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REFORMS OF A GENERALIZED KKMF PRINCIPLE

SEHIE PARK*[1]

[1] The National Academy of Sciences, Republic of Korea, Seoul 06579; and Department of Mathematical Sciences, Seoul National University, Seoul 08826, KOREA

ABSTRACT. In 2005, Ben-El-Mechaiekh, Chebbi, and Florenzano obtained a generalization of Ky Fan's 1984 KKM theorem on the intersection of a family of closed sets on non-compact convex sets in a topological vector space. They also extended the Fan-Browder fixed point theorem to multimaps on non-compact convex sets. In this article, we deduce the better abstract versions of such results from a general KKM theorem on abstract convex spaces in our previous works.

KEYWORDS:KKM theorem, Fan's 1961 KKM lemma, 1984 KKM theorem, Fan-Browder fixed point theorem, abstract convex space, (partial) KKM space.
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1. Introduction

The KKM theory, first called by the author in 1992, is the study on applications of any equivalent or extended formulations of the KKM theorem due to Knaster, Kuratowski, and Mazurkiewicz [8] in 1929. The KKM theorem is one of the most well-known and important existence principles and provides the foundations for many of the modern essential results in diverse areas of mathematical sciences.

The KKM theory was originally devoted to convex subsets of topological vector spaces mainly by Ky Fan and Granas, and later to the so-called convex spaces by Lassonde [9], to c-spaces (or H-spaces) by Horvath [7] and others, to generalized convex (G-convex) spaces mainly by the present author. Since 2006, we proposed new concepts of abstract convex spaces and (partial) KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory. Consequently we have obtained a large number of new results in such frame; see [10, 14, 19].

Recall that a milestone on the history of the KKM theory was erected by Fan in 1961 [4]. His 1961 KKM Lemma (or the Fan-KKM theorem or the KKMF principle [2]) extended the KKM theorem to arbitrary topological vector spaces and was

* *Corresponding author.*
Email address : park35@snu.ac.kr; sehiepark@gmail.com.
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applied to various problems in his subsequent papers. Moreover, his lemma was extended in 1979 and 1984 [5, 6] to his 1984 KKM theorem with a new coercivity (or compactness) condition for noncompact convex sets; see [10, 14, 19].

In 1993, we introduced generalized convex (G-convex) spaces $(E, D; \Gamma)$ [20] and, in 1998, we derived new concept of them removing the original monotonicity restriction; see [10, 14, 19]. Motivated by our original G-convex spaces in 1993, Ben-El-Mechaiekh, Chebbi, Florenzano, and Llinares [1] in 1998 introduced L-spaces (E, Γ) and claimed incorrectly that G-convex spaces are particular to their L-spaces. Since then a number of authors followed the misconception of [1] and published incorrect obsolete articles even after we established the KKM theory on abstract convex spaces in 2006–2010.

In 2005, Ben-El-Mechaiekh, Chebbi, and Florenzano [2] obtained a generalization of Ky Fan's 1984 KKM theorem [6] on the intersection of a family of closed sets on non-compact convex sets in a topological vector space. They also extended the Fan-Browder fixed point theorem to multimaps on non-compact convex sets. This type of studies also followed by many others whom may adequately called the L-space theorists.

In our previous work [18] in 2013, we obtained some results on generalizations on those in [2] based on our theory of abstract convex spaces. The present article is a continuation and supplement of [18] and aims to improve some contents of [18] and [2].

Section 2 is devoted to a short history of KKM type theorems. It begins with the original KKM theorem and ends with one of our most general extension in our recent works on abstract convex theory. In Section 3, we introduce some coercing families extending the one in [2] and show that they are complicated forms of a simple consequence of the coercivity due to S. Y. Chang [3] early in 1989. Section 4 deals with reforms or extensions of some results of [2] and [18].

For preliminaries on the KKM theoretic terminology on abstract convex spaces, see our previous work [18, 19].

2. The KKMF principle and generalizations

In 1929, Knaster, Kuratowski, and Mazurkiewicz [8] obtained the following so-called KKM theorem from the weak Sperner lemma in 1928:

Theorem 2.1. *(KKM [8]) Let A_i $(0 \le i \le n)$ be $n+1$ closed subsets of an n-simplex $p_0 p_1 \cdots p_n$. If the inclusion relation*

$$p_{i_0} p_{i_1} \cdots p_{i_k} \subset A_{i_0} \cup A_{i_1} \cup \cdots \cup A_{i_k}$$

holds for all faces $p_{i_0} p_{i_1} \cdots p_{i_k}$ $(0 \le k \le n, 0 \le i_0 < i_1 < \cdots < i_k \le n)$, then $\bigcap_{i=0}^{n} A_i \neq \emptyset$.

Later it was known that this holds also for open subsets instead of closed ones; see [10, 19].

From 1961, Ky Fan showed that the KKM theorem provides the foundations for many of the modern essential results in diverse areas of mathematical sciences. Actually, a milestone of the history of the KKM theory was erected by Fan [4]. He extended the KKM theorem to arbitrary topological vector spaces and applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a compact convex set into a uniform space.

Lemma 2.2. *(Fan [4]) Let X be an arbitrary set in a Hausdorff topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *The convex hull of a finite subset $\{x_1, \cdots, x_n\}$ of X is contained in $\bigcup_{i=1}^n F(x_i)$.*
(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

This is usually known as the 1961 KKM Lemma of Ky Fan (or the Fan-KKM theorem or the KKMF principle [2]). Later the Hausdorffness of Y was known to be superfluous.

Moreover, Fan [5, 6] in 1984 introduced a KKM theorem with a more general coercivity (or compactness) condition for noncompact convex sets as follows:

Theorem 2.3. *(Fan [6]) In a Hausdorff topological vector space, let Y be a convex set and $\emptyset \neq X \subset Y$. For each $x \in X$, let $F(x)$ be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^n F(x_i)$. If there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x \in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y, then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

From now on, in this Section, *all theorems holds for a topological vector space $E = X$, its nonempty subset D, and $\Gamma : \langle D \rangle \multimap E$ is the convex hull operation.*

The following particular form of Lassonde [[9], Theorem I] for $X = Y$ in 1983 extends the 1984 theorem of Fan:

Theorem 2.4. *(Lassonde [9]) Let D be an arbitrary set in a convex space X, and $F : D \multimap X$ be a multimap having the following properties*

(i) *for each $x \in D$, $F(x)$ is compactly closed in X;*
(ii) *F is a KKM map, that is, for any finite subset $\{x_1, \ldots, x_n\}$ of D,*

$$co\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^n F(x_i);$$

(iii) *For some c-compact subset $L \subset X$, $\bigcap\{F(x) \mid x \in L \cap D\}$ is compact.*

Then $\bigcap\{F(x) | x \in D\} \neq \emptyset$.

In 1989, S. Y. Chang [3] obtained the following theorem which eliminated the concept of c-compact sets in Theorem 2.4

Theorem 2.5. *(Chang [3]) Let D be a nonempty subset of a convex space X and $F : D \multimap X$ be a multimap. Suppose that*

(i) *for each $x \in D$, $F(x)$ is closed in X;*
(ii) *F is a KKM map;*
(iii) *there exist a nonempty compact subset K of X and, for each finite subset N of D, a compact convex subset L_N of X containing N such that*

$$L_N \cap \bigcap\{F(x) \mid x \in L_N \cap D\} \subset K.$$

Then $\bigcap\{F(x) \mid x \in D\} \neq \emptyset$.

For a long period, the present author tried to unify hundreds of generalizations of the KKM type theorems and, finally, obtained the following standard forms in [11, 12]:

Theorem 2.6. *Let $(E, D; G)$ be a partial KKM space [resp. KKM space], and $G : D \multimap E$ be a multimap satisfying*

(1) *G has closed [resp. open] values; and*
(2) *$\Gamma_N \subset G(N)$ for any $N \in \langle D \rangle$ (that is, G is a KKM map).*

Then $\{G(y)\}_{y\in D}$ has the finite intersection property. Further, if

(3) *$\bigcap_{y\in M} \overline{G(y)}$ is compact for some $M \in \langle D \rangle$, then we have*

$$\bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Actually the first part of Theorem 2.6 is a definition.

Consider the following related four conditions due to Luc et al. in 2010 for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y\in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y\in D} G(y) \neq \emptyset$.
(b) $\bigcap_{y\in D} \overline{G(y)} = \overline{\bigcap_{y\in D} G(y)}$ (G is *intersectionally closed-valued*)
(c) $\bigcap_{y\in D} \overline{G(y)} = \bigcap_{y\in D} G(y)$ (G is *transfer closed-valued*).
(d) G is closed-valued.

From the definition of $\mathfrak{KC}$-maps, we have a whole intersection property of the Fan type under certain 'coercivity' conditions. The following is given in [15, 16, 17, 19]:

Theorem 2.7. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, Z)$, and $G : D \multimap Z$ a multimap such that*

(1) *$\overline{G}$ is a KKM map w.r.t. F; and*
(2) *there exists a nonempty compact subset K of Z such that either*
(i) *$K \sup \bigcap\{\overline{G(y)} | y \in M\}$ for some $M \in \langle D \rangle$; or*
(ii) *for each $N \in \langle D \rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, $\overline{(L_N)}$ is compact, and*

$$K \sup \overline{(L_N)} \cap \bigcap\{\overline{G(y)} | y \in D'\}.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap \overline{G(y)} | y \in D\} \neq \emptyset.$$

Furthermore,

(α) *if G is transfer closed-valued, then $\overline{F(E)} \cap K \cap \bigcap\{G(y) | y \in D\} \neq \emptyset$; and*
(β) *if G is intersectionally closed-valued, then $\bigcap\{G(y) | y \in D\} \neq \emptyset$.*

Remark 2.8. 1. The coercivity (ii) is originated from S. Y. Chang [6] in 1989.

2. Taking $\overline{K}$ instead of K, we may assume K is closed and the closure notations in (i) and (ii) can be erased.
3 In our previous work [16, 17, 19], we showed that a particular form of Theorem 2.7 unifies several important KKM type theorems appeared in history.
4 Many particular forms of Theorem 2.7 have equivalent formulations or lead many KKM theoretic results.

3. Various coercing families

In 2005, Ben-El-Mechaiekh, Chebbi, and Florenzano [2] obtained a generalization of Ky Fan's 1984 KKM theorem [6] on the intersection of a family of closed sets on non-compact convex sets in a topological vector space. They also extended the

Fan-Browder fixed point theorem to multimaps on non-compact convex sets. This type of studies also followed by the L-space theorists.

The following is given by Ben-El-Mechaiekh, Chebbi, and Florenzano [[2], Definition 2.1]:

[A] ([2]) Consider a subset X of a Hausdorff topological vector space E and a topological space Z. A family $\{(D_i, K_i)\}_{i\in I}$ of pairs of sets is said to be *coercing* for a map $F : X \multimap Z$ if and only if:

(i) for each $i \in I$, D_i is contained in a compact convex subset of X, and K_i is a compact subset of Z;

(ii) for each $i, j \in I$, there exists $k \in I$ such that $D_i \cup D_j \subset D_k$;

(iii) for each $i \in I$, there exists $k \in I$ with $\bigcap_{x\in D_k} F(x) \subset K_i$.

If I is a singleton, the family is called a *single* coercing family. Note that $(E \supset X; \mathrm{co})$ is a G-convex space and that (ii) will be shown redundant.

Motivated by [2], we defined the following [18]

[B] Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. We say that a map $G : D \multimap Z$ has a *coercing family* $\{(D_i, K_i)\}_{i\in I}$ if and only if

(1) for each $i \in I$, K_i is a compact subset of Z and $D_i \subset D$ such that, for each $N \in \langle D\rangle$, there exist a compact subset L^i_N of E that is Γ-convex relative to $D_i \cup N$;

(2) for each $i \in I$, there exists $k \in I$ with $\bigcap_{y\in D_k} F(y) \subset K_i$.

Remark 3.1. In [2], it is noted that the condition (2) holds *if and only if* the 'dual' map $\Phi : Z \multimap X$ of F, defined by $\Phi(z) = X \setminus F^{-}(z),\ z \in Z$, verifies

$(2)'$ $\forall i \in I,\ \exists k \in I,\ \forall z \in Z \setminus K_i,\quad \Phi(z) \cap C_k \neq \emptyset.$

In [2], there are given several deep examples of condition $(2)'$ related to an exceptional family, an escaping sequence, an attracting trajectory, and others.

The coercivity [**B**] improves [**A**] as follows:

Proposition 3.2. [**A**] $\Longrightarrow$ [**B**].

Proof. Since each D_i is contained in a compact convex subset of $X \subset E$ by [**A**](i) and E is a Hausdorff topological vector space, for each $N \in \langle X\rangle$, there exists a compact convex subset L^i_N of E containing $D_i \cup N$; see Lassonde [9]. Therefore, Condition [**B**](1) holds. Since [**A**](iii) is the same to [**B**](2), all requirements of [**B**] are satisfied. □

Note that Condition [**A**](ii) is redundant for [**B**].

Let us begin with the following particular form of the condition (ii) in Theorem 2.7 with $sG : D \multimap Z$ instead of $G : D \multimap Z$ [18]

[C] Let $(E, D; \Gamma)$ be an abstract convex space, $G : D \multimap E$ a multimap, Z a topological space, and $s : E \longrightarrow Z$ a continuous map such that

(C) there exists a nonempty compact subset K of Z such that, for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and

$$s(L_N) \cap \bigcap_{y\in D'} sG(y) \subset K.$$

Note that $s \in \mathfrak{KC}(E, Z)$. Condition [**C**] appeared as Condition (I) in [18]. The following corrects [[18], Proposition 3.6]:

Proposition 3.3. [**B**] $\Longrightarrow$ [**C**].

Proof. Let $G : D \multimap E$ and $s : E \longrightarrow Z$ be given in [**C**] and let $F = sG : D \multimap Z$. Choose any $i \in I$ by (B)(1), we have K_i and D_i such that, for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset $L_N := L_N^i$ of E relative to $D' = D_i \cup N$. By [**B**](2), we have a $k \in I$ such that

$$\bigcap_{y \in D_k} F(y) = \bigcap_{y \in D_k} sG(y) \subset K_i.$$

Since i was arbitrary, we may assume $k = i$ and $K = K_i$. Moreover, since $D' = D_k \cup N$, we have

$$\bigcap_{y \in D'} sG(y) \subset K \implies s(L_N) \cap \bigcap_{y \in D'} sG(y) \subset K.$$

Hence the coercivity condition [**C**] holds. □

Note that Conditions [**A**], [**B**], and [**C**] are examples of the coercivity (ii) in Theorem 2.7.

4. Generalization of the KKMF principle

In this section, we deduce generalized better forms of the main theorems in [2]:

Theorem 4.1. *Let $(E, D; \Gamma)$ be a partial KKM space. Suppose that*

(1) *$G : D \multimap E$ is a closed-valued KKM map,*
(2) *the coercivity condition [**C**] or [**B**] holds for G.*

Then we have $K \cap \bigcap_{y \in D} G(y) \neq \emptyset$.

Proof. We apply Theorem 2.7 with $F = s$.

(1) Since $s^{-1}G$ is a closed-valued KKM map, Condition [**C**] implies $\Gamma_A \subset R(A) = s^{-1}G(A)$ and $s\Gamma_A \subset sR(A) = G(A)$ for all $A \in \langle D \rangle$. Therefore $\overline{G}$ is a KKM map w.r.t. s.

(2) Condition (2) implies (ii) in Theorem 2.7 with $F = s$ and $G = sR$. Therefore, by the case (ii) of Theorem 2.7, we have

$$s(E) \cap K \cap \bigcap_{y \in D} sR(y) \neq \emptyset.$$

This implies the conclusion. □

Corollary 4.2. *Let E be a Hausdorff topological vector space, Y a convex subset of E, X a non-empty subset of Y, and $F : X \multimap Y$ a KKM map with closed (in Y) values. If F admits a coercing family in the sense of [**A**] without (ii), then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

Proof. Put $E = Y$, $D = X$, and $G = F$ in Theorem 4.1. Since [**A**] $\Longrightarrow$ [**B**] $\Longrightarrow$ [**C**], we have the conclusion by Theorem 4.1. □

The main theorem [[2], Theorem 3.1] of [2] is Corollary 4.2 under the assumption of [**A**] and the compactly closed values of F. However, replacing the topology of Y to compactly generated one (as in k-spaces), we may assume F has closed values.

If $D_i = D$ and $K_i = K$ for all $i \in$, D is contained in a compact convex subset of X and K is a compact subset of Y, then Corollary 4.2 reduces to the 1984 KKM theorem of Ky Fan [6] which in turn generalizes the 1961 KKM Lemma of Ky Fan [4]; see Section 2.

Now we formulate Theorem 4.1 to a fixed point theorem:

Theorem 4.3. *Let* $(X;\Gamma)$ *be a partial KKM space and* $\Phi : X \multimap X$ *be a map with open fibers and non-empty values. If* Φ *admits a coercing family* $[\mathbf{B}]$ *in the sense of Remark 3.1, then the map* $\mathrm{co}_\Gamma\Phi$ *has a fixed point.*

Proof. Suppose that $\mathrm{co}_\Gamma(\Phi)$ has no fixed point, i.e., $x \notin \mathrm{co}_\Gamma(\Phi)(x)$ for all $x \in X$. Define $F : X \multimap X$ by

$$F(x) := \{y \in X \mid x \notin \Phi(y)\}, \quad x \in X.$$

Then F is closed-valued. We claim that F is a KKM map. Suppose that for some $N \in \langle X \rangle$, there exists $z \in \mathrm{co}_\Gamma N$ such that $z \notin F(N)$. Then $N \subset \Phi(z)$ and $z \in \mathrm{co}_\Gamma(\Phi(z))$, which contradicts the assumption that $\mathrm{co}_\Gamma(\Phi)$ has no fixed point. To complete our proof, we remember the coercing family is also a coercing family $[\mathbf{B}]$ with Remark 3.1. Theorem 4.1 implies $\bigcap_{x\in X} F(x) \neq \emptyset$ which contradicts the fact that Φ has non-empty values. □

Corollary 4.4. *Let* X *be a non-empty convex subset of a topological vector space* E *and* $\Phi : X \multimap X$ *be a map with open fibers (in* X*) and non-empty values. If* Φ *admits a coercing family* $[\mathbf{A}]$*(without (ii)) in the sense of Remark 3.1, then the map* $\mathrm{conv}(\Phi)$ *has a fixed point.*

Note that [[2], Theorem 3.2] is Corollary 4.4 under the assumption of $[\mathbf{A}]$ and the compactly open fibers of Φ. However, replacing the topology of Y to compactly generated one (as in k-spaces), we may assume Φ has open fibers. Note that Corollary 4.4 generalize the Fan-Browder fixed point theorem.

As was noted by [2], the results in this section can be used to extend existing results on the solvability of complementarity problems, existence of zero on noncompact domains and existence of equilibria for qualitative games and abstract economies.

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