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CHARACTERIZATIONS OF THE BASIC CONSTRAINT QUALIFICATION AND ITS APPLICATIONS

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ABSTRACT. In convex programming, the basic constraint qualification is a necessary and sufficient constraint qualification for the optimality condition. In this paper, we give characterizations of the basic constraint qualification at each feasible solution. By using the result, we give an alternative method for checking up the basic constraint qualification at every feasible point without subdifferentials and normal cones.

KEYWORDS:convex programming, constraint qualification, alternative theorem
AMS Subject Classification: :90C25, 90C30, 90C46

1. Introduction

In this paper, we consider the following convex programming problem:

$$(\mathrm{P}) \begin{cases} \text{minimize} & f(x), \\ \text{subject to} & g_i(x) \leq 0 \text{ for each } i \in I, \end{cases}$$

where X is a locally convex Hausdorff topological vector space, I is an arbitrary index set, f is an extended real-valued convex function on X, and g_i is an extended real-valued convex function on X for each $i \in I$. Constraint qualifications are essential in mathematical programming, see [1, 2, 3, 4, 5, 8, 9, 10, 11, 12, 13, 14, 15] and references therein. In particular, they ensure the existence of Lagrange multipliers or zero duality gap between (P) and its Lagrangian dual problem. These results have played a critical role in the development of convex programming. Additionally, constraint qualifications for the following optimality condition have been studied by many researchers:

$$\exists \lambda \in \mathbb{R}_+^{(I)} \text{ such that } 0 \in \partial f(x_0) + \sum_{i \in I} \lambda_i \partial g_i(x_0),$$

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where $\mathbb{R}_+^{(I)} = \{\lambda \in \mathbb{R}^I \mid \forall i \in I, \lambda_i \geq 0, \{i \in I \mid \lambda_i \neq 0\} : \text{finite}\}$. One of the best-known constraint qualification for the optimality condition is the Slater constraint qualification. It is easy to check whether the Slater constraint qualification holds or not. However, the Slater constraint qualification is often not satisfied for many problems arising in applications. The lack of a constraint qualification can cause both theoretical and numerical difficulties in applications. Recently, it was shown that the basic constraint qualification (BCQ), which was introduced in [3], is a necessary and sufficient constraint qualification for the optimality condition by Li, Ng and Pong, see [8]. To check the BCQ at a feasible point, however, we have to calculate the subdifferential of all g_i and the normal cone of the feasible set at the point. In this point of view, checking up the BCQ at every feasible points is not so easy.

The purpose of this paper is to give characterizations of the basic constraint qualification at each feasible point, and to give an alternative method to checking up the BCQ at every feasible points. The paper is organized as follows. In section 2, we describe our notation and present preliminary results. In section 3, we give characterizations of the basic constraint qualification at each feasible point, and we give an alternative method for checking up the BCQ at every feasible points. Also we remark that alternative results which are generalizations of Farkas' Lemma are given. In section 4, we explain the usefulness of our result obtained in this paper.

2. Preliminaries

In this section, we describe our notation and present preliminary results. Let X be a locally convex Hausdorff topological vector space over the real-field $\mathbb{R}$, let X^* be the continuous dual space of X, and let $\langle v, x\rangle$ denote the value of a functional $v \in X^*$ at $x \in X$. For a subset A^* of X^*, we denote the weak*-closure, the conical hull and the convex hull of A by $\mathrm{cl}A^*$, $\mathrm{cone}A^*$ and $\mathrm{co}A^*$, respectively. Let f be a function from X to $\mathbb{R} \cup \{+\infty\}$. The effective domain of f, denoted by $\mathrm{dom}f$ is defined by

$$\mathrm{dom}f = \{x \in X \mid f(x) < +\infty\},$$

and the epigraph of f, denoted by $\mathrm{epi}f$ is defined by

$$\mathrm{epi}f = \{(x, r) \in X \times \mathbb{R} \mid x \in \mathrm{dom}f, f(x) \leq r\}.$$

The function f is said to be convex, proper and lower semicontinuous (lsc) if $\mathrm{epi}f$ is a convex set, nonempty set and closed set, respectively. When f is a proper lsc convex function, the conjugate function of f, $f^* : X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(v) = \sup\{\langle v, x\rangle - f(x) \mid x \in X\}.$$

The subdifferential of f at $x \in X$, denoted by $\partial f(x)$, is defined by

$$\partial f(x) = \{v \in X^* \mid f(x) + \langle v, y - x\rangle \leq f(y), \forall y \in X\}.$$

For nonempty convex set $A \subseteq X$, the indicator function $\delta_A : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_A(x) = \begin{cases} 0 & x \in A, \\ +\infty & x \notin A. \end{cases}$$

For any $x \in A$, the normal cone of A at x, denoted by $N_A(x)$, is defined by

$$N_A(x) = \{v \in X^* \mid \langle v, y - x\rangle \leq 0, \forall y \in A\}.$$

For proper lsc convex functions g, $h : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, the infimal convolution of g with h, denoted by $g \oplus h$, is defined by

$$(g \oplus h)(x) = \inf_{x_1 + x_2 = x} \{g(x_1) + h(x_2)\}.$$

It is well known that if $\mathrm{dom} g \cap \mathrm{dom} h \neq \emptyset$, then

$$(g \oplus h)^* = g^* + h^* \text{ and } (g+h)^* = \mathrm{cl}(g^* \oplus h^*). \tag{2.1}$$

The closure operation in the second equation is superfluous if one of g and h is continuous at some $a \in \mathrm{dom} g \cap \mathrm{dom} h$. Then,

$$\mathrm{epi}(g+h)^* = \mathrm{epi} g^* + \mathrm{epi} h^* \text{ and} \tag{2.2}$$

$$\partial(g+h)(x) = \partial g(x) + \partial h(x), \text{ for each } x \in \mathrm{dom} g \cap \mathrm{dom} h, \tag{2.3}$$

see [2].

We denote by $\mathbb{R}_+^{(I)}$ the space of generalized finite sequences $(\lambda_i)_{i \in I}$ such that $\lambda_i \in \mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geq 0\}$ for each $i \in I$, and with only finitely many λ_i different from zero. Let g_i be an extended real-valued proper lsc convex function on X for each $i \in I$, and let $\lambda \in \mathbb{R}_+^{(I)}$. Assume that each g_i is continuous at least at one point of $\bigcap_{i \in I} \mathrm{dom} g_i$, and $0 \times \infty = 0$. Then

$$\mathrm{epi} \left(\sum_{i \in I} \lambda_i g_i \right)^* = \begin{cases} \sum_{i \in I} \lambda_i \mathrm{epi} g_i^* & \sum_{i \in I} \lambda_i > 0, \\ \{0\} \times \mathbb{R}_+ & \sum_{i \in I} \lambda_i = 0, \end{cases} \tag{2.4}$$

$$\partial \sum_{i \in I} \lambda_i g_i(x) = \sum_{i \in I} \lambda_i \partial g_i(x), \forall x \in \bigcap_{i \in I} \mathrm{dom} g_i. \tag{2.5}$$

Definition 2.1. Let I be an arbitrary index set, g_i an extended real-valued proper lsc convex function on X for each $i \in I$, $S = \{x \in X \mid g_i(x) \leq 0, \forall i \in I\}$, and $\bar{x} \in S$. The family $\{g_i \mid i \in I\}$ is said to satisfy the basic constraint qualification (BCQ) at $\bar{x}$ if

$$N_S(\bar{x}) = \mathrm{coneco} \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}),$$

where $I(\bar{x}) = \{i \in I \mid g_i(\bar{x}) = 0\}$.

We introduce the following previous result of the BCQ.

Theorem 2.1. [8] *Let I be an arbitrary index set, g_i an extended real-valued proper lsc convex function on X for each $i \in I$, $S = \{x \in X \mid g_i(x) \leq 0, \forall i \in I\}$, and $\bar{x} \in S$. Then the following statements are equivalent:*

(i) *the family $\{g_i \mid i \in I\}$ satisfies the BCQ at $\bar{x}$,*
(ii) *for each real-valued convex function f, $\bar{x}$ is a minimizer of f in S if and only if there exist a finite subset $J \subseteq I(\bar{x})$ and $(\lambda_i)_{i \in J} \in \mathbb{R}_+^J$, such that*

$$0 \in \partial f(\bar{x}) + \sum_{i \in J} \lambda_i \partial g_i(\bar{x}).$$

By Theorem 2.1, the BCQ is said to be a necessary and sufficient constraint qualification for the optimality condition.

The following results, a set containment characterization and Fenchel duality, are used in our main theorem.

Theorem 2.2. [6] *Let I be an arbitrary index set, g_i an extended real-valued proper lsc convex function on X for each $i \in I$, $v \in X^*$, and $\alpha \in \mathbb{R}$. Assume that $S = \{x \in X \mid g_i(x) \leq 0, \forall i \in I\}$ is nonempty.*

Then the following statements are equivalent:

(i) $\{x \in X \mid g_i(x) \leq 0, \forall i \in I\} \subseteq \{x \in X \mid \langle v, x\rangle \leq \alpha\}$,

(ii) $(v, \alpha) \in \mathrm{clconeco} \bigcup_{i \in I} \mathrm{epi} g_i^*$.

Theorem 2.3. [7] *Let f and g be extended real-valued proper lsc convex functions on X such that* $\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$. *If* $\mathrm{epi} f^* + \mathrm{epi} g^*$ *is w^*-closed, then*

$$\inf_{x \in X} \{f(x) + g(x)\} = \max_{v \in X^*} \{-f^*(-v) - g^*(v)\}.$$

3. Main Result

Throughout this section, we consider the following convex inequality system:

$$\{g_i(x) \leq 0 \mid i \in I\}$$

where I is an arbitrary index set, and g_i an extended real-valued proper lsc convex function on X for each $i \in I$. Let $S = \{x \in X \mid g_i(x) \leq 0, \forall i \in I\}$, and assume that each g_i is continuous at least at one point of $\bigcap_{i \in I} \mathrm{dom} g_i$. We show the following theorem as our main result.

Theorem 3.1. *Let $\bar{x} \in S = \{x \in X \mid g_i(x) \leq 0 \ \forall i \in I\}$. Then the following statements are equivalent:*

(i) *the family $\{g_i \mid i \in I\}$ satisfies the BCQ at $\bar{x}$,*

(ii) *for each real-valued convex function f, $\bar{x}$ is a minimizer of f in S if and only if there exist a finite subset $J \subseteq I(\bar{x})$ and $(\lambda_i)_{i \in J} \in \mathbb{R}^J_+$, such that*

$$0 \in \partial f(\bar{x}) + \sum_{i \in J} \lambda_i \partial g_i(\bar{x}),$$

(iii) *the following inclusion holds:*

$$\left\{ v \,\middle|\, (v, \langle v, \bar{x}\rangle) \in \mathrm{clconeco} \bigcup_{i \in I} \mathrm{epi} g_i^* \right\} \subseteq \left\{ v \,\middle|\, (v, \langle v, \bar{x}\rangle) \in \mathrm{coneco} \bigcup_{i \in I} \mathrm{epi} g_i^* \right\},$$

(iv) *for each extended real-valued proper lsc convex function f with* $\mathrm{epi} f^* + \mathrm{epi} \delta_S^*$ *is w^*-closed, exactly one of the following two statements is true:*

(a) *there exists $x \in X$ such that* $\begin{cases} f(x) < f(\bar{x}), \\ g_i(x) \leq 0, \text{ for each } i \in I, \end{cases}$

(b) *there exists $\lambda \in \mathbb{R}^{(I)}_+$ such that*

$$\begin{cases} f(x) + \sum_{i \in I} \lambda_i g_i(x) \geq f(\bar{x}) \text{ for each } x \in X, \\ \lambda_i g_i(\bar{x}) = 0 \text{ for each } i \in I, \end{cases}$$

(v) *for each $v \in X^*$, exactly one of the following two statements is true:*

(a) *there exists $x \in X$ such that* $\begin{cases} \langle v, x\rangle < \langle v, \bar{x}\rangle, \\ g_i(x) \leq 0 \text{ for each } i \in I, \end{cases}$

(b) *there exists $\lambda \in \mathbb{R}^{(I)}_+$ such that*

$$\begin{cases} \langle v, x\rangle + \sum_{i \in I} \lambda_i g_i(x) \geq \langle v, \bar{x}\rangle, \text{ for each } x \in X, \\ \lambda_i g_i(\bar{x}) = 0 \text{ for each } i \in I. \end{cases}$$

Proof. By Theorem 2.1, (i) and (ii) are equivalent.

We show that (ii) implies (iii). Assume that the statement (ii) holds, and let $v \in X^*$ satisfying $(v, \langle v, \bar{x}\rangle) \in \mathrm{clconeco} \bigcup_{i \in I} \mathrm{epi} g_i^*$. Then, by Theorem 2.2,

$$\{x \in X \mid g_i(x) \le 0, \forall i \in I\} \subseteq \{x \in X \mid \langle v, x\rangle \le \langle v, \bar{x}\rangle\}.$$

This shows that $\bar{x}$ is a global minimizer of $-v$ in S. By the statement (ii), there exist a finite subset $J \subseteq I(\bar{x})$ and $(\lambda_i)_{i \in J} \in \mathbb{R}_+^J$, such that

$$0 \in \partial(-v)(\bar{x}) + \sum_{i \in J} \lambda_i \partial g_i(\bar{x}),$$

that is, $v \in \sum_{i \in J} \lambda_i \partial g_i(\bar{x})$. For each $i \in J \subseteq I(\bar{x})$, we show that $w \in \partial g_i(\bar{x})$ if and only if $(w, \langle w, \bar{x}\rangle) \in \mathrm{epi} g_i^*$. Actually,

$$\begin{aligned} w \in \partial g_i(\bar{x}) &\iff \forall y \in X, g_i(y) \ge g_i(\bar{x}) + \langle w, y - \bar{x}\rangle \\ &\iff \forall y \in X, g_i(y) \ge \langle w, y - \bar{x}\rangle \\ &\iff \forall y \in X, \langle w, \bar{x}\rangle \ge \langle w, y\rangle - g_i(y) \\ &\iff \langle w, \bar{x}\rangle \ge g_i^*(w) \\ &\iff (w, \langle w, \bar{x}\rangle) \in \mathrm{epi} g_i^*. \end{aligned}$$

Hence,

$$(v, \langle v, \bar{x}\rangle) \in \sum_{i \in J} \lambda_i \mathrm{epi} g_i^* \subseteq \mathrm{coneco} \bigcup_{i \in I} \mathrm{epi} g_i^*.$$

Next, we prove that (iii) implies (iv). Assume that (iii) holds, and let f be an extended real-valued proper lsc convex function with $\mathrm{epi} f^* + \mathrm{epi} \delta_S^*$ is w^*-closed. It is clear that (a) and (b) in (iv) do not hold simultaneously. If (a) does not hold, then for each $x \in S$, $f(x) \ge f(\bar{x})$, that is, $\bar{x}$ is a global minimizer of f in S. By Theorem 2.3,

$$f(\bar{x}) = \min_{x \in S} f(x) = \min_{x \in X} \{f(x) + \delta_S(x)\} = \max_{v \in X^*} \{-f^*(-v) - \delta_S^*(v)\},$$

that is, there exists $v_0 \in X^*$ such that $f(\bar{x}) = -f^*(-v_0) - \delta_S^*(v_0)$. Hence,

$$\begin{aligned} f(\bar{x}) &= -f^*(-v_0) - \delta_S^*(v_0) \\ &= -f^*(-v_0) - \sup_{x \in X} \{\langle v_0, x\rangle - \delta_S(x)\} \\ &= -f^*(-v_0) + \inf_{x \in S} \langle -v_0, x\rangle \\ &\le -f^*(-v_0) + \langle -v_0, \bar{x}\rangle \\ &\le -(\langle -v_0, \bar{x}\rangle - f(\bar{x})) + \langle -v_0, \bar{x}\rangle \\ &= f(\bar{x}). \end{aligned}$$

This shows that $f(\bar{x}) + f^*(-v_0) = \langle -v_0, \bar{x}\rangle$, that is, $-v_0 \in \partial f(\bar{x})$. Additionally, we can see that $\inf_{x \in S} \langle -v_0, x\rangle = \langle -v_0, \bar{x}\rangle$, hence we have

$$\{x \in X \mid g_i(x) \le 0, \forall i \in I\} \subseteq \{x \in X \mid \langle v_0, x\rangle \le \langle v_0, \bar{x}\rangle\}.$$

By Theorem 2.2 and the statement (iii),

$$(v_0, \langle v_0, \bar{x}\rangle) \in \mathrm{coneco} \bigcup_{i \in I} \mathrm{epi} g_i^*.$$

Hence, there exist $\lambda \in \mathbb{R}_+^{(I)}$ and $(a_i, b_i) \in \mathrm{epi} g_i^*$ for each $i \in I$ such that

$$(v_0, \langle v_0, \bar{x}\rangle) = \sum_{i \in I} \lambda_i (a_i, b_i).$$

For each $i \in I$ and $x \in X$, $\langle a_i, x\rangle - g_i(x) \leq b_i$. Therefore,

$$\langle v_0, x\rangle - \sum_{i\in I} \lambda_i g_i(x) \leq \langle v_0, \bar{x}\rangle .$$

Since $-v_0 \in \partial f(\bar{x})$, for each $x \in X$,

$$f(\bar{x}) + \langle -v_0, x - \bar{x}\rangle \leq f(x).$$

This shows that

$$f(x) + \sum_{i\in I} \lambda_i g_i(x) \geq f(\bar{x}).$$

Since $\bar{x} \in S$, we can check easily that $\lambda_i g_i(\bar{x}) = 0$ for each $i \in I$, hence (b) of (iv) holds.

It is clear that (iv) implies (v).

Finally, we prove that (v) implies (i). Assume that (v) holds. At first, we show the following inclusion:

$$N_S(\bar{x}) \supseteq \text{coneco} \bigcup_{i\in I(\bar{x})} \partial g_i(\bar{x})$$

Actually, let $i \in I(\bar{x})$ and $v \in \partial g_i(\bar{x})$, then for each $x \in S$,

$$\langle v, x - \bar{x}\rangle = g_i(\bar{x}) + \langle v, x - \bar{x}\rangle \leq g_i(x) \leq 0.$$

This shows that $v \in N_S(\bar{x})$, that is, $\partial g_i(\bar{x}) \subseteq N_S(\bar{x})$. Since $N_S(\bar{x})$ is a convex cone, the above inclusion holds. Next, we show the following inclusion:

$$N_S(\bar{x}) \subseteq \text{coneco} \bigcup_{i\in I(\bar{x})} \partial g_i(\bar{x}).$$

Let $v \in N_S(\bar{x})$, then, $\bar{x}$ is a global minimizer of $-v$ in S. Hence, the statement (a) in (v) for $-v$ does not hold. By the statement (b) in (v), there exists $\lambda \in \mathbb{R}^{(I)}_+$ such that for each $x \in X$,

$$\langle -v, x\rangle + \sum_{i\in I} \lambda_i g_i(x) \geq \langle -v, \bar{x}\rangle ,$$

and $\lambda_i g_i(\bar{x}) = 0$ for each $i \in I$. This shows that $(\sum_{i\in I} \lambda_i g_i)^*(v) \leq \langle v, \bar{x}\rangle$. Since $\sum_{i\in I} \lambda_i g_i(\bar{x}) + (\sum_{i\in I} \lambda_i g_i)^*(v) \leq \langle v, \bar{x}\rangle$, we can see that $v \in \partial(\sum_{i\in I} \lambda_i g_i)(\bar{x})$. By the equation (2.5) and the complementarity condition,

$$v \in \partial \left(\sum_{i\in I} \lambda_i g_i\right)(\bar{x}) = \sum_{i\in I} \lambda_i \partial g_i(\bar{x}) \subseteq \text{coneco} \bigcup_{i\in I(\bar{x})} \partial g_i(\bar{x}).$$

This shows that (i) holds. This completes the proof. □

Remark 3.1. By (iii) in Theorem 3.1, an alternative method for checking up the BCQ at every feasible points is given. The method requires a convex cone depents on constraint functions and feasible points, but does not require any subdifferentials and any normal cones, see examples in Section 4.

Remark 3.2. By using Theorem 3.1, we can show Farkas' Lemma. Let $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Put $X = \mathbb{R}^n$, $I = \{1, \ldots, m\}$, $g_i = \langle a_i, \cdot\rangle$ where $a_i = (a_{i1}, \ldots, a_{in})^T$, $i \in I$, and $\bar{x} = 0 \in S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0 \ \forall i \in I\}$. Then, we can see that the statement (iii) of Theorem 3.1 always holds, that is,

$$\left\{ v \in \mathbb{R}^n \,\middle|\, (v, 0) \in \text{clconeco} \bigcup_{i=1}^{m} \text{epi} g_i^* \right\} \subseteq \left\{ v \in \mathbb{R}^n \,\middle|\, (v, 0) \in \text{coneco} \bigcup_{i=1}^{m} \text{epi} g_i^* \right\}.$$

The proof is given as follows: at first, we can see that

$$\text{coneco}\bigcup_{i=1}^{m}\text{epi}g_i^* = \left\{\sum_{i=1}^{m}\lambda_i(a_i,\beta_i)\,\middle|\,\lambda_i \geq 0, \sum_{i=1}^{m}\lambda_i > 0, \beta_i \geq 0\right\} \cup \{(0,0)\}$$

and

$$\text{clconeco}\bigcup_{i=1}^{m}\text{epi}g_i^* = \left\{\sum_{i=1}^{m}\lambda_i(a_i,\beta_i)\,\middle|\,\lambda_i \geq 0, \beta_i \geq 0\right\}$$

hold. When $(v,0) = \sum_{i=1}^{m}\lambda_i(a_i,\beta_i) \in \text{clconeco}\bigcup_{i=1}^{m}\text{epi}g_i^*$ for some non-negative λ_i and β_i, it is clear that $\lambda_i\beta_i = 0$ for all $i \in I$. If all λ_i are 0 then $v = 0$, otherwise $\sum_{i=1}^{m}\lambda_i > 0$. Therefore $(v,0) \in \text{coneco}\bigcup_{i=1}^{m}\text{epi}g_i^*$ holds. From Theorem 3.1, the statement (v) holds. When $v = -b$, exactly one of the following two statements is true:

(a) there exists $x \in \mathbb{R}^n$ such that $\langle b, x\rangle > 0$ and $Ax \leq 0$,
(b) there exists $y = (y_1, \ldots, y_m) \in \mathbb{R}_+^m$ such that $A^T y = b$.

This is a variation of Farkas' Lemma. From this observation, each (iv) and (v) of Theorem 3.1 can be considered as a kind of alternative results.

4. Examples and Applications

In this section, we explain the usefulness of our results by some examples and applications. At first, we give three examples and we check up the BCQ at every feasible by using the given alternative method.

Example 4.1. Let $g_1 : \mathbb{R} \longrightarrow \mathbb{R}$ be a function as follows:

$$g_1(x) = \begin{cases} \frac{1}{2}x^2 - x & x \in (-\infty, 0], \\ 0 & x \in (0,1), \\ \frac{1}{2}(x-1)^2 & x \in [1, +\infty). \end{cases}$$

Then $S = [0,1]$, and we can calculate the Fenchel conjugate of g_1 as follows:

$$g_1^*(v) = \begin{cases} \frac{1}{2}(v+1)^2 & v \in (-\infty, -1], \\ 0 & v \in (-1,0), \\ \frac{1}{2}v^2 + v & v \in [0, +\infty). \end{cases}$$

Furthermore,

$$\{v \in \mathbb{R} \mid (v, vx) \in \text{clconeco epi}g_1^*\} = \begin{cases} (-\infty, 0] & x = 0, \\ \{0\} & x \in (0,1), \\ [0, +\infty) & x = 1, \end{cases}$$

and

$$\{v \in \mathbb{R} \mid (v, vx) \in \text{coneco epi}g_1^*\} = \begin{cases} (-\infty, 0] & x = 0, \\ \{0\} & x \in (0,1]. \end{cases}$$

By Theorem 3.1, the BCQ holds at every point of $[0,1)$, however the BCQ does not hold at 1. By Figure 4.1, it is easy to check whether the BCQ holds or not at every feasible point.

Example 4.2. Let $g_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function as follows:

$$g_2(x_1, x_2) = g(x_1) + g(x_2),$$

where

$$g(t) = \begin{cases} \frac{1}{2}(t+1)^2 & t \in (-\infty, -1], \\ 0 & t \in (-1,1), \\ \frac{1}{2}(t-1)^2 & t \in [1, +\infty). \end{cases}$$

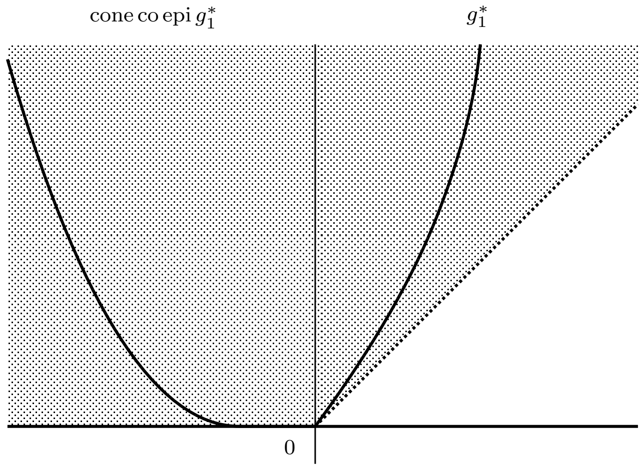

FIGURE 1.

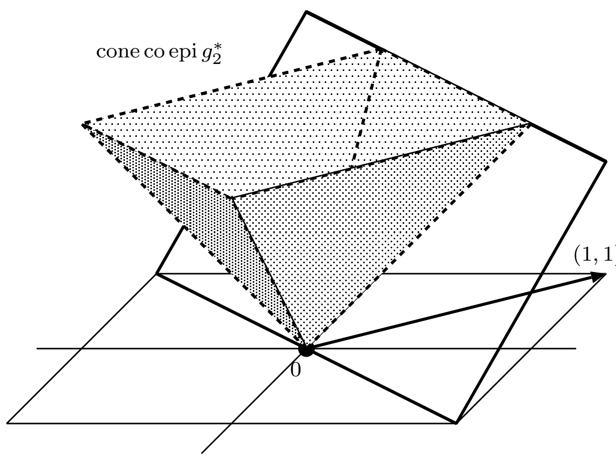

FIGURE 2.

Then, $S = [-1, 1]^2$, $g_2^*(v_1, v_2) = \frac{1}{2}v_1^2 + |v_1| + \frac{1}{2}v_2^2 + |v_2|$,

$$\text{coneco epi}g_2^* = \{(v_1, v_2, r) \in \mathbb{R}^3 \mid |v_1| + |v_2| < r\} \cup \{(0, 0, 0)\},$$

and

$$\text{clconeco epi}g_2^* = \{(v_1, v_2, r) \in \mathbb{R}^3 \mid |v_1| + |v_2| \leq r\}.$$

Hence, the BCQ holds at every point in the interior of S, however the BCQ does not hold at every point in the boundary of S by using Theorem 3.1. See Figure 4.2.

Example 4.3. Let $g_3 : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function as follows:

$$g_3(x) = \frac{1}{8}(\langle v_0, x\rangle - |\langle v_0, x\rangle|)^2 + \frac{1}{2}(\langle w_0, x\rangle + |\langle w_0, x\rangle|),$$

where $v_0,\, w_0 \in \mathbb{R}^n \setminus \{0\}$ and $\langle v_0, w_0\rangle = 0$. Then,

$$S = \{x \in \mathbb{R}^n \mid g_3(x) \leq 0\} = \{sv_0 + tw_0 \mid s \geq 0, t \leq 0\}$$

and

$$g_3^*(v) = \frac{1}{2}(\langle v_0, v\rangle)^2 + \delta_{\{sv_0+tw_0 \mid s\leq 0, t\in[0,1]\}}(v).$$

Hence,

$$\begin{aligned}\text{coneco epi}g_3^* &= \{(sv_0 + tw_0, r) \in \mathbb{R}^{n+1} \mid r > 0, s \in (-\infty, 0], t \in [0, +\infty)\}\\ &\quad \cup\{(tw_0, r) \in \mathbb{R}^{n+1} \mid r \geq 0, t \in [0, +\infty)\},\end{aligned}$$

and

$$\text{clconeco epi}g_3^* = \{(sv_0 + tw_0, r) \in \mathbb{R}^{n+1} \mid r \geq 0, s \in (-\infty, 0], t \in [0, +\infty)\}.$$

Therefore, the BCQ holds at every point in the union of the interior of S and $\{\lambda v_0 \mid \lambda > 0\}$, however the BCQ does not hold at every point in $\{tw_0 \mid t \in (-\infty, 0]\}$, by using Theorem 3.1.

When $n \leq 2$, as we saw in Example 4.1 and Example 4.2, it is possible to check up the BCQ on the feasible solution S by illustrating coneco$\bigcup_{i\in I}$ epig_i^*. When $n \geq 3$, it is not easy to illustrate coneco$\bigcup_{i\in I}$ epig_i^* in general, but Example 4.3 is a special case in which the BCQ can be checked up without illustrating coneco$\bigcup_{i\in I}$ epig_i^*. When every g_i are sublinear, the BCQ can also be checked up without illustrating coneco$\bigcup_{i\in I}$ epig_i^*, by using just $\partial g_i(0)$, see the following result:

Theorem 4.1. *Let I be an index set, g_i be a real-valued sublinear function on X for each $i \in I$, $S = \{x \in X \mid g_i(x) \leq 0, \forall i \in I\}$, and $\bar{x} \in S$. Then the following statements are equivalent:*

(i) *$\{g_i \mid i \in I\}$ satisfies the BCQ at $\bar{x}$,*
(ii) *the following inclusion holds:*

$$\{v \mid (v, \langle v, \bar{x}\rangle) \in \text{clconeco}\bigcup_{i\in I}\text{epi}\delta_{\partial g_i(0)}\} \subseteq \{v \mid (v, \langle v, \bar{x}\rangle) \in \text{coneco}\bigcup_{i\in I}\text{epi}\delta_{\partial g_i(0)}\}.$$

Proof. Since g_i is sublinear, we have

$$g_i^* = \delta_{\partial g_i(0)}.$$

By Theorem 3.1, (i) and (ii) are equivalent. □

Example 4.4. Let $g_4 : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function as follows:

$$g_4(x) = \|x\| + \langle v_0, x\rangle\,,$$

where $v_0 \in \mathbb{R}^n$ with $\|v_0\| = 1$ and $n \geq 2$. Then, g_4 is a sublinear function, and $S = \{x \in \mathbb{R}^n \mid g_4(x) \leq 0\} = \{tv_0 \mid t \leq 0\}$ and the interior of S is empty. We can calculate the subdifferential of g_4 at 0 as follows:

$$\partial g_4(0) = \{v \in \mathbb{R}^n \mid \|v - v_0\| \leq 1\}.$$

Additionally, for each $\bar{x} \in S$,

$$\{v \in \mathbb{R}^n \mid (v, \langle v, \bar{x}\rangle) \in \text{coneco epi}\delta_{\partial g_4(0)}\} = \{(0, 0)\}$$

and

$$\{v \in \mathbb{R}^n \mid (v, \langle v, \bar{x}\rangle) \in \text{clconeco epi}\delta_{\partial g_4(0)}\} = \{tv_0 \mid t \leq 0\}.$$

Therefore, by Theorem 4.1, the BCQ does not hold at every points in S.

Furthermore, we give the following sufficient condition of the BCQ for a sublinear inequality system:

Theorem 4.2. *Let I be an index set, g_i be a real-valued sublinear function on X for each $i \in I$, $S = \{x \in X \mid g_i(x) \leq 0, \forall i \in I\}$, and assume that S is nonempty. If* $\operatorname{coneco} \bigcup_{i\in I} \partial g_i(0)$ *is w^*-closed, then $\{g_i \mid i \in I\}$ satisfy the BCQ at every points in S.*

Proof. Let $\bar{x} \in S$ and let $v \in X^*$ with $(v, \langle v, \bar{x}\rangle) \in \operatorname{clconeco} \bigcup_{i\in I} \operatorname{epi}\delta_{\partial g_i(0)}$. We may assume that $v \neq 0$, because $(0, 0) \in \operatorname{coneco} \bigcup_{i\in I} \operatorname{epi}\delta_{\partial g_i(0)}$. Then there exists a net $\{(v_\alpha, \beta_\alpha) \mid \alpha \in D\} \subseteq \operatorname{coneco} \bigcup_{i\in I} \operatorname{epi}\delta_{\partial g_i(0)}$ such that

$$(v_\alpha, \beta_\alpha) \longrightarrow (v, \langle v, \bar{x}\rangle).$$

Additionally, for each $\alpha \in D$, there exists $\lambda^\alpha \in \mathbb{R}^{(I)}_+$ and $(x_i^\alpha, \gamma_i^\alpha) \in \operatorname{epi}\delta_{\partial g_i(0)}$ for each $i \in I$ such that

$$(v_\alpha, \beta_\alpha) = \sum_{i\in I} \lambda_i^\alpha (x_i^\alpha, \gamma_i^\alpha).$$

Since $\operatorname{epi}\delta_{\partial g_i(0)} = \partial g_i(0) \times [0, +\infty)$ for each $i \in I$, $v_\alpha \in \operatorname{coneco} \bigcup_{i\in I} \partial g_i(0)$ and $\beta_\alpha \in [0, +\infty)$. This shows that $v \in \operatorname{clconeco} \bigcup_{i\in I} \partial g_i(0)$ and $\langle v, \bar{x}\rangle \in [0, +\infty)$. By the assumption, $v \in \operatorname{coneco} \bigcup_{i\in I} \partial g_i(0)$. Hence there exist $\lambda \in \mathbb{R}^{(I)}_+$ and $v_i \in \partial g_i(0)$ for each $i \in I$ such that

$$v = \sum_{i\in I} \lambda_i v_i.$$

For each $i \in I$,

$$\delta_{\partial g_i(0)}(v_i) = 0 \leq \left\langle \frac{v}{\sum_{i\in I} \lambda_i}, \bar{x} \right\rangle,$$

that is

$$\left(v_i, \left\langle \frac{v}{\sum_{i\in I} \lambda_i}, \bar{x} \right\rangle \right) \in \operatorname{epi}\delta_{\partial g_i(0)}.$$

Therefore

$$(v, \langle v, \bar{x}\rangle) \in \operatorname{coneco} \bigcup_{i\in I} \operatorname{epi}\delta_{\partial g_i(0)}.$$

By Theorem 4.2, $\{g_i \mid i \in I\}$ satisfies the BCQ at $\bar{x}$. This completes the proof. □

Example 4.5. Let $g_5 : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function as follows:

$$g_5(x) = |\, \langle v_0, x\rangle \,|,$$

where $v_0 \in \mathbb{R}^n \setminus \{0\}$. Then, $S = \{x \in \mathbb{R}^n \mid \langle v_0, x\rangle = 0\}$, $\partial g_5(0) = \{tv_0 \mid t \in [-1, 1]\}$, and

$$\operatorname{coneco} \partial g_5(0) = \{tv_0 \mid t \in \mathbb{R}\}.$$

Since $\operatorname{coneco} \partial g_5(0)$ is closed, the BCQ holds at every points in S by Theorem 4.2.

5. Conclusion

In this paper, we have studied the basic constraint qualification as a sufficient condition for the optimality condition. In Theorem 3.1, we have given equivalent conditions of the BCQ at each feasible solution. Especially, we have given an alternative method for checking up the BCQ at every feasible points without subdifferentials and normal cones at feasible solutions, although the BCQ was defined by using the subdifferentials and the normal cones. We have explained the usefulness of the method to check up the BCQ by using examples in Section 4, and we have applied the main theorem for a sublinear inequality system in Theorem 4.1 and Theorem 4.2.

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References

1. R. I. Boţ, Conjugate Duality in Convex Optimization, Lecture Notes in Economics and Mathematical Systems, Vol. 637, Springer-Verlag Berlin, Heidelberg, 2010
2. M. A. Goberna, V. Jeyakumar and M. A. López, Necessary and sufficient constraint qualifications for solvability of systems of infinite convex inequalities, Nonlinear Anal. 68, 2008, 1184–1194.
3. J. Hiriart-Urruty and C. Lemarechal, Convex Analysis and Minimization Algorithms I., Vol. 305, Springer-Verlag Berlin, Heidelberg, 1993
4. V. Jeyakumar, Constraint qualifications characterizing Lagrangian duality in convex optimization, J. Optim. Theory Appl. 136, 2008, 31–41.
5. V. Jeyakumar, N. Dinh and G. M. Lee, A new closed cone constraint qualification for convex optimization. Research Report AMR 04/8, Department of Applied Mathematics, University of New South Wales, 2004
6. V. Jeyakumar, A. M. Rubinov, B. M. Glover, and Y. Ishizuka, Inequality systems and global optimization, J. Math. Anal. Appl. 202, 1996, 900–919.
7. V. Jeyakumar, A. M. Rubinov and Z. Y. Wu, Generalized Fenchel's conjugation formulas and duality for abstract convex functions, J. Optim. Theory Appl. 132, 2007, 441–458.
8. C. Li, K. F. Ng and T. K. Pong, Constraint qualifications for convex inequality systems with applications in constrained optimization, SIAM J. Optim. 19, 2008, 163–187.
9. S. Suzuki and D. Kuroiwa, On set containment characterization and constraint qualification for quasiconvex programming, J. Optim. Theory Appl. 149, 2011, 554–563.
10. S. Suzuki and D. Kuroiwa, Optimality conditions and the basic constraint qualification for quasiconvex programming, Nonlinear Anal. 74, 2011, 1279–1285.
11. S. Suzuki and D. Kuroiwa, Necessary and sufficient conditions for some constraint qualifications in quasiconvex programming, Nonlinear Anal. 75, 2012, 2851–2858.
12. S. Suzuki and D. Kuroiwa, Necessary and sufficient constraint qualification for surrogate duality, J. Optim. Theory Appl. 152, 2012, 366-367.
13. S. Suzuki and D. Kuroiwa, Some constraint qualifications for quasiconvex vector-valued systems, J. Global Optim. 55, 2013, 539–548.
14. S. Suzuki and D. Kuroiwa, Generators and constraint qualifications for quasiconvex inequality systems, J. Nonlinear Convex Anal. 18, 2017, 2101–2121.
15. S. Suzuki and D. Kuroiwa, Duality Theorems for Separable Convex Programming without Qualifications, J. Optim. Theory Appl. 172, 2017, 669–683.