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RADIUS OF THE PERTURBATION OF THE OBJECTIVE FUNCTION PRESERVES THE KKT CONDITION IN CONVEX OPTIMIZATION

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ABSTRACT. The problem to find the maximum radius of the perturbation of the objective function which preserves the KKT condition at a feasible point is studied. The maximum radius of the problem is described, and certain values concerned with the extreme direction of a positive polar cone of the union of the subdifferentials of the active constraint functions at the point are observed.

KEYWORDS: convex optimization problem, KKT optimality condition, the basic constraint qualification, extreme direction
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1. Introduction

We study stability of KKT condition for the following convex optimization problem:

$$\text{(P)} \quad \begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, i \in I \end{array} \tag{1.1}$$

where I is a non-empty index set, $f, g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ are proper convex functions, $i \in I$, and assume that the constraint set $S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \ \forall i \in I\}$ is not empty. Under some regularity condition for the family of constraint functions $\{g_i, i \in I\}$, which is called constraint qualification, if $\bar{x}$ is a minimizer of (P), then the KKT condition holds, that is, there exists a finite $J \subset I(\bar{x})$ and

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$\lambda_j \geq 0$, $j \in J$, called KKT multipliers, such that

$$0 \in \partial f(\bar{x}) + \sum_{j \in J} \lambda_j \partial g_j(\bar{x}), \tag{1.2}$$

where $I(\bar{x}) = \{i \in I \mid g_i(\bar{x}) = 0\}$. The most famous constraint qualification is the Slater condition (1.3). Assume that I is a non-empty finite index set, and all g_i are real-valued convex functions. If the Slater condition

$$\exists x_0 \in \mathbb{R}^n \ s.t. \ g_i(x_0) < 0, \ \forall i \in I, \tag{1.3}$$

is satisfied, then the KKT condition holds for any real-valued convex function f. There are many results about constraint qualification for the KKT condition, and the basic constraint qualification (BCQ in short) is called a necessary and sufficient constraint qualification from the following result:

Theorem 1.1 ([4]; cf. [3])**.** *Let I be a non-empty index set, $g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper convex functions, $i \in I$, and assume that $\bar{x} \in S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \ \forall i \in I\}$. Then the following two statements are equivalent:*

(i) *The family $\{g_i : i \in I\}$ satisfies the BCQ at $\bar{x}$, that is,*

$$N_S(\bar{x}) = \operatorname{cone} \operatorname{co} \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x})$$

holds,

(ii) *For each convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\bar{x}$ is a minimizer of (1.1) with $\operatorname{cl} S$, the closure of S, in place of S if and only if there exists a finite subset $J \subset I(\bar{x})$ and $\lambda_j \geq 0$, $j \in J$, such that (1.2) holds.*

Constraint qualifications guarantees the KKT condition holds when $\bar{x}$ is a minimizer of (1.1) for every convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. However, the conditions do not always hold. If $\{g_i, i \in I\}$ does not satisfy the BCQ at $\bar{x} \in S$, then there exists a convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\bar{x}$ is a minimizer of (1.1) with $\operatorname{cl} S$ in place of S, but there does not exist a finite subset $J \subset I(\bar{x})$ and $\lambda_j \geq 0$, $j \in J$, such that (1.2) holds. We are interested in when the KKT condition holds at $\bar{x}$ under the luck of such constraint qualifications, for example, see [6].

The purpose of this paper is to consider the following problem under the KKT condition holds but the BCQ does not hold at $\bar{x}$ for given f, g_i in (1.1):

$$\begin{array}{ll} \text{Maximize} & r \geq 0 \\ \text{subject to} & \text{the KKT condition holds at } \bar{x} \text{ in (1.1) with} \\ & \text{the objective function } f + \langle c, \cdot \rangle \text{ whenever } \|c\| \leq r. \end{array} \tag{1.4}$$

The problem is the same to find the maximum radius of the perturbation of the objective function which preserves the KKT condition at $\bar{x}$. The problem is motivated from [5]; under some conditions,

$$\partial f(\bar{x}) \subset \operatorname{int} (\operatorname{cone} \operatorname{co} \{u_1, \cdots, \ u_n\})$$

holds for any $u_i \in -\partial g_{t_i}(\bar{x})$, $i = 1, \ldots, n$. The paper is organized as follows: In section 2, we describe preliminary results about the notions of extreme point and extreme direction, and give a characterization of extreme direction which is a similar result to extreme point. In section 3, we describe the maximum radius of the problem (1.4), and observe the maximum value and other related values include the notion of extreme direction. Finally we give a conclusion in section 4.

2. Preliminaries

For a convex set $C \subset \mathbb{R}^n$, $x \in C$ is called an extreme point if there does not exist $x_1, x_2 \in C$, $\lambda \in (0,1)$ such that $x_1 \neq x_2$ and $x = (1-\lambda)x_1 + \lambda x_2$, or equivalently, $C \setminus \{x\}$ is convex. Denote $\operatorname{ext} C$ the set of all extreme points of C. For a convex cone $C \subset \mathbb{R}^n$, $x \in C$ is called an extreme direction if $x \neq 0$ and for all $x_1, x_2 \in C$ such that $x = x_1 + x_2$, we have $x_1, x_2 \in \mathbb{R}_+ x$, where $\mathbb{R}_+ x = \{tx \mid t \geq 0\}$. Denote $\operatorname{extd} C$ the set of all extreme directions of C. We obtain a similar result to extreme point for extreme direction as follows:

Proposition 2.1. *Assume that convex cone C is pointed, that is, $C \cap (-C) = \{0\}$. For any $x \in C \setminus \{0\}$, $x \in \operatorname{extd} C$ if and only if $C \setminus \mathbb{R}_+ x$ is convex.*

Proof. Assume that $x \in \operatorname{extd} C$. For any $x_1, x_2 \in C \setminus \mathbb{R}_+ x$ and $\alpha \in (0,1)$, it is clear that $(1-\alpha)x_1 + \alpha x_2 \in C$. If $(1-\alpha)x_1 + \alpha x_2 \in \mathbb{R}_+ x$, then $(1-\alpha)x_1 + \alpha x_2 = tx$ for some $t \geq 0$. If $t > 0$, since

$$\frac{1-\alpha}{t}x_1 + \frac{\alpha}{t}x_2 = x, \quad \frac{1-\alpha}{t}x_1 \in C, \quad \frac{\alpha}{t}x_2 \in C$$

and $x \in \operatorname{extd} C$, then $\frac{1-\alpha}{t}x_1, \frac{\alpha}{t}x_2 \in \mathbb{R}_+ x$ and $x_1, x_2 \in \mathbb{R}_+ x$. This is a contradiction. If $t = 0$, since $(1-\alpha)x_1 + \alpha x_2 = 0$ and C is pointed, then $x_1 = x_2 = 0 \in \mathbb{R}_+ x$, which contradicts to $x_1, x_2 \in C \setminus \mathbb{R}_+ x$.

Conversely, assume that $C \setminus \mathbb{R}_+ x$ is convex. Let $x_1, x_2 \in C$ such that $x = x_1 + x_2$. If $x_1, x_2 \notin \mathbb{R}_+ x$, since $x_1, x_2 \in C \setminus \mathbb{R}_+ x$,

$$x = 2\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) \in C \setminus \mathbb{R}_+ x.$$

This contradicts to $x \in \mathbb{R}_+ x$. If one of two is in $\mathbb{R}_+ x$, for example $x_1 \in \mathbb{R}_+ x$, then $x_1 = tx$ for some $t \geq 0$, that is $x_2 = x - x_1 = (1-t)x$. If $1 - t < 0$, since $x_2 \in C \cap (-C)$ and C is pointed, we have $x_2 = 0 \in \mathbb{R}_+ x$ and if $1 - t \geq 0$, then $x_2 \in \mathbb{R}_+ x$. □

Remark 2.2. The assumption pointed in this result is essential. Take a non-zero vector x_0 and define $C = \mathbb{R}x_0$. Clearly $x_0 \in C$, C is a convex cone, and $C \setminus \mathbb{R}_+ x_0$ is convex, however x_0 is not any extreme direction of C because $x_0 = 3x_0 + (-2)x_0$, $3x_0 \in C$, $-2x_0 \in C$, but $-2x_0 \notin \mathbb{R}_+ x_0$.

Denote the positive polar cone of $A \subset \mathbb{R}^n$ as $A^+ = \{b \in \mathbb{R}^n \mid \langle b, a \rangle \geq 0, \forall a \in A\}$. Then the following result holds, see [1]:

Proposition 2.3. *Let $D \subset \mathbb{R}^n$ be a closed pointed convex cone. Then $D = (\operatorname{extd} D^+)^+$. Let $D \subset \mathbb{R}^n$ be a closed pointed convex cone with nonempty interior. Then $D = (\operatorname{extd} D^+)^+$.*

3. Main results

Let I be a non-empty index set, $f, g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i \in I$, be proper convex functions. Assume that the KKT condition (1.2) holds at $\bar{x} \in S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0,\ \forall i \in I\}$. Define

$$K = -\operatorname{cone}\operatorname{co} \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}),$$

where $I(\bar{x}) = \{i \in I \mid g_i(\bar{x}) = 0\}$, and $\operatorname{cone} A$ and $\operatorname{co} A$ are the conical and convex hull of $A \subset \mathbb{R}^n$, respectively. Then it is easy to verify that the KKT condition (1.2) holds at $\bar{x}$ if and only if

$$\partial f(\bar{x}) \cap K \neq \emptyset \tag{3.1}$$

holds. At first, we observe problem (1.4) in the following situation:

$$\partial f(\bar{x}) \cap K \neq \emptyset \quad \text{and} \quad \partial f(\bar{x}) \cap \operatorname{int} K = \emptyset. \tag{3.2}$$

In this case, we have the optimal value of problem (1.4) is 0 from the following result:

Theorem 3.1. *Assume that (3.2) holds and* $\operatorname{int} K \neq \emptyset$. *For every* $r > 0$ *there exists* $c \in \mathbb{R}^n$ *such that* $\|c\| \leq r$ *and* $(\partial f(\bar{x}) + c) \cap K = \emptyset$. *The optimal value of problem (1.4) is 0.*

Proof. We give a proof of the first part of this theorem by using the separation theorem. From $\partial f(\bar{x}) \cap \operatorname{int} K = \emptyset$, we can choose non-zero $a \in \mathbb{R}^n$ such that for all $y \in \partial f(\bar{x})$ and $k \in \operatorname{int} K$,

$$\langle a, y \rangle \leq 0 < \langle a, k \rangle .$$

This shows that $0 \leq \langle a, k \rangle$ for each $k \in K$. We may assume that $\|a\| = 1$. Therefore, for every $r > 0$, $c = -ra$ satisfies $\|c\| = r$ and $(\partial f(\bar{x}) + c) \cap K = \emptyset$ holds because

$$\langle a, y + c \rangle = \langle a, y \rangle - r \leq -r < 0$$

for all $y \in \partial f(\bar{x})$. The second part of this theorem is easy to show from the first part of this theorem, the assumption $\partial f(\bar{x}) \cap K \neq \emptyset$, and the fact $\partial(f + \langle c, \cdot \rangle)(\bar{x}) = \partial f(\bar{x}) + c$. □

Next, we observe problem (1.4) in the following situation:

$$\partial f(\bar{x}) \cap \operatorname{int} K \neq \emptyset. \tag{3.3}$$

Theorem 3.2. *Assume that (3.3) holds,* K *is closed and pointed, and* $\partial f(\bar{x})$ *is compact. Then the optimal value of problem (1.4) is equal to*

$$r_0 := \inf_{d \in K^+, \|d\|=1} \sup_{y \in \partial f(\bar{x})} \langle y, d \rangle . \tag{3.4}$$

Moreover if f *is differentiable at* $\bar{x}$, *then*

$$r_0 = \inf_{d \in \operatorname{extd} K^+, \|d\|=1} \langle \nabla f(\bar{x}), d \rangle . \tag{3.5}$$

Proof. It is clear that $r_0 > 0$ from the assumption. In general, the optimal value of problem (1.4) is as follows:

$$v = \sup\{r > 0 \mid (\partial f(\bar{x}) + c) \cap K \neq \emptyset, \forall c \in \mathbb{R}^n(\|c\| \leq r)\}. \tag{3.6}$$

Also $(K^+)^+ = K$ holds because K is a closed pointed convex cone from Proposition 2.3. At first, we show $r_0 \geq v$. For any $r \in \mathbb{R}(0 < r < v)$, there exist r' such that $r < r'$ and for all $c \in \mathbb{R}^n(\|c\| \leq r')$, $(\partial f(\bar{x}) + c) \cap K \neq \emptyset$. Take $y' \in \partial f(\bar{x})$ satisfying $y' + c \in K$. For all $d \in K^+(\|d\| = 1)$, $\langle y' + c, d \rangle \geq 0$, that is

$$\sup_{y \in \partial f(\bar{x})} \langle y, d \rangle \geq \langle y', d \rangle \geq \langle -c, d \rangle .$$

Since c is arbitrary, we have $\sup_{y \in \partial f(\bar{x})} \langle y, d \rangle \geq r' > r$. Also d and r are arbitrary, then

$$r_0 = \inf_{d \in K^+(\|d\|=1)} \sup_{y \in \partial f(\bar{x})} \langle y, d \rangle \geq v.$$

On the other hand, we show $(\partial f(\bar{x}) + c) \cap K \neq \emptyset$ for all $c \in \mathbb{R}^n(\|c\| \leq r_0)$. Otherwise, there exists $c_0 \in \mathbb{R}^n(\|c_0\| \leq r_0)$ such that

$$(\partial f(\bar{x}) + c_0) \cap K = \emptyset.$$

Since $\partial f(\bar{x}) + c_0$ is compact convex, by using the strong separation theorem, there exists $d_0 \in K^+ \setminus \{0\}$ such that

$$\sup_{y \in \partial f(\bar{x})} \langle y + c_0, d_0 \rangle < 0.$$

We may assume that $\|d_0\| = 1$, then we have

$$r_0 \leq \sup_{y \in \partial f(\bar{x})} \langle y, d_0 \rangle < \langle -c_0, d_0 \rangle \leq \| - c_0\| \leq r_0,$$

which is a contradiction. Therefore, $(\partial f(\bar{x}) + c) \cap K \neq \emptyset$ for all $c \in \mathbb{R}^n(\|c\| \leq r_0)$ and then we have $r_0 \leq v$.

In general we have

$$r_0 = \inf_{d \in K^+, \|d\|=1} \sup_{y \in \partial f(\bar{x})} \langle y, d \rangle \leq \inf_{d \in \operatorname{extd} K^+, \|d\|=1} \sup_{y \in \partial f(\bar{x})} \langle y, d \rangle =: r_1. \tag{3.7}$$

Assume that f is differentiable at $\bar{x}$. We show $(\nabla f(\bar{x}) + c) \cap K \neq \emptyset$ for all $c \in \mathbb{R}^n(\|c\| \leq r_1)$. Otherwise, there exists $c_0 \in \mathbb{R}^n(\|c_0\| \leq r_0)$ such that

$$(\nabla f(\bar{x}) + c_0) \cap K = \emptyset.$$

Since $K = (\operatorname{extd} K^+)^+$, there exists $d_0 \in \operatorname{extd} K^+ \setminus \{0\}$ such that

$$\langle \nabla f(\bar{x}) + c_0, d_0 \rangle < 0.$$

In the same way to the above, we have a contradiction. This shows that $r_1 \leq v$ and finally we have $r_1 = v$. □

Remark 3.1. (i) When K is finitely generated, it is well-known that the number of extreme direction is finite except for the difference in length. Therefore, the number of elements of $\{d \in \operatorname{extd} K^+ \mid \|d\| = 1\}$ is finite, but even in this situation, the number of elements of $\{d \in K^+ \mid \|d\| = 1\}$ is infinite. This means that to determine r_1 is easier than to determine r_0.

(ii) The value r_1 is an upper bound of the problem (1.4), but it is not feasible of the problem in general, see Example 3.2. When f is differentiable, r_1 becomes the optimum value of the problem (1.4).

(iii) For any fixed $y \in \partial f(\bar{x}) \cap \operatorname{int} K$, put $r_y = \inf_{d \in \operatorname{extd} K^+, \|d\|=1} \langle y, d \rangle$. In a similar way to the proof, we have $0 < r_y \leq r_0$. Put

$$r_2 := \sup_{y \in \partial f(\bar{x}) \cap \operatorname{int} K} r_y = \sup_{y \in \partial f(\bar{x}) \cap \operatorname{int} K} \inf_{d \in \operatorname{extd} K^+, \|d\|=1} \langle y, d \rangle$$

then r_2 is a lower bound of the problem (1.4). This means that every perturbation of the objective function radius r_2 preserves the KKT condition at $\bar{x}$. But r_2 is not equal to r_0 in general, see Example 3.2.

(iv) The compactness assumption of $\partial f(\bar{x})$ is redundant when f is continuous at $\bar{x}$. In this case, since $\sup_{y \in \partial f(\bar{x})} \langle y, d \rangle = f'(\bar{x}, d)$, which is the directional derivative of f at $\bar{x}$ in direction d defined as $\lim_{t \downarrow 0}(f(\bar{x} + td) - f(\bar{x}))/t$, then the optimal value of (1.4) is equal to:

$$r_0 = \inf_{d \in K^+, \|d\|=1} f'(\bar{x}, d).$$

(v) The closedness assumption of K is redundant when $I(\bar{x})$ is finite and all g_i, $i \in I(\bar{x})$ are differentiable.

Example 3.2. Assume that $I(\bar{x}) = \{1, 2\}$, and $\nabla g_1(\bar{x}) = (-1, 0)$, $\nabla g_2(\bar{x}) = (0, -1)$, Then we can see that

$$K = \operatorname{cone} \operatorname{co} \{(1, 0), (0, 1)\} = K^+,$$

and

$$\operatorname{extd} K^+ = \operatorname{cone}\{(1,0),(0,1)\} \setminus \{(0,0)\}.$$

If $\partial f(\bar{x}) = \operatorname{co}\{(1,0),(0,1)\}$, then

$$r_0 = \inf_{d \in K^+, \|d\|=1} \sup_{y \in \partial f(\bar{x})} \langle y, d \rangle = \frac{1}{\sqrt{2}},$$

$$r_1 = \inf_{d \in \operatorname{extd} K^+, \|d\|=1} \sup_{y \in \partial f(\bar{x})} \langle y, d \rangle = 1,$$

$$r_2 = \sup_{y \in \partial f(\bar{x}) \cap \operatorname{int} K} \inf_{d \in \operatorname{extd} K^+, \|d\|=1} \langle y, d \rangle = \frac{1}{2}.$$

Corollary 3.3. *Assume that I is finite, f and g_i, $i \in I$, are differentiable at $\bar{x}$, and $\nabla f(\bar{x}) \in -\operatorname{int} \operatorname{cone} \operatorname{co}\{\nabla g_i(\bar{x}) \mid i \in I(\bar{x})\}$. Then $\{d \in \operatorname{extd} K^+ \mid \|d\| = 1\}$ is finite and for every $c \in \mathbb{R}^n$ satisfying $\|c\| \leq r_1$, KKT condition (1.2) at $\bar{x}$ in (1.1) with the objective function $f + \langle c, \cdot \rangle$ holds, where*

$$r_1 = \min\{\langle \nabla f(\bar{x}), d \rangle \mid d \in \operatorname{extd} K^+, \|d\| = 1\}.$$

Proof. The convex cone $\operatorname{cone} \operatorname{co}\{\nabla g_i(\bar{x}) \mid i \in I(\bar{x})\}$ is closed because it is finitely generated. □

We give an example of the corollary as follows:

Example 3.4. Assume that $I(\bar{x}) = \{1,2,3,4\}$, and $\nabla g_1(\bar{x}) = (-1,-1,0)$, $\nabla g_2(\bar{x}) = (0,-1,0)$, $\nabla g_3(\bar{x}) = (0,0,-1)$, $\nabla g_4(\bar{x}) = (-1,0,-1)$. Then we can see that

$$K = \operatorname{cone} \operatorname{co}\{(1,1,0),(0,1,0),(0,0,1),(1,0,1)\},$$

$$K^+ = \operatorname{cone} \operatorname{co}\{(1,0,0),(0,1,0),(0,0,1),(-1,1,1)\},$$

and

$$\operatorname{extd} K^+ = \operatorname{cone}\{(1,0,0),(0,1,0),(0,0,1),(-1,1,1)\} \setminus \{(0,0,0)\}.$$

When f is differentiable at $\bar{x}$ and $\nabla f(\bar{x}) = (y_1, y_2, y_3) \in \operatorname{int} K$, then the optimal value of (1.4) is equal to

$$\begin{aligned} r_1 &= \min\{\langle \nabla f(\bar{x}), d \rangle \mid d \in \operatorname{extd} K^+, \|d\| = 1\} \\ &= \min\left\{y_1, y_2, y_3, (-y_1 + y_2 + y_3)/\sqrt{3}\right\}. \end{aligned}$$

4. Conclusion

We have studied the problem to find the maximum radius of the perturbation of the objective function which preserves the KKT condition at a feasible point. The situation changed when the subdifferential of the objective function at the feasible point meets the interior of a convex cone which was generated by the subdifferentials of the active constraint functions at the point, or not. We have described the maximum radius of the problem in each cases, in Theorem 3.1 and Theorem 3.2. Also we observe the maximum value and other related values include the notion of extreme direction, of which a characterization was given in Section 2. Finally we have applied Theorem 3.2 to a differentiable convex minimization problem in which the maximum radius was simply expressed and an example was given.

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