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COMMON FIXED POINT THEOREMS FOR SOME CONTRACTIVE CONDITION WITH φ-MAPPING IN COMPLEX VALUED B-METRIC SPACES

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ABSTRACT. In this paper, by using the concept of φ-mappings introduced by Mohanta and Maitra [8], we can prove the existence and the uniqueness of common fixed points for some generalized contractive mappings in complex-valued b-metric spaces. Our results extend and improve the results of Tripathi and Dubey [12] and many others.

KEYWORDS:common fixed point, complex-valued metric space, complex valued b-metric space.
AMS Subject Classification: :46C05, 47D03, 47H09, 47H10, 47H20.

1. Introduction

The concept of a metric space was introduced by Frechet in 1906 [7]. Many mathematicians studied the existence and the uniqueness of fixed points by using the Banach contraction principle. The principle was also proved in some generalized metric spaces, see [5].

Fixed point theorems in metric spaces have been studied extensively by many researchers as in [13, 6] and [11]. In 1989, Bakhtin [3] introduced the notion of b-metric spaces. After that, many researchers extended fixed point theorems from metric spaces to b-metric spaces, for example in [1, 2]

In 2011, A. Azam, B. Fisher and M. Khan [2] introduced the notion of complex valued metric spaces and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition. A complex valued metric space is a generalization of the classical metric space. Bhatt et al. [4] have proved a common fixed point theorem for weakly compatible mappings in a complex valued metric space. In 2013, Mohanta and Maitra [8], introduced the concept of common fixed points with φ-mapping in complex valued metric spaces, In 2017, Zada et. al. [14], proved common fixed point theorems in complex valued metric spaces with $(E.A)$ and (CLR) properties.

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The aim of this paper is to introduce some contractive conditions of two mappings by using the concept of φ-mappings and prove the existence and the uniqueness of common fixed points in complex valued b-metric spaces. Therefore, our results are comprehensive the results of [8] and [12].

2. Preliminaries

In this section, we present some definitions and lemmas for using in section 3, and define the definition of b-metric space in the complex plane.

Definition 2.1. Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a metric if for $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.
The pair (X, d) is called a metric space, and d is called a metric on X.

Next, we suppose the definition of b-metric space, this space is generalized than metric spaces.

Definition 2.2. [3] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
The pair (X, d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (X, d).

The following is some example for b-metric spaces.

Example 2.3. [3] Let (X, d) be a metric space. The funcion $\rho(x, y)$ is defined by $\rho(x, y) = (d(x, y))^2$. Then (X, ρ) is a b-metric space with coefficient $s = 2$. This can be seen from the nonnegativity property and triangle inequality of metric to prove the property (iii).

There is a completeness property in real number but on order relation is not welll-defined in complex numbers. Before giving the definition of complex valued metric spaces and complex-valued b-metric spaces, we define partial order in complex numbers (see [9]). Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define partial order relation $\preccurlyeq$ on $\mathbb{C}$ as follows;

$$z_1 \preccurlyeq z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

This means that we would have $z_1 \preccurlyeq z_2$ if and only if one of the following conditions holds:
(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.
If one of the conditions $(ii), (iii)$, and (iv) holds, then we write $z_1 \prec z_2$. From the above partial order relation we have the following remark.

Remark 2.4. We can easily check the following:
(i) If $a, b \in \mathbb{R}, 0 \leq a \leq b$ and $z_1 \preccurlyeq z_2$ then $az_1 \preccurlyeq bz_2, \forall z_1, z_2 \in \mathbb{C}$.
(ii) If $0 \preccurlyeq z_1 \prec z_2$ then $|z_1| < |z_2|$.
(iii) If $z_1 \preccurlyeq z_2$ and $z_2 \prec z_3$ then $z_1 \prec z_3$.
(iv) If $z \in \mathbb{C}$, for $a, b \in \mathbb{R}$ and $a \leq b$, then $az \preccurlyeq bz$.

A b-metric on a b-metric sapce is a funcion having real value. Based on the definition of partial order on complex number, real-valued b-metric can be generalized into compleex-valued b-metric as folllows.

Definition 2.5. [2] Let X be a nonempty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in \mathbb{C}$, the following conditions are satisfied:
(i) $0 \preccurlyeq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \preccurlyeq d(x, y) + d(y, z)$.
Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Next, we give the definition of complex valued b-metric space.

Definition 2.6. [11] Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if, for all $x, y, z \in \mathbb{C}$, the following conditions are satisfied:
(i) $0 \preccurlyeq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \preccurlyeq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a complex valued b-metric space. We see that if $s = 1$ then (X, d) is complex valued metric space which is defined in Definition 2.5. The following example is some example of complex valued b-metric space.

Example 2.7. [11] Let $X = \mathbb{C}$. Define the mapping $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$ for all $x, y \in X$. Then $(\mathbb{C}, d)$ is complex valued b-metriic space with $s = 2$.

Definition 2.8. [10] Let (X, d) be a complex valued b-metric space.
(i) A point $x \in X$ is called interior point of set $A \subseteq X$ if there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in Y : d(x, y) \prec r\} \subseteq A.$$

(ii) A point $x \in X$ is called limit point of a set A if for every $0 \prec r \in \mathbb{C}, B(x, r) \cap (A - x) \neq \emptyset$
(iii) A subset $A \subseteq X$ is open if each element of A is an interior point of A.
(iv) A subset $A \subseteq X$ is closed if each limit point of A is contained in A.

Definition 2.9. [10] Let (X, d) be complex valued b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$.
(i) The sequence $\{x_n\}$ is converges to $x \in X$ if for every $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, x) \prec r$. Thus x is the limit of (x_n) and we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(ii) The sequence $\{x_n\}$ is said to be a Cauchy sequence if for ever $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, x_{n+m}) \prec r$, where $m \in \mathbb{N}$.
(iii) If for every Cauchy sequence in$X$ is convergent, then (X, d) is said to be a complete complex valued b-metric space.

Definition 2.10. [8] Let $P = \{z \in \mathbb{C} : re(z) \geqslant 0 \text{ and } Im(z) \geqslant 0\}$. A nondecreasing mapping $\varphi : P \to P$ is called a φ-mapping if

(i) $\varphi(0) = 0$ and $0 \prec \varphi(z) \prec z$ for $z \in P - \{0\}$;
(ii) $\varphi(z) \prec z$ for every $z \succ 0$;
(iii) $\lim_{n\to\infty} \varphi^n(z) = 0$ for every $z \in P - \{0\}$.

Lemma 2.11. [10] *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.12. [10] *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

3. Main Results

In this section, we define some contraction by using a φ-mapping, and prove the existence and uniqueness of common fixed point theorem in a complete complex valued b-metric space.

Theorem 3.1. *Let (X,d) be a complete complex valued b-metric space and the mappings $S,T:X\to X$ are self mappings satisfying the condition*

$$d(Sx,Ty)\preccurlyeq\varphi[\lambda\frac{d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)}{d(x,Ty)+d(y,Sx)}+\mu d(x,y)] \tag{3.1}$$

for all $x,y\in X$, where $d(x,Ty)+d(y,Sx)\neq 0$ and λ,μ are nonnegative reals with the condition $\lambda+\mu<1$. If φ is continuous then S and T has a unique common fixed point.

Proof. Let $x_0\in X$ be an arbitrary point. We define

$$\begin{aligned} x_{2n+1} &= Sx_{2n} \text{ and} \\ x_{2n+2} &= Tx_{2n+1}, n=0,1,2,3,... \end{aligned}$$

By equations (3.1) and (3.2), we consider

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) &= d(Sx_{2n},Tx_{2n+1}) \\ &\preccurlyeq \varphi[\lambda\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tx_{2n+1})+d(x_{2n+1},Tx_{2n+1})d(x_{2n+1},Sx_{2n})}{d(x_{2n},Tx_{2n+1})+d(x_{2n+1},Sx_{2n})} \\ &\quad +\mu d(x_{2n},x_{2n+1})] \\ &\preccurlyeq \varphi[\lambda\frac{d(x_{2n},x_{2n+1})d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+2})d(x_{2n+1},x_{2n+1})}{d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})} \\ &\quad +\mu d(x_{2n},x_{2n+1})]. \end{aligned} \tag{3.2}$$

From $\lambda+\mu<1$ and (3.2), we have

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) &= \varphi[\lambda\frac{d(x_{2n},x_{2n+1})d(x_{2n},x_{2n+2})}{d(x_{2n},x_{2n+2})}+\mu d(x_{2n},x_{2n+1})] \\ &\preccurlyeq \varphi[\lambda d(x_{2n},x_{2n+1})+\mu d(x_{2n},x_{2n+1})] \\ &= \varphi[(\lambda+\mu)d(x_{2n},x_{2n+1})] \\ &\preccurlyeq \varphi[d(x_{2n},x_{2n+1})]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} d(x_{2n},x_{2n+1}) &= \varphi[\lambda\frac{d(x_{2n-1},x_{2n})d(x_{2n-1},x_{2n+1})}{d(x_{2n-1},x_{2n+1})}+\mu d(x_{2n-1},x_{2n})] \\ &\preccurlyeq \varphi[\lambda d(x_{2n-1},x_{2n})+\mu d(x_{2n-1},x_{2n})] \\ &= \varphi[(\lambda+\mu)d(x_{2n-1},x_{2n})] \\ &\preccurlyeq \varphi[d(x_{2n-1},x_{2n})]. \end{aligned}$$

By mathematical induction, implies that

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) &\preccurlyeq \varphi(\varphi(\varphi(\cdots\varphi(d(x_0,x_1))))) \\ &= \varphi^{2n+1}d(x_0,x_1). \end{aligned} \tag{3.3}$$

From (3.3) and Definition 2.10, we conclude that

$$d(x_{n+1},x_{n+2}) \preccurlyeq \varphi^{n+1}d(x_0,x_1) \tag{3.4}$$

So, for $m>n$ and Definition 2.6, we consider

$$\begin{aligned} d(x_n,x_{n+m}) &\preccurlyeq s[d(x_n,x_{n+1})+d(x_{n+1},x_{n+m})] \\ &\preccurlyeq sd(x_n,x_{n+1})+sd(x_{n+1},x_{n+m}) \\ &\preccurlyeq sd(x_n,x_{n+1})+s[s(d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+m}))] \\ &\preccurlyeq sd(x_n,x_{n+1})+s^2(d(x_{n+1},x_{n+2})+s^2d(x_{n+2},x_{n+m}) \\ &\preccurlyeq sd(x_n,x_{n+1})+s^2(d(x_{n+1},x_{n+2})+s^2d(x_{n+2},x_{n+m}) \end{aligned}$$

$$
\begin{aligned}
& +s^{n+m-1}d(x_{n+m-2},x_{n+m-1})+...+s^{m}d(x_{n+m-1},x_{n+m}) \\
\precsim\ & sd(x_n,x_{n+1})+s^2(d(x_{n+1},x_{n+2})+s^3d(x_{n+2},x_{n+3})+s^3d(x_{n+3},x_{n+m}) \\
& +...+s^{n+m-1}d(x_{n+m-2},x_{n+m-1})+s^{m}d(x_{n+m-1},x_{n+m}) \\
\precsim\ & s\varphi^n d(x_0,x_1)+s^2\varphi^{n+1}(d(x_0,x_1)+s^3d(x_0,x_1)+s^3\varphi^{n+2}d(x_0,x_1) \\
& +...+s^{n+m-1}\varphi^{n+m-2}d(x_0,x_1)s^m\varphi^{n+m-1}d(x_0,x_1) \\
=\ & [s\varphi^n+s^2\varphi^{n+1}+s^3+s^3\varphi^{n+2}+s^{n+m-1}\varphi^{n+m-2}+s^m\varphi^{n+m-1}]d(x_0,x_1).
\end{aligned} \tag{3.5}
$$

From remark 2.4 (ii), we have

$$
|d(x_n,x_{n+m})| \leqslant [s\varphi^n+s^2\varphi^{n+1}+s^3+s^3\varphi^{n+2}+\cdots+s^{n+m-1}\varphi^{n+m-2}+s^m\varphi^{n+m-1}]|d(x_0,x_1)|. \tag{3.6}
$$

From (3.4), (3.6) and Taking $n\to\infty$, it follows that $|d(x_n,x_{n+m})|\to\infty$.

By Lemma 2.12, implies that $\{x_n\}$ is a cauchy sequence in X. Since X is complete, there exists $u\in X$ such that $x_n\to u$. Now, we show that u is a fixed point of T and S. Consider,

$$
\begin{aligned}
d(Su,x_{2n+2}) &= d(Su,Tx_{2n+1}) \\
&\precsim \varphi(\lambda\frac{d(u,Su)d(u,Tx_{2n+1})+d(x_{2n+1},Tx_{2n+1})d(x_{2n+1},Su)}{d(u,Tx_{2n+1})+d(x_{2n+1},Su)}+\mu d(u,x_{2n+1})) \\
&\precsim \varphi(\lambda\frac{d(u,Su)d(u,x_{2n+2})+d(x_{2n+1},x_{2n+2})d(x_{2n+1},Su)}{d(u,x_{2n+2})+d(x_{2n+1},Su)}+\mu d(u,x_{2n+1})).
\end{aligned} \tag{3.7}
$$

From φ is a continuous, (3.7), $x_n\to u$ as $n\to\infty$ and Definition 2.6 (1), we have

$$
\begin{aligned}
d(Su,u) &= \lim_{n\to\infty} d(Su,x_{2n+1}) \\
&\precsim \lim_{n\to\infty}\varphi(\lambda\frac{d(u,Su)d(u,x_{2n+2})+d(x_{2n+1},x_{2n+2})d(x_{2n+1},Su)}{d(u,x_{2n+2})+d(x_{2n+1},Su)}+\mu d(u,x_{2n+1})) \\
&= \varphi(\lim_{n\to\infty}(\lambda\frac{d(u,Su)d(u,x_{2n+2})+d(x_{2n+1},x_{2n+2})d(x_{2n+1},Su)}{d(u,x_{2n+2})+d(x_{2n+1},Su)}+\mu d(u,x_{2n+1}))) \\
&= \varphi(\lambda\frac{d(u,Su)d(u,u)+d(u,u)d(u,Su)}{d(u,u)+d(u,Su)}+\mu d(u,u)) \\
&= \varphi(0)=0.
\end{aligned} \tag{3.8}
$$

Thus $u=Su$. Hence u is a fixed point of S. Next, we show that u is a fixed point of T. Consider,

$$
\begin{aligned}
d(x_{2n+1},Tu) &= d(Sx_{2n},Tu) \\
&\precsim \varphi(\lambda\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tu)+d(u,Tu)d(u,Sx_{2n})}{d(x_{2n},Tu)+d(u,Sx_{2n})}+\mu d(x_{2n},u)) \\
&= \varphi(\lambda\frac{d(x_{2n},x_{2n+1})d(x_{2n},Tu)+d(u,Tu)d(u,x_{2n+1})}{d(x_{2n},Tu)+d(u,x_{2n+1})}+\mu d(x_{2n},u)).
\end{aligned} \tag{3.9}
$$

From φ is a continuous, (3.9), $x_n\to u$ as $n\to\infty$ and Definition 2.6 (1), we have

$$
\begin{aligned}
d(u,Tu) &= \lim_{n\to\infty} d(x_{2n+1},Tu) \\
&\precsim \lim_{n\to\infty}\varphi(\lambda\frac{d(x_{2n},x_{2n+1})d(x_{2n},Tu)+d(u,Tu)d(u,x_{2n+1})}{d(x_{2n},Tu)+d(u,x_{2n+1})}+\mu d(x_{2n},u)) \\
&= \varphi(\lim_{n\to\infty}(\lambda\frac{d(x_{2n},x_{2n+1})d(x_{2n},Tu)+d(u,Tu)d(u,x_{2n+1})}{d(x_{2n},Tu)+d(u,x_{2n+1})}+\mu d(x_{2n},u))) \\
&= \varphi(\lambda\frac{d(u,u)d(u,Tu)+d(u,Tu)d(u,u)}{d(u,Tu)+d(u,u)}+\mu d(u,u)) \\
&= \varphi(0)=0.
\end{aligned} \tag{3.10}
$$

Thus $u = Tu$. Hence u is a fixed point of T. Therefore, u is a common fixed point of S and T. Finally, we prove the uniqueness of common fixed point of S and T. Suppose that v is a common fixed point of S and T. So $Sv = v = Tv$. Now, we show that $u = v$. Assume that $u \neq v$, we consider

$$\begin{aligned} d(u,v) &= d(Su,Tv) \\ &\precsim \varphi(\lambda\frac{d(u,Su)d(u,Tv)+d(v,Tv)d(v,Su)}{d(u,Tv)+d(v,Su)}+\mu d(u,v)) \\ &= \varphi(\lambda\frac{d(u,u)d(u,v)+d(v,v)d(v,u)}{d(u,v)+d(v,u)}+\mu d(u,v)) \\ &= \varphi(\mu d(u,v)). \end{aligned} \tag{3.11}$$

Since $\mu < 1$, we have $\mu d(u,v) < d(u,v)$. By Definition 2.10 (2) and φ is a nondecreasing, we have

$$d(u,v) \precsim \varphi(\mu d(u,v)) \precsim \varphi(d(u,v)) \prec d(u,v). \tag{3.12}$$

From remark 2.4 (ii), taking absolute value of both side, we have

$$|d(u,v)| < |d(u,v)|.$$

It is a contradiction. We can conclude that $u = v$. Therefore u is a uniqueness common fixed point of S and T. □

From Theorem 3.1, we have the parallel result with the result of Dubey et. al [12] as following.

Corollary 3.2. *Let (X,d) be a complete complex valued b-metric space and the mappings $S,T : X \to X$ satisfy the condition*

$$d(Sx,Ty) \precsim \lambda\frac{d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)}{d(x,Ty)+d(y,Sx)}+\mu d(x,y) \tag{3.13}$$

for all $x,y \in X$, where $d(x,Ty)+d(y,Sx) \neq 0$ and λ,μ are nonnegative reals with $\lambda+\mu < 1$. If either S or T is continuous and the pair (S,T) is compatible, then S and T has a unique common fixed point.

Proof. If $\varphi = I$ is an identity mapping, then (3.1) reduces to (3.13) and suppose one of S and T is continuous and the pair (S,T) is compatible, then S and T has a unique common fixed point. This completes the proof. □

Theorem 3.3. *Let (X,d) be a complete complex valued b-metric space and the mappings $S,T : X \to X$ are self mappings satisfying the condition*

$$d(S^n x,T^n y) \precsim \varphi[\lambda\frac{d(x,S^n x)d(x,T^n y)+d(y,T^n y)d(y,S^n x)}{d(x,T^n y)+d(y,S^n x)}+\mu d(x,y)] \tag{3.14}$$

for all $x,y \in X$, $n \geq 1$, where $d(x,Ty)+d(y,Sx) \neq 0$ and λ,μ are nonnegative reals with the condition $\lambda+\mu < 1$. If φ is continuous then S and T has a unique common fixed point.

Proof. Suppose $A = S^n$ and $B = T^n$, by Theorem 3.1, there exists a common fixed point u of A and B, such that

$$Au = u = Bu.$$

Thus $S^n u = u$ and $T^n u = u$. We claim that $Su = u$. Assume that $Su \neq u$, we have

$$\begin{aligned} d(Su,u) &= d(S(S^n u),T^n u) \\ &= d(S^n(Su),T^n u) \\ &\precsim \varphi\left[\lambda\frac{d(Su,S^n(Su))d(Su,T^n u)+d(u,T^n u)d(u,S^n(Su))}{d(Su,T^n u)+d(u,S^n(Su))}+\mu d(Su,u)\right] \\ &= \varphi\left[\lambda\frac{d(Su,S(S^n u))d(Su,T^n u)+d(u,T^n u)d(u,S(S^n u))}{d(Su,T^n u)+d(u,S(S^n u))}+\mu d(Su,u)\right] \\ &= \varphi\left[\lambda\frac{d(Su,Su)d(Su,u)+d(u,u)d(u,Su)}{d(Su,u)+d(u,Su)}+\mu d(Su,u)\right] \end{aligned}$$

$$= \varphi\left[\mu d(Su,u)\right].$$

From Definition 2.10, we have $d(Su,u) \prec \mu d(Su,u)$. A contradiction, because $\mu < 1$. Hence, $Su = u$. Next, we claim that $Tu = u$. Assume that $Tu \neq u$, we have

$$\begin{aligned} d(u,Tu) &= d(S^n u, T(T^n u)) \\ &= d(S^n u, T^n(Tu)) \\ &\precsim \varphi\left[\lambda \frac{d(u,S^n u)d(u,T^n(Tu)) + d(Tu,T^n(Tu))d(Tu,S^n u)}{d(u,T^n(Tu)) + d(Tu,S^n u)} + \mu d(u,Tu)\right] \\ &= \varphi\left[\lambda \frac{d(u,S^n u)d(u,T(T^n u)) + d(Tu,T(T^n u))d(Tu,S^n u)}{d(u,T(T^n u)) + d(Tu,S^n u)} + \mu d(u,Tu)\right] \\ &= \varphi\left[\lambda \frac{d(u,u)d(u,Tu) + d(Tu,Tu)d(Tu,u)}{d(u,Tu) + d(Tu,u)} + \mu d(u,Tu)\right] \\ &= \varphi\left[\mu d(u,Tu)\right]. \end{aligned}$$

From Definition 2.10, we have $d(u,Tu) \prec \mu d(u,Tu)$. A contradiction, because $\mu < 1$. Hence, $Tu = u$. Hence u is a common fixed point of S and T.

Finally, we show that u is a unique fixed point of S and T. Let v be a common fixed point of S and T, thus $S^n v = v = T^n v$. We must show that $u = v$. Assume that $u \neq v$, we have

$$\begin{aligned} d(u,v) &= d(S^n u, T^n v) \\ &\precsim \varphi\left[\lambda \frac{d(u,S^n u)d(u,T^n v) + d(v,T^n v)d(v,S^n u)}{d(u,T^n v) + d(v,S^n u)} + \mu d(u,v)\right] \\ &= \varphi\left[\lambda \frac{d(u,u)d(u,v) + d(v,v)d(v,u)}{d(u,v) + d(v,u)} + \mu d(u,v)\right] \\ &= \varphi\left[\mu d(u,v)\right]. \end{aligned}$$

From Definition 2.10, we have $d(u,v) \prec \mu d(u,v)$. A contradiction, because $\mu < 1$. Hence, $u = v$. Therefore, u is a unique common fixed point of S and T. □

Example 3.4. Let $X = \mathbb{C}$. Define a function $d : X \times X \to \mathbb{C}$ such that

$$d(z_1,z_2) = |x_1 - x_2|^2 + i|y_1 - y_2|^2,$$

where $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$.

From Example 18 in [9], it implied that (X,d) is a complete complex valued b-metric space with $s = 2$. Now, we define two self-mappings $S,T : X \to X$ as follows:

$$Sz = \begin{cases} 0 & \text{if } a,b \in \mathbb{Q}, \\ 2 & \text{if } a \in \mathbb{Q}^C, b \in \mathbb{Q} \\ 2i & \text{if } a,b \in \mathbb{Q}^C \\ 2+2i & \text{if } a \in \mathbb{Q}, b \in \mathbb{Q}^C \end{cases} \quad \text{and } Tz = \begin{cases} 0 & \text{if } a,b \in \mathbb{Q}, \\ 1 & \text{if } a \in \mathbb{Q}^C, b \in \mathbb{Q} \\ i & \text{if } a,b \in \mathbb{Q}^C \\ 1+i & \text{if } a \in \mathbb{Q}, b \in \mathbb{Q}^C \end{cases}$$

where $z = a + bi \in X$. We see that $S^n z = 0 = T^n z$ for $n > 1$, so

$$d(S^n x, T^n y) = 0 \precsim \lambda \frac{d^2(x,y)}{1 + d(x,y)} + \mu d(y,T^n y) + \rho d(x,S^n x),$$

for all $x,y \in X$ and $\lambda,\mu,\rho \geq 0$ with $2(\lambda + \rho) + \mu < 1$. So all conditions of Theorem 3.3 are satisfied to get a unique common fixed point 0 of S and T.

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