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HIGH CONVERGENCE ORDER SOLVERS IN BANACH SPACE

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ABSTRACT. The local convergence of an eighth order solver is established using only the first derivative for Banach space valued operators. Earlier studies have used up to the ninth order derivatives, which limit the applicability of the solver. The results are tested using numerical experiments.

KEYWORDS: Banach space, Newton-type, local convergence, Fréchet derivative.
AMS Subject Classification: 65F08, 37F50, 65N12.

1. Introduction

Let $\Omega \subset \mathcal{B}_1$ be nonempty, open, and $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces. One of the greatest challenges in Computational Mathematics is to find a solution x_* of the equation [1, 2, 3, 4, 6, 7, 11, 12, 13, 14, 15, 16, 17]

$$\mathcal{F}(x) = 0, \tag{1.1}$$

where $\mathcal{F} : \Omega \longrightarrow \mathcal{B}_2$ is Fréchet differentiable operator.

In this study, we are concerned with the local convergence of the Newton-type solver given as

$$\begin{aligned} & x_0 \in \Omega, \\ y_n &= x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \\ z_n &= x_n - [\frac{1}{4}I + \frac{1}{2}\mathcal{F}'(y_n)^{-1}\mathcal{F}(x_n) + \frac{1}{4}(\mathcal{F}'(y_n)^{-1}\mathcal{F}'(x_n))^2]\mathcal{F}'(x_n)^{-1}\mathcal{F}(y_n) \\ x_{n+1} &= z_n - [\frac{1}{2}I + \frac{1}{2}(\mathcal{F}'(y_n)^{-1}\mathcal{F}'(x_n))^2]\mathcal{F}'(x_n)^{-1}\mathcal{F}(z_n). \end{aligned} \tag{1.2}$$

Methods (1.2) was studied in [6], but for the case $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^k$ (k a natural number). Using conditions on ninth order derivative, and Taylor series(although

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these derivatives do not appear in solver (1.2)), the eighth convergence order was established. The hypotheses on higher order derivatives limit the usage of solver (1.2).

As an academic example: Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Define $\mathcal{F}$ on Ω by

$$\mathcal{F}(x) = x^3 \log x^2 + x^5 - x^4$$

Then, we have $x_* = 1$, and

$$\mathcal{F}'(x) = 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2,$$
$$\mathcal{F}''(x) = 6x \log x^2 + 20x^3 - 12x^2 + 10x,$$
$$\mathcal{F}'''(x) = 6 \log x^2 + 60x^2 = 24x + 22.$$

Obviously $\mathcal{F}'''(x)$ is not bounded on Ω. So, the convergence of solver (1.2) not guaranteed by the analysis in [6, 7, 8, 9, 11, 15].

Other problems with the usage of solver (1.2) are: no information on how to choose x_0; bounds on $\|x_n - x_*\|$ and information on the location of x_*. All these are addressed in this paper by only using conditions on the first derivative, and in the more general setting of Banach space valued operators. That is how, we expand the applicability of solver (1.2). To avoid the usage of Taylor series and high convergence order derivatives, we rely on the computational order of convergence (COC) or the approximate computational order of convergence (ACOC) [1, 6, 10].

The layout of the rest of the paper includes: the local convergence in Section 2, and the example in Section 3.

2. Ball convergence

We introduce some scalar functions and parameters for the convenience of our convergence analysis of solver (1.2). Let $w_0 : [0, \infty) \longrightarrow [0, \infty)$ be an increasing and continuous function with $w_0(0) = 0$. Suppose that equation

$$w_0(t) = 1 \tag{2.1}$$

has at least one positive solution. Denote by ρ_1 the smallest such solution. Let $w : [0, \rho_1) \longrightarrow [0, \infty)$ and $w_1 : [0, \rho_1) \longrightarrow [0, \infty)$ be increasing and continuous functions with $w(0) = 0$. Define functions ψ_1 and $\bar{\psi}_1$ on the interval $[0, \rho_1)$ by

$$\psi_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1 - w_0(t)}$$

and

$$\bar{\psi}_1(t) = \psi_1(t) - 1.$$

We have $\bar{\psi}_1(0) = -1$ and $\bar{\psi}_1(t) \longrightarrow \infty$ as $t \longrightarrow \rho_1^-$. The intermediate value theorem assures that equation $\bar{\psi}_1(t) = 0$ has at least one solution in $(0, \rho_1)$. Denote by R_1 the smallest such solution. Suppose that equation

$$w_0(\psi_1(t)t) = 1 \tag{2.2}$$

has at least one positive solution. Denote by ρ_2 the smallest such solution. Set $\rho_0 = \min\{\rho_1, \rho_2\}$. Define functions ψ_2 and $\bar{\psi}_2$ on $[0, \rho_0)$ by

$$\psi_2(t) = \left\{ \frac{\int_0^1 w((1-\theta)\psi_1(t)t)d\theta}{1 - w_0(\psi_1(t)t)} \right. + \frac{(w_0(\psi_1(t)t) + w_0(t)) \int_0^1 w_1(\theta\psi_1(t)t)d\theta}{(1 - w_0(\psi_1(t)t))(1 - w_0(t))}$$

$$+\frac{1}{4}\left[\frac{(w_0(\psi_1(t)t)+w_0(t))^2}{(1-w_0(\psi_1(t)t))^2}\right.$$
$$\left.+\ \frac{2(w_0(\psi_1(t)t)+w_0(t))}{1-w_0(\psi_1(t)t)}\right]$$
$$\left.\times\ \frac{\int_0^1 w_1(\theta\psi_1(t)t)d\theta}{1-w_0(t)}\right\}$$

and $\bar{\psi}_2(t)=\psi_2(t)-1$. We get $\bar{\psi}_2(0)=-1$ and $\bar{\psi}_2(t)\longrightarrow\infty$ as $t\longrightarrow\rho_0^-$. Denote by R_2 the smallest solution of equation $\bar{\psi}_2(t)=0$ in $(0,\rho_2)$. Suppose that

$$w_0(\psi_3(t)t)=1 \tag{2.3}$$

has at least one positive solution. Denote by ρ_3 the smallest such solution. Set $\rho=\min\{\rho_2,\rho_3\}$. Define functions ψ_3 and $\bar{\psi}_3$ on the interval $[0,\rho)$ by

$$\begin{aligned}\psi_3(t) \quad = \quad & \left\{\frac{\int_0^1 w((1-\theta)\psi_2(t)t)d\theta}{1-w_0(\psi_2(t)t)}\right.\\ & +\frac{(w_0(\psi_2(t)t)+w_0(t))\int_0^1 w_1(\theta\psi_2(t)t)d\theta}{(1-w_0(\psi_2(t)t))(1-w_0(t))}\\ & +\frac{1}{2}\left[\frac{(w_0(\psi_1(t)t)+w_0(t))^2}{(1-w_0(\psi_1(t)t))^2}\right.\\ & \left.+\ \frac{2(w_0(\psi_1(t)t)+w_0(t))w_1(t)}{(1-w_0(t))(1-w_0(\psi_1(t)t))}\right]\\ & \left.\times\ \frac{\int_0^1 w_1(\theta\psi_2(t)t)d\theta}{1-w_0(t)}\right\}\psi_2(t),\end{aligned}$$

and $\bar{\psi}_3(t)=\psi_3(t)-1$. We get $\bar{\psi}_3(0)=-1$ and $\bar{\psi}_3(t)\longrightarrow\infty$ as $t\longrightarrow\rho^-$. Denote by R_3 smallest solution of equation $\bar{\psi}_3(t)=0$ in $(0,\rho)$. Define a radius of convergence R by

$$R=\min\{R_m\}, m=1,2,3. \tag{2.4}$$

It follows that for each $t\in[0,R)$

$$0 \leq w_0(t)<1, \tag{2.5}$$
$$0 \leq w_0(\psi_1(t)t)<1, \tag{2.6}$$
$$0 \leq w_0(\psi_2(t)t)<1 \tag{2.7}$$

and

$$0\leq\psi_m(t)<1. \tag{2.8}$$

Let $\mathcal{B}(\mathcal{B}_1,\mathcal{B}_2)=\{G:\mathcal{B}_1\longrightarrow\mathcal{B}_2$ be bounded and linear$\}$, $T(x,d)=\{y\in\mathcal{B}_1: \|y-x\|<d; d>0\}$ and $\bar{T}(x,d)=\{y\in\mathcal{B}_1:\|y-x\|\leq d; d>0\}$. We shall use the conditions (C) in the local convergence analysis of solver (1.2) that follows:

- (c1) $\mathcal{F}:\Omega\longrightarrow\mathcal{B}_2$ a continuously differentiable operator in the sense of Fréchet, and there exists $p\in\Omega$ such that $\mathcal{F}(p)=0$, and $\mathcal{F}'(p)^{-1}\in\mathcal{B}(\mathcal{B}_2,\mathcal{B}_1)$.
- (c2) There exists function $w_0:[0,\infty)\longrightarrow[0,\infty)$ continuous, and increasing with $w_0(0)=0$ such that for each $x\in\Omega$

$$\|\mathcal{F}'(p)^{-1}(F'(x)-F'(p))\|\leq w_0(\|x-p\|).$$

Set $\Omega_0=\Omega\cap T(p,\rho_1)$ where ρ_1 is given in (2.1).

(c3) There exist functions $w : [0, \rho_0) \longrightarrow [0, \infty), w_1 : [0, \rho_0) \longrightarrow [0, \infty)$ continuous, and increasing such that for each $x, y \in \Omega_0$

$$\|\mathcal{F}'(p)^{-1}(\mathcal{F}'(y)] - \mathcal{F}'(x))\| \leq w(\|y - x\|)$$

and

$$\|\mathcal{F}'(p)^{-1}\mathcal{F}'(x))\| \leq w_1(\|x - p\|.$$

(c4) $\bar{T}(p, R) \subseteq \Omega$ where R is defined by (2.4) and ρ_1, ρ_2, ρ_3 are given in (2.1)–(2.3), respectively.

(c5) There exists $R_1 \geq R$ such that $\int_0^1 w_0(\theta R_1)d\theta < 1$.
Set $\Omega_1 = \Omega \cap \bar{T}(x_*, R_1)$.

Next, the convergence of solver (1.2) follows using preceding notation and the conditions (C).

Theorem 2.1. *Suppose that the conditions (C) hold. Then, the sequence $\{x_n\}$ starting at $x_0 \in T(p, R) - \{p\}$, and generated by solver (1.2) is well defined, remains in $T(p, R)$ for each $n = 0, 1, 2, \ldots,$ and converges to p. Moreover the following error bounds hold*

$$\|y_n - p\| \leq \psi_1(\|x_n - p\|)\|x_n - p\| \leq \|x_n - p\| < r, \tag{2.9}$$

$$\|z_n - p\| \leq \psi_2(\|x_n - p\|)\|x_n - p\| \leq \|x_n - p\|, \tag{2.10}$$

$$\|x_{n+1} - p\| \leq \psi_3(\|x_n - p\|)\|x_n - p\| \leq \|x_n - p\|, \tag{2.11}$$

where functions ψ_i are given previously and R is defined in (2.4). Furthermore, the limit point p is the only solution of equation $\mathcal{F}(x) = 0$ in the set Ω_1.

Proof. We shall use a mathematical induction based proof. Let $x \in T(p, R) - \{p\}$. By (2.4), (c1) and (c2), we get that

$$\|\mathcal{F}'(p)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(p))\| \leq w_0(\|x - p\|) < w_0(R) < 1, \tag{2.12}$$

so by the Banach lemma on invertible operators [15, 16], we have that $\mathcal{F}'(x)^{-1} \in \mathcal{B}(\mathcal{B}_2, \mathcal{B}_1)$, and

$$\|\mathcal{F}'(x)^{-1}\mathcal{F}'(p)\| \leq \frac{1}{1 - w_0(\|x - p\|)}. \tag{2.13}$$

This also shows that y_0 is well defined. Using (2.4), (2.8) (for $m = 1$), (c3), (2.13) and (1.2), we obtain in turn that

$$\begin{aligned} \|y_0 - p\| &\leq \|x_0 - p - \mathcal{F}'(x_0)^{-1}\mathcal{F}'(x_0)\| \\ &\leq \|\mathcal{F}'(x_0)^{-1}(\mathcal{F}(p)\| \\ &\quad \times \|\int_0^1 \mathcal{F}'(p)(\mathcal{F}'(p + \theta(x_0 - p)) - \mathcal{F}'(x_0))(x_0 - p)d\theta\| \\ &\leq \frac{\int_0^1 w((1 - \theta)\|x_0 - p\|)d\theta\|x_0 - p\|}{1 - w_0(\|x_0 - p\|)} \\ &= \psi_1(\|x_0 - p\|\|x_0 - p\| \leq \|x_0 - p\| < R, \end{aligned} \tag{2.14}$$

so (2.9) holds for $n = 0$ and $y_0 \in T(p, R)$, so z_0 and x_1 are well defined. By the second substep of solver (1.2) for $n = 0$, we can write by (c1) that

$$\mathcal{F}(x) = \mathcal{F}(x) - \mathcal{F}(p) = \int_0^1 \mathcal{F}'(p + (\theta(x - p))d\theta(x - p). \tag{2.15}$$

Then, by the second condition (c3), and (1.2) we get respectively, that

$$\|\mathcal{F}'(p)^{-1}\mathcal{F}(x)\| \leq \int_0^1 w_1(\theta\|x - p\|)d\theta\|x - p\|, \tag{2.16}$$

and

$$\begin{aligned} z_0 - p &= (y_0 - p - \mathcal{F}'(y_0)^{-1}\mathcal{F}(y_0)) + \mathcal{F}'(y_0)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(y_0))\mathcal{F}'(x_0)^{-1}\mathcal{F}(y_0) \\ &\quad + \frac{1}{4}[(\mathcal{F}'(y_0)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(y_0)))^2 \\ &\quad - 2(\mathcal{F}'(y_0)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(y_0))]\mathcal{F}'(x_0)^{-1}\mathcal{F}(y_0). \end{aligned} \tag{2.17}$$

Using (2.4), (2.8) (for $m = 2$), (2.13) (for $x = y_0$), (2.14) and (2.16) (for $x = y_0$) and (2.17), we obtain in turn that

$$\begin{aligned} \|z_0 - p\| &\leq \|y_0 - p - \mathcal{F}'(y_0)^{-1}\mathcal{F}(y_0)\| \\ &\quad + \|\mathcal{F}'(y_0)^{-1}\mathcal{F}'(p)\|[\|\mathcal{F}'(p)^{-1}(\mathcal{F}'(y_0) - \mathcal{F}'(p))\| \\ &\quad + \|\mathcal{F}'(p)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(p))\| \\ &\quad \times \|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(p)\|\|\mathcal{F}'(p)^{-1}\mathcal{F}(y_0)\| \\ &\quad + \frac{1}{4}[\|\mathcal{F}'(y_0)^{-1}\mathcal{F}'(p)\|^2(\|\mathcal{F}'(p)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(p))\| \\ &\quad + (\|\mathcal{F}'(p)^{-1}(\mathcal{F}'(y_0) - \mathcal{F}'(p))\|)^2 \\ &\quad + 2\|\mathcal{F}'(y_0)^{-1}\mathcal{F}'(p)\|(\|\mathcal{F}'(p)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(p))\| \\ &\quad + \|\mathcal{F}'(p)^{-1}(\mathcal{F}'(y_0) - \mathcal{F}'(p))\|)] \\ &\quad \times \|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(p)\|\|\mathcal{F}'(p)^{-1}\mathcal{F}(y_0)\| \\ &\leq \left\{ \frac{\int_0^1 w((1-\theta)\|y_0 - p\|)d\theta)}{1 - w_0(\|y_0 - p\|)} \right. \\ &\quad \left. + \frac{\int_0^1 w_1(\theta\|y_0 - p\|)d\theta(w_0(\|y_0 - p\|) + w_0(\|x_0 - p\|))}{(1 - w_0(\|x_0 - p\|))(1 - w_0(\|y_0 - p\|)} \right] \\ &\quad + \frac{1}{4}\left[\frac{(w_0(\|x_0 - p\|) + w_0(\|y_0 - p\|))^2}{(1 - w_0(\|y_0 - p\|))^2} \right. \\ &\quad \left. + \frac{2(w_0(\|x_0 - p\|) + w_0(\|y_0 - p\|))}{1 - w_0(\|y_0 - p\|)} \right] \frac{\int_0^1 w_1(\theta\|y_0 - p\|)d\theta}{1 - w_0(\|x_0 - p\|)} \Bigg\} \|y_0 - p\| \\ &\leq \psi_2(\|x_0 - p\|)\|x_0 - p\| \leq \|x_0 - p\| < R, \end{aligned} \tag{2.18}$$

which shows (2.11) for $n = 0$, and $z_0 \in T(p, R)$. Moreover, by the third substep of solver (1.2) for $n = 0$, we have that

$$\begin{aligned} x_1 - p &= (z_0 - p - \mathcal{F}'(z_0)^{-1}\mathcal{F}(z_0)) + (\mathcal{F}'(z_0)^{-1} - \mathcal{F}'(x_0)^{-1})\mathcal{F}(z_0) \\ &\quad + \frac{1}{2}[I - (\mathcal{F}'(y_0)^{-1}\mathcal{F}'(x_0))^2]\mathcal{F}'(x_0)^{-1}\mathcal{F}(z_0) \\ &= (z_0 - p - \mathcal{F}'(z_0)^{-1}\mathcal{F}(z_0)) \\ &\quad + \mathcal{F}'(z_0)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(p))\mathcal{F}'(x_0)^{-1}\mathcal{F}(z_0) \\ &\quad + \frac{1}{2}[(I - \mathcal{F}'(y_0)^{-1}\mathcal{F}'(x_0))^2 + 2(I - \mathcal{F}'(y_0)^{-1}\mathcal{F}'(x_0))\mathcal{F}'(y_0)^{-1}\mathcal{F}'(x_0)] \\ &\quad \times \mathcal{F}'(x_0)^{-1}\mathcal{F}(z_0). \end{aligned} \tag{2.19}$$

Using (2.4), (2.8) (for $m = 3$), (2.13) (for $x = y_0, z_0$), (2.16) (for $x = z_0$), (2.18) and (2.20), we get in turn that

$$\begin{aligned} \|x_1 - p\| &\leq \|z_0 - p - \mathcal{F}'(z_0)^{-1}\mathcal{F}(z_0)\| \\ &\quad + \|\mathcal{F}'(z_0)^{-1}\mathcal{F}'(p)\|[\|\mathcal{F}'(p)^{-1}(\mathcal{F}'(z_0) - \mathcal{F}'(p))\| \\ &\quad + \|\mathcal{F}'(p)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(p))\|] \end{aligned}$$

$$
\begin{aligned}
& \times\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(p)\|\|\mathcal{F}'(p)^{-1}\mathcal{F}'(z_0)\| \\
& +\frac{1}{2}[\|(\mathcal{F}'(y_0)^{-1}(\mathcal{F}'(y_0)-\mathcal{F}'(x_0))\|^2 \\
& +2\|\mathcal{F}'(y_0)^{-1}(\mathcal{F}'(y_0)-\mathcal{F}'(x_0))\|\|\mathcal{F}'(y_0)^{-1}\mathcal{F}'(p)\| \\
& \times\|\mathcal{F}'(p)^{-1}\mathcal{F}'(x_0)\|] \\
& \times\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(p)\|\|\mathcal{F}'(p)^{-1}\mathcal{F}(z_0)\| \\
\leq\ & \Bigg\{\frac{\int_0^1 w((1-\theta)\|z_0-p\|)d\theta}{1-w_0(\|z_0-p\|)} \\
& +\frac{(w_0(\|z_0-p\|)+w_0(\|x_0-p\|))\int_0^1 w_1(\theta\|z_0-p\|)d\theta}{(1-w_0(\|z_0-p\|)(1-w_0(\|x_0-p\|))} \\
& \frac{1}{2}\Bigg[\frac{(w_0(\|x_0-p\|)+w_0(\|y_0-p\|))^2}{(1-w_0(\|y_0-p\|))^2} \\
& \frac{2(w_0(\|x_0-p\|)+w_0(\|y_0-p\|))}{1-w_0(\|x_0-p\|)} \\
& \times\frac{w_1(\|z_0-p\|)}{1-w_0(\|y_0-p\|)}\Bigg] \\
& \frac{\int_0^1 w_1(\theta\|z_0-p\|)d\theta}{1-w_0(\|x_0-p\|)}\Bigg\}\|z_0-p\| \\
\leq\ & \psi_3(\|x_0-p\|)\|x_0-p\| \leq \|x_0-p\| < R, \qquad (2.20)
\end{aligned}
$$

so (2.11) holds for $n=0$ and $x_1 \in T(p,R)$. The induction for (2.11) is completed, if x_m, y_m, z_m, x_{m+1} replace x_0, y_0, z_0, x_1 in the preceding estimates. Then,in view of the estimate

$$\|x_{m+1}-p\| \leq \lambda\|x_m-p\| \leq \|x_m-p\| < R, \qquad (2.21)$$

where $\lambda=\psi_3(\|x_0-p\|)\in[0,1)$, we deduce that $x_{m+1}\in T(p,R)$, and $\lim_{m\longrightarrow\infty}x_m = p$. Further for the uniqueness part, let $p_* \in \Omega_1$ with $\mathcal{F}(p_*)=0$. Define $G=\int_0^1 \mathcal{F}'(p+\theta(p_*-p))d\theta$. Then, using (c5), we get

$$\|\mathcal{F}'(p)^{-1}(G-\mathcal{F}'(p))\| \leq \int_0^1 w_0(\theta\|p_*-p\|)d\theta \leq \int_0^1 w_0(\theta R)d\theta < 1,$$

so G^{-1} exists, and from

$$0=\mathcal{F}(p)-\mathcal{F}(p_*)=G(p-p_*),$$

we derive $p=p_*$.

□

Remark 2.1. (a) In the case when $w_0(t)=L_0t, w(t)=Lt$ and $\Omega_0=\Omega$, the radius $\rho_A=\frac{2}{2L_0+L}$ was obtained by Argyros in [4] as the convergence radius for Newton's solver under condition (2.7)-(2.9). Notice that the convergence radius for Newton's solver given independently by Rheinboldt [16] and Traub [17] is given by

$$\rho_{TR}=\frac{2}{3L}<\rho_A.$$

As an example, let us consider the function $F(x)=e^x-1$. Then $\alpha^*=0$. Set $\Omega=B(0,1)$. Then, we have that $L_0=e-1<L=e$, so $\rho_{TR}=0.24252961<\rho_A=0.324947231$.

(b) The local results can be used for projection solvers such as Arnoldi's solver, the generalized minimum residual solver(GMREM), the generalized conjugate solver(GCM) for combined Newton/finite projection solvers and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [2, 3, 4].

(c) The results can be also be used to solve equations where the operator F' satisfies the autonomous differential equation [2, 3, 10, 13]:

$$F'(x) = P(F(x)),$$

where $P : \mathcal{B}_2 \longrightarrow \mathcal{B}_2$ is a known continuous operator. Since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$ and $x^* = 0$.

(d) It is worth noticing that solvers (1.2) is not changing when we use the conditions of the preceding Theorem instead of the stronger conditions used in [15]. Moreover, we can compute the computational order of convergence (COC) defined as

$$\xi = \ln\left(\frac{\|x_{n+1} - x_*\|}{\|x_n - x_*\|}\right) / \ln\left(\frac{\|x_n - x_*\|}{\|x_{n-1} - x_*\|}\right)$$

or the approximate computational order of convergence (ACOC) [5, 6]

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence, but not higher order derivatives are used.

3. Numerical example

We present the following example to test the convergence criteria.

Example 3.1. Let $\mathcal{B}_1 = \mathcal{B}_1 = \mathbb{R}^3$, $\Omega = U(0,1), x_* = (0,0,0)^T$ and define $\mathcal{F}$ on Ω by

$$\mathcal{F}(x) = \mathcal{F}(u_1, u_2, u_3) = (e^{u_1} - 1, \frac{e-1}{2}{u_2}^2 + u_2, u_3)^T. \tag{3.1}$$

For the points $u = (u_1, u_2, u_3)^T$, the Fréchet derivative is given by

$$\mathcal{F}'(u) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the norm of the maximum of the rows $x_* = (0,0,0)^T$ and since $\mathcal{F}'(x_*) = diag(1,1,1)$, we get by conditions (H) $w_0(t) = (e-1)t$, $w(t) = e^{\frac{1}{e-1}}t, w_1(t) = e^{\frac{1}{e-1}}$.

$$R_1 = 0.382692,\ R_2 = 0.227598, R_3 = 169362.$$

Example 3.2. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1], \Omega = \bar{U}(0,1)$. Define function F on Ω by

$$F(w)(x) = w(x) - 5\int_0^1 x\theta w(\theta)^3 d\theta.$$

Then, the Fréchet-derivative is given by

$$F'(w(\xi))(x) = \xi(x) - 15\int_0^1 x\theta w(\theta)^2 \xi(\theta) d\theta,\ \text{ for each } \xi \in \Omega.$$

Then, we have that $x^* = 0, w_0(t) = L_0 t, w(t) = Lt, w_1(t) = 2, L_0 = 7.5 < L = 15$. Then, the radius of convergence are given by

$$R_1 = 0.0667,\ R_2 = 0.0395822 = \rho,\ R_3 = 0.0297337.$$

Example 3.3. Returning back to the motivational example given at the introduction of this study, we can choose $w_0(t) = w(t) = 96.662907t$, $w_1(t) = 1.0631$. Then, the radius of convergence are given by

$$R_1 = 0.00689682,\ R_2 = 0.00457799,\ R_3 = 1 = 0.00378481.$$

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