n v) \\
&= \lambda_1 d(u, v) + \lambda_2 d(u, u) + \lambda_3 d(v, v) + \lambda_4 d(v, u) \\
&\quad + \lambda_5 d(u, v)
\end{aligned}$$

$\therefore (1 - \lambda_1 - \lambda_4 - \lambda_5) d(u, v) \precsim 0$.
By Remark 2.4, taking absolute value on both side, we have

$$(1 - \lambda_1 - \lambda_4 - \lambda_5)|d(u, v)| \leq 0.$$

Since $\Sigma_{i=1}^{5} \lambda_i < 1$, $(1 - \lambda_1 - \lambda_4 - \lambda_5) > 0$, then $|d(u, v)| = 0$. It follows that $u = v$. Therefore, u is a unique fixed point of T. This complete the proof. □

From Thorem 3.4, we can reduce to the following corollary, as follows:

Corollary 3.5. *Let (X, d) be a complete complex valued b-metric space with the constant $s \geq 1$. Let $T : X \to X$ be a mapping (for some fixed n) satisfying*

$$d(T^n x, T^n y) \precsim a d(x, T^n x) + b d(y, T^n y) + c d(x, y)$$

for all $x, y \in X$ where a, b, c are nonnegative real number with $s(a + b + c) < 1$. Then T has a uniqe fixed point in X.

Proof. We put $\lambda_4 = \lambda_5 = 0$, $a = \lambda_2$, $b = \lambda_3$ and $c = \lambda_1$ By theorem 3.1, T has a unique fixed point. This complete the proof. □

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