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SOLUTIONS FOR THE ORDERED VARIATIONAL INCLUSION PROBLEMS IN BANACH SPACES

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ABSTRACT. In this study, we consider the ordered variational inclusion problems in ordered Banach spaces involving the weak RRD-multivalued mappings. By using the technique of relaxed resolvent operators, we suggest an iterative algorithm and prove the existence of solutions of ordered variational inclusion problems. Also, we prove the convergence of the sequences generated by an iterative algorithm.

KEYWORDS: Ordered variational inclusion problems, Algorithm, Weak-RRD multivalued mappings, Ordered Banach spaces.
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1. Introduction

Most of the problems related to variational inequalities, variational inclusions, complementarity problems and equilibrium problems are solved by the maximal monotone operators and their generalizations such as H-monotonicity, H-accretivity, penalization, regularization and many more in these fields *see*, [1, 2, 6, 7, 9, 13, 25]. Some splitting methods are based on the resolvent operator of the form $[I+\lambda M]^{-1}$, where M is a multivalued monotone mapping, λ is a positive constant and I is an identity mapping.

Generalized nonlinear ordered variational inclusions have wide applications to many fields including for example, mathematical physics, optimization and control theory, nonlinear programming, economics and engineering sciences. Li [18] studied a class of generalized nonlinear ordered variational inequalities in ordered Banach spaces. One year later, the same author [19] studied another class of general nonlinear ordered variational inequalities in the same setting. He continued to his researches in this field by studying a class of nonlinear inclusion problems

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for ordered RME set-valued mappings in ordered Hilbert spaces. The introduction of the concept of ordered (α, λ)-NODM set-valued mappings was first made by Li [21]. He considered a class of nonlinear variational inclusion problems involving (α, λ)-NODM set-valued mappings and proved an existence theorem for solutions of such a class of problems. He defined the resolvent operator associated with an α-NODM set-valued mapping and proved that it is a comparison Lipschitz continuous mapping. Based on the resolvent operator appeared in [21], the author suggested an iterative algorithm and studied the convergence analysis of the sequence generated by his proposed iterative algorithm. With inspiration and motivation from this work, during the past six years, many investigators have shown interest in introducing various kinds of ordered set-valued mappings in the setting of different ordered spaces and defined the resolvent operators associated with them. They used the resolvent operators defined in their papers for solving various classes of ordered variational inequalities/inclusions.

In 2001, Huang and Fang [11] introduced the concepts of generalized m-accretive mapping and studied the properties of resolvent operator with the generalized m-accretive mappings. Essentially, using the resolvent operator techniques, one can show that the variational inclusions are commensurate to the fixed point problems. This equivalent formulation has played a great job in designing some exotic techniques for solving variational inclusions and related optimization problems.

Inspired and motivated by the recent research works, [1]-[30], In this paper, we consider a relaxed resolvent operator $[(I-R)+\lambda M]^{-1}$, where R is a single valued mapping, I is an identity mapping, and show that the relaxed resolvent operator is a comparison mapping with respect to the operator $\oplus$. Finally, we prove the existence of solutions of ordered variational inclusion problems by using the weak-RRD multivalued mappings and also discuss the convergence of an iterative sequences generated by the algorithms.

2. Preliminaries

Let $(X, \|\cdot\|)$ be an ordered Banach spaces whose dual X^* is endowed with the dual norm, denoted also by $\|\cdot\|$. Let d be the metric induced by the norm $\|\cdot\|, 2^X$ (respectively $C(X)$) be the family of nonempty (respectively compact) subsets of X, and $\mathfrak{D}(\cdot,\cdot)$ be the Hausdorff metric on $C(X)$ defined by,

$$\mathfrak{D}(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\},$$

where $A, B \in C(X), d(x,B) = \inf_{y\in B} d(x,y)$ and $d(A,y) = \inf_{x\in A} d(x,y)$.

Definition 2.1. Let X be a real Banach space with norm $\|\cdot\|$ and θ be the zero element in X. A nonempty closed convex subset C of X is said to be a cone if

(i) $C + C \subset C$,
(ii) for any $x \in C$ and any $\lambda > 0, \lambda x \in C$.

Also said to be a pointed cone, if for any $x \in C$ and $-x \in C$ then $x = \theta$.

Definition 2.2. [18] Let C be a cone in real Banach space X. A cone C is called normal if there exists a constant $\lambda_{C_N} > 0$ such that for $\theta \le x \le y$, $\|x\| \le \lambda_{C_N}\|y\|$, where λ_{C_N} is a normal constant of C.

Definition 2.3. [29] A relation $\le$ is said to be partial ordered in X, if for any elements $x, y \in X, x \le y$, then $x - y \in C$. The real Banach space X endowed with the ordered relation $\le$ defined by C is called an ordered real Banach space.

Definition 2.4. [29] For arbitrary elements $x, y \in X$, x and y are called comparable to each other, if $x \leq y$ (or $y \leq x$) holds. And denoted by $x \propto y$ for $x \leq y$ and $y \leq x$.

Lemma 2.5. ([18, 23], *Lemma 2.1*) *Let C be a normal cone of an ordered real Banach space X and $\leq$ be a partial ordered relation defined by the cone C. For arbitrary elements $x, y \in X, lub\{x, y\}$ and $glb\{x, y\}$ express the least upper bound and greatest lower bound of the set $\{x, y\}$. Suppose $lub\{x, y\}$ and $glb\{x, y\}$ exists, some binary operations can be defined as follows:*

(i) $x \vee y = lub\{x, y\}$;
(ii) $x \wedge y = glb\{x, y\}$;
(iii) $x \oplus y = (x - y) \vee (y - x)$;
(iv) $x \odot y = (x - y) \wedge (y - x)$;
(v) *If $x \propto y$, then $lub\{x, y\}$ and $glb\{x, y\}$ exist, $x - y \propto y - x$ and $\theta \leq (x - y) \vee (y - x)$;*
(vi) *If $x \vee y = lub\{x, y\}, x \wedge y = glb\{x, y\}, x \oplus y = (x - y) \vee (y - x), x \odot y = (x - y) \wedge (y - x)$.*

The operators $\vee, \wedge, \oplus$ and $\odot$ are called OR, AND, XOR and XNOR operations, respectively. Then here in after relations survive:

(1) $x \oplus y = y \oplus x, x \oplus x = \theta, x \odot y = y \odot x = -(x \oplus y) = -(y \oplus x)$;
(2) *For a real λ, $(\lambda x) \oplus (\lambda y) =\mid \lambda \mid (x \oplus y)$;*
(3) $x \odot \theta \leq \theta$, *if* $x \propto \theta$;
(4) $\theta \leq x \oplus y$ *if* $x \propto y$;
(5) *if $x \propto y$ then $x \oplus y = \theta$ if and only if $x = y$;*
(6) $(x + y) \odot (u + v) \geq (x \odot u) + (y \odot v)$;
(7) $(x + y) \odot (u + v) \geq (x \odot v) + (y \odot u)$;
(8) *let $(x + y) \vee (u + v)$ exist, and if $x \propto u, v$ and $y \propto u, v$, then*
$$(x + y) \oplus (u + v) \leq (x \oplus u + y \oplus v) \wedge (x \oplus v + y \oplus u);$$
(9) *if $x \leq y$ and $u \leq v$, then $x + u \leq y + v$:*
(10) $x \vee y = x + y - (x \wedge y)$;
(11) $\alpha x \oplus \beta x =\mid \alpha - \beta \mid x = (\alpha \oplus \beta) \mid x$, *if* $x \propto \theta$, $\forall x, y, u, v \in X$ *and* $\alpha, \beta, \gamma \in \mathbb{R}$.

Proposition 2.6. [8] *If $x \propto y$, then $lub\{x, y\}$ and $glb\{x, y\}$ exist $x - y \propto y - x$ and $\theta \leq (x - y) \vee (y - x)$.*

Proposition 2.7. [8] *If for any positive integer n if $x \propto y_n$ and $y_n \longrightarrow y^*$ $(n \longrightarrow \infty)$ then $x \propto y^*$.*

Proposition 2.8. ([23], *Theorem 2.5*) *Let X be a positive Hilbert space and $x, y, z, w \in X$. Then we have the following statements:*

(1) *If $x \leq y$, $\theta \leq z$, then $\langle y, z\rangle \geq \langle x, z\rangle$;*
(2) *If $\theta \leq z$, then $\langle x \vee y, z\rangle \geq \langle x, z\rangle \vee \langle y, z\rangle, \langle x, z\rangle \wedge \langle y, z\rangle \geq \langle x \wedge y, z\rangle$;*
(3) *If $\theta \leq z$, then $\langle x + y, z\rangle \geq \langle x, z\rangle \vee \langle y, z\rangle + \langle x \wedge y, z\rangle$;*
(4) *If $\theta \leq z$, then $\langle x \vee y, z\rangle \geq \langle x, z\rangle + \langle y, z\rangle - \langle x, z\rangle \wedge \langle y, z\rangle$;*
(5) *If $\theta \leq z$, then $\langle x \oplus y, z\rangle \geq \langle x, z\rangle \oplus \langle y, z\rangle$.*

Lemma 2.9. ([17, 19], *Lemma 1.13*) *Let X be an ordered real Banach space and C be a normal cone in X with normal constant λ_{C_N}. Then for arbitrary elements $x, y \in X$, the following relations hold:*

(1) $\|\theta \oplus \theta\| = \|\theta\| = \theta$;

(2) $\|x \wedge y\| \leq \|x\| \wedge \|y\| \leq \|x\| + \|y\|$;
(3) $\|x \oplus y\| \leq \|x - y\| \leq \lambda_{C_N}\|x \oplus y\|$;
(4) *if* $x \propto y$, *then* $\|x \oplus y\| = \|x - y\|$;
(5) $\lim_{x \longrightarrow x_0} \|A(x) - A(x_0)\| = 0$, *if and only if*

$$\lim_{x \longrightarrow x_0} A(x) \oplus A(x_0) = \theta.$$

Definition 2.10. [18] Let X be a real ordered Banach space and $R : X \longrightarrow X$ be a single valued mapping, Then

(i) R is called comparison, if for each $x, y \in X$, $x \propto y$ then $R(x) \propto R(y)$, $x \propto R(x)$ and $y \propto R(y)$;
(ii) R is called strongly comparison, if R is a comparison and $R(x) \propto R(y)$ if and only if $x \propto y$ for $x, y \in X$.

Definition 2.11. [21] Let X be a real ordered Banach space and $R, B : X \longrightarrow X$ be two single valued mappings, Then R and B is said to be comparable to each other, if for each $x, \in X$, $R(x) \propto B(x)$ (denoted by $R \propto B$). Obviously, if R is comparable, then $R \propto I$, where I is the identity mapping.

Definition 2.12. ([18], Definition 2.10) Let X be a real ordered Banach space and C be the normal cone with normal constant C_{λ_N} in X. A mapping $R : X \longrightarrow X$ is called β-ordered compression mapping if R is a comparison and there exists a constant $0 < \beta < 1$ such that

$$(R(x) \oplus R(y)) \leq \beta(x \oplus y).$$

Lemma 2.13. [17] *Let C be a normal cone in X. If for $x, y \in X$, they can be compared to each other, then the following condition holds:*

$$(x + y) \vee ((-x) + (-y)) \leq (x \vee (-x)) + (y \vee (-y)).$$

Definition 2.14. [20, 23, 24] Let X be a real ordered Banach space and $M : X \longrightarrow 2^X$ be a multivalued mapping. Let $R : X \longrightarrow X$ be a strong comparison and β-ordered compression. Then,

(i) M is called comparison mapping, if for any $v_x \in M(x)$, $x \propto y$, $x \propto v_x$, then for any $v_x \in M(x)$, $v_y \propto M(y)$ and $v_x \propto v_y$, for all $x, y \in X$;
(ii) M is called ordered rectangular mapping, if for each $x, y \in X, v_x \in M(x)$ and $v_y \in M(y)$ such that

$$\langle v_x \odot v_y, -(x \oplus y)\rangle = 0;$$

(iii) M is called γ_R-ordered rectangular mapping with respect to R, if there exists a constant $\gamma_R > 0$, for any $x, y \in X$ there exists $v_x \in M(R(x))$ and $v_y \in M(R(y))$ such that

$$\langle v_x \odot v_y, -(R(x) \oplus R(y))\rangle \geq \gamma_R\|R(x) \oplus R(y)\|^2,$$

where v_x and v_y are said to be γ_R-elements, *respectively*;
(iv) M is called weak comparison mapping with respect to R, if for any $x, y \in X$, $x \propto y$ then there exists $v_x \in M(R(x))$ and $v_y \in M(R(y))$ such that $x \propto v_x$, $y \propto v_y$ and $v_x \propto v_y$, where v_x and v_y are said to be weak comparison elements, *respectively*;

(v) M is called λ-weak ordered different comparison mapping with respect to R, if there exists a constant $\lambda > 0$, for any $x, y \in X$ there exists $v_x \in M(R(x))$, $v_y \in M(R(y))$ such that $\lambda(v_x - v_y) \propto (x - y)$, where v_x and v_y are said to be λ-elements, *respectively*;

(vi) A comparison mapping M is called λ-ordered strongly monotonic increasing, if for $x \geq y$ there exists a constant $\lambda > 0$ such that

$$\lambda(v_x - v_y) \geq x - y,\ \forall x, y \in X, v_x \in M(x), v_y \in M(y);$$

(vii) A weak comparison mapping M is said to be a (γ_R, λ)-weak-RRD multivalued mapping with respect to B if M is a γ_R-ordered rectangular and λ-weak ordered different comparison mapping with respect to B and $(R+\lambda M)X = X$ for $\lambda > 0$ and there exists $v_x \in M(R(x))$ and $v_y \in M(R(y))$ such that v_x and v_y are (γ_R, λ)-elements, *respectively*.

Remark 2.15. Let X be a real Hilbert space. Then we have the followings:

(i) Every λ-ordered monotone mapping is a λ-weak ordered different comparison mapping.

(ii) The γ_I-ordered rectangular mapping is an ordered rectangular mapping, where I is the identity mapping,

(iii) An ordered RME mapping is a λ-weak RRD-mapping.

Definition 2.16. Let X be an ordered real Banach space. A multivalued mapping $A : X \longrightarrow 2^X$ is said to be $\mathfrak{D}$-Lipschitz continuous if for each $x, y \in X, x \propto y$ there exists a constant $\delta_A > 0$ such that

$$\mathfrak{D}(A(x), A(y)) \leq \delta_A \|x \oplus y\|.$$

Definition 2.17. Let X be an ordered real Banach space. Let $M : X \longrightarrow 2^X$ be a multivalued mapping, $R : X \longrightarrow X$ be a single valued mapping and $I : X \longrightarrow X$ be the identity mapping. Then a weak comparison mapping M is said to be a (γ',λ)-weak RRD multivalued mapping with respect to $(I - R)$, if M is a γ'-ordered rectangular and λ-weak ordered different comparison mapping with respect to $(I-R)$ and $[(I-R)+\lambda M](X) = X$ for $\lambda > 0$ and there exists $v_x \in M((I-R)(x))$ and $v_y \in M((I-R)(y))$ such that v_x and v_y are (γ', λ)-elements, *respectively*.

Example 2.18. Let $X = \mathbb{R}$ and let $R : X \longrightarrow X$ be a mapping defined by

$$R(x) = \frac{x}{2}, \forall x \in X$$

and a multivalued mapping $M : X \longrightarrow 2^X$ is defined by $M(x) = 1$ for $x = 0$ and $M(x) = \{\frac{x}{3}\}$ for $x \neq 0$. Then we can easily check that R is 1-ordered compression and M is $(\frac{1}{7}, 1)$- weak RRD mapping with respect to R.

3. Formulation of Problems

Let X be an ordered Banach space and $A, B, T, G : X \longrightarrow C(X)$ be multivalued mappings. Suppose that $M : X \longrightarrow 2^X$ be a multivalued mapping and $N : X \times X \times X \longrightarrow X$ be a single valued mapping. We consider the following problems for finding $u \in X, v \in A(u), w \in B(u), q \in T(u)$ and $z \in G(u)$ such that

$$\rho \in N(v, w, q) + \tau M(z),\ \text{for some } \rho \in X \text{ and } \tau > 0. \tag{3.1}$$

Problem (3.1) is called an ordered variational inclusion problem.

Special Cases:

(i) If $N(v,w,q) \equiv 0$ and G is a single valued mapping, then (3.1) reduces to the following problem of finding $u \in X$ such that

$$\rho \in \tau M(u), \text{ for some } \rho \in X \text{ and } \tau > 0, \tag{3.2}$$

studied by Li *et al.* [23].

(ii) If $B = T \equiv 0$ (zero mappings) and A, G are single valued mappings, then (3.1) reduces to the following problem of finding $u \in X$ such that

$$\rho \in N(u) + \tau M(u), \text{ for some } \rho \in X \text{ and } \tau > 0, \tag{3.3}$$

studied by Li *et al.* [24].

(iii) If $\rho = 0, \tau = 1,\ T \equiv 0$ (a zero mapping) and G is a single valued mapping, then (3.1) reduces to the following problem of finding $u \in X, v \in A(u), w \in B(u)$ such that

$$0 \in N(v,w) + M(u), \tag{3.4}$$

studied by Verma [30].

(iv) If $\rho = 0, \tau = 1,\ T \equiv 0$ (a zero mapping) and $N(v,w) = f(v) - p(w)$, then (3.1) reduces to the following problem of finding $u \in X, v \in A(u), w \in B(u), z \in G(u)$ such that

$$0 \in f(v) - p(w) + M(z), \tag{3.5}$$

where $f, p : X \longrightarrow X$ are single valued mappings, studied by Huang [10].

Definition 3.1. ([23], Theorem 3.3) Let X be an ordered Banach space, C be a normal cone with normal constant λ_{C_N} and $M : X \longrightarrow 2^X$ be a weak-RRD-multivalued mapping. Let $I : X \longrightarrow X$ be the identity mapping and $R : X \longrightarrow X$ be a single valued mapping. The relaxed resolvent operator $J_{M,\lambda}^{(I-R)} : X \longrightarrow X$, associated with I, R and M is defined by

$$J_{M,\lambda}^{(I-R)}(x) = [(I-R) + \lambda M]^{-1}(x), \forall x \in X, \tag{3.6}$$

where $\lambda > 0$ is a constant.

Proposition 3.2. ([23], *Theorem 3.3) Let X be an ordered Banach space, $R : X \longrightarrow X$ be a β-ordered compression mapping and $M : X \longrightarrow 2^X$ be a multivalued ordered rectangular mapping. Then relaxed resolvent operator $J_{M,\lambda}^{(I-R)} : X \longrightarrow X$ is a single valued for all $\lambda > 0$.*

Proposition 3.3. *Let X be an ordered Banach space and C be a normal cone with normal constant λ_{C_N} in X. Let $\leq$ be an ordering relation defined by the cone C, the operator $\oplus$ be a XOR operator. Let $M : X \longrightarrow 2^X$ be a (γ_R, λ)-weak-RRD-multivalued mapping with respect to $J_{M,\lambda}^{(I-R)}$. Let $R : X \longrightarrow X$ be a single valued mapping and $I : X \longrightarrow X$ be the identity mapping. Then the resolvent operator $J_{M,\lambda}^{(I-R)} : X \longrightarrow X$ is a comparable.*

Proof. Let M be a (γ_R, λ)-weak RRD-multivalued mapping with respect to $J_{M,\lambda}^{(I-R)}$. That is, M is γ_R-ordered rectangular and λ-weak ordered different comparison mapping with respect to $J_{M,\lambda}^{(I-R)}$, so that $x \propto J_{M,\lambda}^{(I-R)}(x)$. For any $x, y \in X$, let $x \propto y$ and let

$$v_x = \frac{1}{\lambda}(x - (I-R)(J_{M,\lambda}^{(I-R)}(x))) \in M(J_{M,\lambda}^{(I-R)}(x)) \tag{3.7}$$

and

$$v_y = \frac{1}{\lambda}(y - (I-R)(J_{M,\lambda}^{(I-R)}(y))) \in M(J_{M,\lambda}^{(I-R)}(y)). \tag{3.8}$$

Using (3.7) and (3.8), we have

$$\begin{aligned} v_x - v_y &= \frac{1}{\lambda}(x - (I-R)(J_{M,\lambda}^{(I-R)}(x))) - \frac{1}{\lambda}(y - (I-R)(J_{M,\lambda}^{(I-R)}(y))) \\ &= \frac{1}{\lambda}(x - y + (I-R)(J_{M,\lambda}^{(I-R)}(y) - J_{M,\lambda}^{(I-R)}(x))). \end{aligned}$$

Since M is a λ-weak ordered different comparison mapping with respect to $J_{M,\lambda}^{(I-R)}$, we have

$$\begin{aligned} \theta &\leq \lambda(v_x - v_y) - (x - y) \\ &= (x - y) + (I-R)(J_{M,\lambda}^{(I-R)}(y)) - (I-R)(J_{M,\lambda}^{(I-R)}(x)) - (x - y) \\ &= (I-R)(J_{M,\lambda}^{(I-R)}(y)) - (I-R)(J_{M,\lambda}^{(I-R)}(x)). \end{aligned}$$

If $y \leq x$ then $\lambda(v_x - v_y) - (x - y) \in C$ and if $x \leq y$ then $(x - y) - \lambda(v_x - v_y) \in C$. Therefore, from Lemma 2.5

$$J_{M,\lambda}^{(I-R)}(x) \propto J_{M,\lambda}^{(I-R)}(y).$$

The proof is completed. □

Definition 3.4. Let X be a real ordered Banach space, C be a normal cone with a normal constant γ_{C_N} in X. A mapping $N : X \times X \times X \longrightarrow X$ is called (μ_N, η_N, ξ_N)-ordered compression mapping, if $x \propto y, u \propto v$ and $p \propto q$, then $N(x,u,p) \propto N(y,v,q)$ and there exist the constants $\mu_N, \eta_N, \xi_N > 0$ such that

$$N(x,u,p) \oplus N(y,v,q) \leq \mu_N(x \oplus y) + \eta_N(u \oplus v) + \xi_N(p \oplus q).$$

Lemma 3.5. *Let X be an ordered Banach space and C be a normal cone with normal constant λ_{C_N} in X. Let $\leq$ be an ordering relation defined by the cone C. Let $M : X \longrightarrow 2^X$ be a (γ_R, λ)-weak-RRD multivalued mapping with respect to $J_{M,\lambda}^{(I-R)}$. Let $R : X \longrightarrow X$ be a comparison and β-ordered compression mapping and I be the identity mapping. If $\lambda\gamma_R > \beta + 1 > 0$, and $v_x \in M(J_{M,\lambda}^{(I-R)}(x))$ and $v_y \in M(J_{M,\lambda}^{(I-R)}(y))$ are γ_R and λ-elements, respectively. Then relaxed resolvent operator $J_{M,\lambda}^{(I-R)}$ of M is a comparison and*

$$\|J_{M,\lambda}^{(I-R)}(x) \oplus J_{M,\lambda}^{(I-R)}(y)\| \leq \frac{1}{\lambda\gamma_R - \beta - 1}\|x \oplus y\|.$$

Proof. Let M be a (γ_R, λ)-weak RRD multivalued mapping with respect to $J_{M,\lambda}^{(I-R)}$. That is M is a γ_R-ordered rectangular and λ-weak ordered different comparison mapping with respect to $J_{M,\lambda}^{(I-R)}$. Then for any $x, y \in X,\ \lambda > 0$, set $u_x = J_{M,\lambda}^{(I-R)}(x)$, $u_y = J_{M,\lambda}^{(I-R)}(y)$ and let

$$v_x = \frac{1}{\lambda}(x - (I-R)(J_{M,\lambda}^{(I-R)}(x))) \in M(J_{M,\lambda}^{(I-R)}(x))$$

and

$$v_y = \frac{1}{\lambda}(y - (I-R)(J_{M,\lambda}^{(I-R)}(y))) \in M(J_{M,\lambda}^{(I-R)}(y)).$$

Since R is β-ordered compression mapping and from Lemma 2.5, we have

$$\begin{aligned} v_x \oplus v_y &= \frac{1}{\lambda}((x - (I-R)(u_x)) \oplus (y - (I-R)(u_y))) \\ &\leq \frac{1}{\lambda}(x \oplus y + (I-R)(u_x) \oplus (I-R)(u_y)) \end{aligned}$$

$$\leq \frac{1}{\lambda}(x \oplus y + u_x \oplus u_y + R(u_x) \oplus R(u_y))$$
$$\leq \frac{1}{\lambda}(x \oplus y + u_x \oplus u_y + \beta(u_x \oplus u_y))$$
$$\leq \frac{1}{\lambda}(x \oplus y + (1+\beta)(u_x \oplus u_y)).$$

Since M is γ_R-ordered rectangular mapping with respect to $J_{M,\lambda}^{(I-R)}$, we have

$$\begin{aligned}
\gamma_R \|u_x \oplus u_y\|^2 &\leq \langle v_x \odot v_y, -(u_x \oplus u_y)\rangle \\
&\leq \langle v_x \oplus v_y, u_x \oplus u_y\rangle \\
&\leq \langle \frac{1}{\lambda}(x \oplus y + (1+\beta)(u_x \oplus u_y)), u_x \oplus u_y\rangle \\
&\leq \frac{1}{\lambda}\langle x \oplus y, (u_x \oplus u_y)\rangle + \frac{1+\beta}{\lambda}\langle (u_x \oplus u_y), (u_x \oplus u_y)\rangle \\
&\leq \frac{1}{\lambda}\|x \oplus y\|\|u_x \oplus u_y\| + \frac{1+\beta}{\lambda}\|u_x \oplus u_y\|^2.
\end{aligned}$$

It follows that

$$(\gamma_R - \frac{1+\beta}{\lambda})\|u_x \oplus u_y\| \leq \frac{1}{\lambda}\|x \oplus y\|$$

and consequently, we have

$$\|J_{M,\lambda}^{(I-R)}(x) \oplus J_{M,\lambda}^{(I-R)}(y)\| \leq \frac{1}{\lambda\gamma_R - \beta - 1}\|x \oplus y\|.$$

This completes the proof. □

4. Main Results

In this section, we will show the convergence of the approximation sequences generated by iterative algorithm for finding the solution of problem (3.1).

Algorithm 4.1. *Let $A, B, T, G : X \longrightarrow C(X)$ be the multivalued mappings, $R : X \longrightarrow X$ be a single valued mapping and $I : X \longrightarrow X$ be the identity mapping. Suppose that $N : X \times X \times X \longrightarrow X$ is a single valued mapping and $M : X \longrightarrow C(X)$ is a multivalued mapping. For any given initial $u_0 \in X, v_0 \in A(u_0), w_0 \in B(u_0), q_0 \in T(u_0), z_0 \in G(u_0)$, let*

$$u_1 = u_0 - z_0 + J_{M,\lambda}^{(I-R)}[(I-R)(z_0) + \frac{\lambda}{\tau}(\rho - N(v_0, w_0, q_0))].$$

Since $v_0 \in A(u_0) \in C(X), w_0 \in B(u_0) \in C(X), q_0 \in T(u_0) \in C(X)$ and $z_0 \in G(u_0) \in C(X)$, by Lemma 2.9 there exist $v_1 \in A(u_1) \in C(X), w_1 \in B(u_1) \in C(X), q_1 \in T(u_1) \in C(X)$ and $z_1 \in G(u_1) \in C(X)$ and suppose that $u_0 \propto u_1, v_0 \propto v_1, w_0 \propto w_1, q_0 \propto q_1$ and $z_0 \propto z_1$ such that

$$\|v_1 \oplus v_0\| = \|v_1 - v_0\| \leq \mathfrak{D}(A(u_1), A(u_0)),$$
$$\|w_1 \oplus w_0\| = \|w_1 - w_0\| \leq \mathfrak{D}(B(u_1), B(u_0)),$$
$$\|q_1 \oplus q_0\| = \|q_1 - q_0\| \leq \mathfrak{D}(T(u_1), T(u_0)),$$
$$\|z_1 \oplus z_0\| = \|z_1 - z_0\| \leq \mathfrak{D}(G(u_1), G(u_0)).$$

Continuing the above process inductively, we can define the iterative sequences $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{q_n\}$ and $\{z_n\}$ with the supposition that $u_n \propto u_{n+1}, v_n \propto v_{n+1}, w_n \propto$

$w_{n+1}, q_n \propto q_{n+1}$ and $z_n \propto z_{n+1}$ for all $n \in N$. We define the following iterative schemes:

$$u_{n+1} = u_n - z_n + J_{M,\lambda}^{(I-R)}[(I-R)(z_n) + \frac{\lambda}{\tau}(\rho - N(v_n, w_n, q_n))] \tag{4.1}$$

$$\begin{aligned} &v_{n+1} \in A(u_{n+1}), \|v_{n+1} \oplus v_n\| = \|v_{n+1} - v_n\| \leq \mathfrak{D}(A(u_{n+1}), A(u_n)), \\ &w_{n+1} \in B(u_{n+1}), \|w_{n+1} \oplus w_n\| = \|w_{n+1} - w_n\| \leq \mathfrak{D}(B(u_{n+1}), B(u_n)), \\ &q_{n+1} \in T(u_{n+1}), \|q_{n+1} \oplus q_n\| = \|q_{n+1} - q_n\| \leq \mathfrak{D}(T(u_{n+1}), T(u_n)), \\ &z_{n+1} \in G(u_{n+1}), \|z_{n+1} \oplus z_n\| = \|z_{n+1} - z_n\| \leq \mathfrak{D}(G(u_{n+1}), G(u_n)) \end{aligned} \tag{4.2}$$

where λ, τ are constants and $\rho \in X$.

Now, we convert our problem (3.1) into a fixed point problem.

Lemma 4.2. *Let $u \in X, v \in A(u), w \in B(u), q \in T(u)$ and $z \in G(u)$ be the solution of ordered variational inclusion problems (3.1) involving weak RRD-multivalued mappings if and only if (u, v, w, q, z) satisfies the following relation*

$$u = u - z + J_{M,\lambda}^{(I-R)}[(I-R)(z) + \frac{\lambda}{\tau}(\rho - N(v, w, q))]$$

where

$$J_{M,\lambda}^{(I-R)} = [(I-R) + \lambda M]^{-1}$$

and λ, τ are constants and $\rho \in X$.

Proof. This directly follows from the definition of relaxed resolvent operator $J_{M,\lambda}^{(I-R)}$ and the conditions of comparison mappings, *respectively.* □

Theorem 4.3. *Let X be a real ordered Banach space and C be a normal cone with normal constant λ_{C_N} in X. Let $R : X \longrightarrow X$ be a comparison, β-ordered compression mapping and $I : X \longrightarrow X$ be the identity mapping. Let $N : X \times X \times X \longrightarrow X$ be the (μ_N, η_N, ξ_N)-ordered compression mapping with constants $\mu_N, \eta_N, \xi_N > 0$. Let $A, B, T, G : X \longrightarrow C(X)$ be the multivalued mappings such that A, B, T and G are $\mathfrak{D}$-Lipschitz continuous mapping with constants $\delta_A, \delta_B, \delta_T$ and δ_G, respectively. Suppose that $M : X \longrightarrow C(X)$ is a (γ_R, λ)-weak RRD-multivalued mapping such that the following conditions are satisfied:*

$$\lambda_{C_N}[\tau\lambda\gamma_R(1+\delta_G) + \lambda\mu_N\delta_A + \lambda\eta_N\delta_B + \lambda\xi_N\delta_T] < \lambda\tau\gamma_R + \tau(1+\beta)(\lambda_{C_N} - 1). \tag{4.3}$$

Then the iterative sequences $\{u_n\}, \{v_n\}, \{w_n\}, \{q_n\}$ and $\{z_n\}$ generated by Algorithm 4.1 converge strongly to u, v, w, q and z, respectively and (u, v, w, q, z) is a solution of ordered variational inclusion problems (3.1) involving weak RRD-multivalued mappings, where $u \in X, v \in A(u), w \in B(u), q \in T(u)$ and $z \in G(u)$.

Proof. Let $h(u_n) = [(I-R)(z_n) + \frac{\lambda}{\tau}(\rho - N(v_n, w_n, q_n))]$. Using Algorithm 4.1 and Lemma 2.5, we obtain

$$\begin{aligned} 0 &\leq u_{n+1} \oplus u_n \\ &= (u_n - z_n + J_{M,\lambda}^{(I-R)}(h(u_n))) \oplus (u_{n-1} - z_{n-1} + J_{M,\lambda}^{(I-R)}(h(u_{n-1}))) \\ &\leq u_n \oplus u_{n-1} + z_n \oplus z_{n-1} + (J_{M,\lambda}^{(I-R)}(h(u_n)) \oplus J_{M,\lambda}^{(I-R)}(h(u_{n-1}))). \end{aligned} \tag{4.4}$$

Using Definition 2.2, Lemma 3.5 and from (4.4), we have

$$\begin{aligned} &\|u_{n+1} \oplus u_n\| \\ &\leq \lambda_{C_N}\|u_n \oplus u_{n-1} + z_n \oplus z_{n-1} + (J_{M,\lambda}^{(I-R)}(h(u_n)) \oplus J_{M,\lambda}^{(I-R)}(h(u_{n-1})))\| \end{aligned}$$

$$\leq \lambda_{C_N}[\|u_n \oplus u_{n-1}\| + \|z_n \oplus z_{n-1}\| + \|J^{(I-R)}_{M,\lambda}(h(u_n)) \oplus J^{(I-R)}_{M,\lambda}(h(u_{n-1}))\|]$$
$$\leq \lambda_{C_N}[\|u_n \oplus u_{n-1}\| + \mathfrak{D}(G(u_n), G(u_{n-1})) + \|J^{(I-R)}_{M,\lambda}(h(u_n)) \oplus J^{(I-R)}_{M,\lambda}(h(u_{n-1}))\|]$$
$$\leq \lambda_{C_N}[\|u_n \oplus u_{n-1}\| + \delta_G\|u_n \oplus u_{n-1}\| + \frac{1}{\lambda\gamma_R - \beta - 1}\|h(u_n) \oplus h(u_{n-1})\|]. \quad (4.5)$$

Since R is a β-ordered compression mapping and N is a (μ_N, η_N, ξ_N)-ordered compression mappings and A, B, T, G are $\mathfrak{D}$-Lipschitz continuous with respect to the constants $\delta_A, \delta_B, \delta_T$ and δ_G, *respectively*, we have

$$\|h(u_n) \oplus h(u_{n-1})\| = \|[(I - R)(z_n) + \frac{\lambda}{\tau}(\rho - N(v_n, w_n, q_n))]$$

$$\oplus [(I - R)(z_{n-1}) + \frac{\lambda}{\tau}(\rho - N(v_{n-1}, w_{n-1}, q_{n-1}))]\|$$
$$\leq \|(I - R)(z_n) \oplus (I - R)(z_{n-1})\| + \frac{\lambda}{\tau}\|(\rho - N(v_n, w_n, q_n))$$
$$\oplus (\rho - N(v_{n-1}, w_{n-1}, q_{n-1}))\|$$
$$\leq \|z_n \oplus z_{n-1}\| + \|R(z_n) \oplus R(z_{n-1})\| + \frac{\lambda}{\tau}[\mu_N\|v_n \oplus v_{n-1}\|$$
$$+ \eta_N\|w_n \oplus w_{n-1}\| + \xi_N\|q_n \oplus q_{n-1}\|]$$
$$\leq \mathfrak{D}(G(u_n), G(u_{n-1})) + \|R(z_n) \oplus R(z_{n-1})\| + \frac{\lambda}{\tau}[\mu_N\|v_n \oplus v_{n-1}\|$$
$$+ \eta_N\|w_n \oplus w_{n-1}\| + \xi_N\|q_n \oplus q_{n-1}\|]$$
$$\leq \delta_G\|u_n \oplus u_{n-1}\| + \beta\mathfrak{D}(G(u_n), G(u_{n-1})) + \frac{\lambda}{\tau}[\mu_N\mathfrak{D}(A(u_n), A(u_{n-1}))$$
$$+ \eta_N\mathfrak{D}(B(u_n), B(u_{n-1})) + \xi_N\mathfrak{D}(T(u_n), T(u_{n-1}))]$$
$$\leq \delta_G\|u_n \oplus u_{n-1}\| + \beta\delta_G\|u_n \oplus u_{n-1}\| + \frac{\lambda}{\tau}[\mu_N\delta_A\|u_n \oplus u_{n-1}\|$$
$$+ \eta_N\delta_B\|u_n \oplus u_{n-1}\| + \xi_N\delta_T\|u_n \oplus u_{n-1}\|]$$
$$\leq [(1 + \beta)\delta_G + \frac{\lambda}{\tau}[\mu_N\delta_A + \eta_N\delta_B + \xi_N\delta_T]]\|u_n \oplus u_{n-1}\|,$$

which implies that

$$\|h(u_n) \oplus h(u_{n-1})\| \leq [(1 + \beta)\delta_G + \frac{\lambda}{\tau}[\mu_N\delta_A + \eta_N\delta_B + \xi_N\delta_T]]\|u_n \oplus u_{n-1}\|. \quad (4.6)$$

Using (4.5) and (4.6), we have

$$\|u_{n+1} \oplus u_n\|$$
$$\leq \lambda_{C_N}[1 + \delta_G + \frac{1}{\lambda\gamma_R - \beta - 1}((1 + \beta)\delta_G + \frac{\lambda}{\tau}(\mu_N\delta_A + \eta_N\delta_B + \xi_N\delta_T))]\|u_n \oplus u_{n-1}\|.$$

By Lemma 2.9, we have

$$\|u_{n+1}-u_n\| = \|u_{n+1}\oplus u_n\| \leq \lambda_{C_N}[1+\delta_G+\frac{1}{\lambda\gamma_R - \beta - 1}((1+\beta)\delta_G+\frac{\lambda}{\tau}(\mu_N\delta_A+\eta_N\delta_B$$
$$+\xi_N\delta_T))]\|u_n \oplus u_{n-1}\|,$$

that is,

$$\|u_{n+1} - u_n\| \leq \Theta\|u_n - u_{n-1}\| \quad (4.7)$$

where

$$\Theta = \lambda_{C_N}[1 + \delta_G + \frac{1}{\lambda\gamma_R - \beta - 1}((1+\beta)\delta_G + \frac{\lambda}{\tau}(\mu_N\delta_A + \eta_N\delta_B + \xi_N\delta_T))].$$

By condition (4.3), we have $0 < \Theta < 1$. Thus $\{u_n\}$ is a Cauchy sequence in X and since X is a complete space, there exists $u \in X$ such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$. From (4.2) and $\mathfrak{D}$-Lipschitz continuity of A, B, T, G, we have

$$\begin{aligned}
\|v_{n+1} - v_n\| &\leq \mathfrak{D}(A(u_{n+1}), A(u_n)) \leq \delta_A\|u_{n+1} - u_n\|,\\
\|w_{n+1} - w_n\| &\leq \mathfrak{D}(B(u_{n+1}), B(u_n)) \leq \delta_B\|u_{n+1} - u_n\|,\\
\|q_{n+1} - q_n\| &\leq \mathfrak{D}(T(u_{n+1}), T(u_n)) \leq \delta_T\|u_{n+1} - u_n\|,\\
\|z_{n+1} - z_n\| &\leq \mathfrak{D}(G(u_{n+1}), G(u_n)) \leq \delta_G\|u_{n+1} - u_n\|. \qquad (4.8)
\end{aligned}$$

It is clear from (4.8) that $\{v_n\}, \{w_n\}, \{q_n\}$ and $\{z_n\}$ are also Cauchy sequences in X and so there exist v, w, q, z in X such that $v_n \longrightarrow v, w_n \longrightarrow w, q_n \longrightarrow q$ and $z_n \longrightarrow z$ as $n \longrightarrow \infty$. By using the continuity of operators $A, B, T, G, J^{(I-R)}_{M,\lambda}$ and Algorithm 4.1, we have

$$u = u - z + J^{(I-R)}_{M,\lambda}[(I-R)z + \frac{\lambda}{\tau}(\rho - N(v, w, q))].$$

From Lemma 4.2, we conclude that (u, v, w, q, z) is a solution of problems (3.1). It remain to show that $v \in A(u), w \in B(u), q \in T(u)$ and $z \in G(u)$. In fact

$$\begin{aligned}
d(v, A(u)) &\leq \|v - v_n\| + d(v_n, A(u))\\
&\leq \|v - v_n\| + \mathfrak{D}(A(u_n), A(u))\\
&\leq \|v - v_n\| + \delta_A\|u_n - u\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

Hence $v \in A(u)$. Similarly we can show that $w \in B(u), q \in T(u), z \in G(u)$. This completes the proof. □

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