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BALANCED MAPPINGS AND AN ITERATIVE SCHEME IN COMPLETE GEODESIC SPACES

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ABSTRACT. In this paper, we define a balanced mapping by a maximizer of a certain function generated by a finite number of mappings without regard to their order and find its fundamental properties in a complete CAT(1) space. Furthermore, we approximate a fixed point of a balanced mapping which is generated by a finite number of quasinonexpansive and Δ-demiclosed mappings by using Mann's iterative scheme.

KEYWORDS: Common fixed point, CAT(1) space, quasinonexpansive, Mann type, iteration
AMS Subject Classification: 47H09

1. Introduction

In the study of nonlinear analysis, we approximate a fixed point of many kinds of mappings. We focus on a balanced mapping which is generated by a finite number of mappings without regard to their order. Hasegawa and Kimura [2] defined it by proving the following theorem in the setting of complete CAT(0) spaces. We will extend its definition in the setting of complete CAT(1) spaces.

Theorem 1.1. (Hasegawa–Kimura [2]) *Let X be a complete* CAT(0) *space. Let T^k be a nonexpansive mapping from X to X for every $k = 1, 2, \dots, N$. Let $\alpha^k \in [0, 1]$ for every $k = 1, 2, \dots, N$ such that $\sum_{k=1}^{N} \alpha^k = 1$. Let x be a point of X. Then the set*

$$\operatorname*{argmin}_{y \in X} \sum_{k=1}^{N} \alpha^k d(T^k x, y)^2$$

consists of one point.

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We know that there are various kinds of iterative schemes which is effective to find fixed points of nonexpansive mappings. We pay attention to Mann's [3] iterative scheme. A number of authors have proved approximation theorems by using that scheme. Reich [7] proved it in a Banach space. Dhompongsa and Panyanak [1] proved it in a CAT(0) space. Kimura, Saejung, and Yotkaew [4] proved it by using a quainonexpansive and Δ-demiclosed mapping in a CAT(1) space. We particularly note that Hasegawa and Kimura [2] proved the convergence of Mann type iteration by using a balanced mapping.

Theorem 1.2. (Hasegawa–Kimura [2]) *Let X be a complete* CAT(0) *space. Let T^k be a nonexpansive mapping from X to X for every $k = 1, 2, \dots, N$ such that $F = \bigcap_{k=1}^{N} F(T^k) \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\left\{\alpha_n^k\right\}, \{\beta_n\} \subset [a, 1-a]$ for every $k = 1, 2, \dots, N$ and $n \in \mathbb{N}$ such that $\sum_{k=1}^{N} \alpha_n^k = 1$. Define U_n be a mapping from X to X by*

$$U_n x = \operatorname*{argmin}_{y \in X} \sum_{k=1}^{N} \alpha_n^k d(T^k x, y)^2$$

for every $x \in X$ and $n \in \mathbb{N}$. For a given point $x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by

$$x_{n+1} = \beta_n x_n \oplus (1 - \beta_n) U_n x_n$$

for every $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a point in F.

In this paper, we define a balanced mapping in a complete CAT(1) space and prove a convergence theorem of Mann type iteration by using it. Namely, our results are a modified version of the results by Hasegawa and Kimura [2] in a complete CAT(1) space.

2. Preliminaries

Let X be a metric space and $\{x_n\}$ a sequence in X. An element $z \in X$ is said to be an asymptotic center of $\{x_n\} \subset X$ if

$$\limsup_{n \to \infty} d(x_n, z) = \inf_{x \in X} \limsup_{n \to \infty} d(x_n, x).$$

Moreover, we say $\{x_n\}$ Δ-converges to a Δ-limit z if z is the unique asymptotic center of any subsequences of $\{x_n\}$. For $x, y \in X$, a mapping $c : [0, l] \to X$ is called a geodesic if c satisfies

$$c(0) = x, c(l) = y, \text{ and } d(c(u), c(v)) = |u - v|$$

for every $u, v \in [0, l]$. An image of $[x, y]$ of c is called a geodesic segment joining x and y. For $r > 0$, X is said to be an r-geodesic space if for every $x, y \in X$ with $d(x, y) < r$, there exists a geodesic c joining x and y. Moreover, if such a geodesic segment is unique for each pair of points, then X is said to be a uniquely r-geodesic space.

Let X be a uniquely π-geodesic space. For a triangle $\triangle(x, y, z) \subset X$ such that $d(x, y) + d(y, z) + d(z, x) < 2\pi$, let a comparison triangle $\triangle(\bar{x}, \bar{y}, \bar{z})$ in two-dimensional unit sphere $\mathbb{S}^2$ be such that each corresponding edge has the same length as that of the original triangle. X is called a CAT(1) space if every $p, q \in \triangle(x, y, z)$ and their corresponding points $\bar{p}, \bar{q} \in \triangle(\bar{x}, \bar{y}, \bar{z})$ satisfy that

$$d(p, q) \leq d_{\mathbb{S}^2}(\bar{p}, \bar{q}),$$

where $d_{\mathbb{S}^2}$ is the spherical metric on $\mathbb{S}^2$.

Let X be a CAT(1) space. For every $x, y \in X$ with $d(x,y) < \pi$ and $\alpha \in [0,1]$, if $z \in [x,y]$ satisfies that $d(y,z) = \alpha d(x,y)$ and $d(x,z) = (1-\alpha)d(x,y)$, then we denote z by $z = \alpha x \oplus (1-\alpha)y$. A subset $C \subset X$ is called π-convex if $\alpha x \oplus (1-\alpha)y \in C$ for every $x, y \in C$ with $d(x,y) < \pi$ and $\alpha \in [0,1]$.

Let X be a CAT(1) space and let T be a mapping from X to X such that the set $F(T) = \{z \in X : z = Tz\}$ of fixed points of T is not empty. If $d(Tx,p) \leq d(x,p)$ for every $x \in X$ and $p \in F(T)$, then we call T a quasinonexpansive mapping.

T is said to be a strongly quasinonexpansive mapping if T is a quasinonexpansive mapping, and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ whenever $\{x_n\} \subset X$ satisfies $\sup_{n\in\mathbb{N}} d(x_n, p) < \pi/2$ and $\lim_{n\to\infty}(\cos d(x_n,p)/\cos d(Tx_n,p)) = 1$ for every $p \in F(T)$.

Let X be a CAT(1) space and let T be a mapping from X to X such that $F(T) \neq \emptyset$. T is said to be a Δ-demiclosed mapping if $z \in F(T)$ whenever $\{x_n\}$ Δ-converges to z and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Let X be a complete CAT(1) space and let $C \subset X$ be a nonempty closed π-convex subset such that $d(x,C) = \inf_{y\in C} d(x,y) < \pi/2$ for every $x \in X$. Then for every $x \in X$, there exists a unique point $x_0 \in C$ satisfying

$$d(x, x_0) = \inf_{y\in C} d(x,y).$$

We define the metric projection P_C from X onto C by $P_C x = x_0$.

We introduce some lemmas used for our results.

Lemma 2.1. (Kimura and Satô [5]) *Let X be a* CAT(1) *space. For every $x, y, z \in X$ with $d(x,y) + d(y,z) + d(z,x) < 2\pi$ and $\alpha \in [0,1]$, the following inequality holds:*

$$\cos d(x,w)\sin d(y,z) \geq \cos d(x,y)\sin(\alpha d(y,z)) + \cos d(x,z)\sin((1-\alpha)d(y,z)),$$

where $w = \alpha y \oplus (1-\alpha)z$.

Lemma 2.2. (Kimura and Satô [6]) *Let X be a* CAT(1) *space. For every $x, y, z \in X$ with $d(x,y) + d(y,z) + d(z,x) < 2\pi$ and $\alpha \in [0,1]$, the following inequality holds:*

$$\cos d(x,w) \geq \alpha\cos d(x,y) + (1-\alpha)\cos d(x,z),$$

where $w = \alpha y \oplus (1-\alpha)z$.

Lemma 2.3. (Kimura and Satô [6]) *Let X be a* CAT(1) *space and y_0, y_1 and y elements of X such that $d(y_0,y) + d(y_1,y) + d(y_0,y_1) < 2\pi$. Then we have*

$$\cos d\left(\frac{1}{2}y_0 \oplus \frac{1}{2}y_1, y\right)\cos\frac{d(y_0,y_1)}{2} \geq \min\{\cos d(y_0,y), \cos d(y_1,y)\}.$$

3. Balanced mapping in CAT(1) spaces

In this section, we define a balanced mapping and find its fundamental properties in a CAT(1) space. We begin with the following theorem which guarantees that the balanced mapping can be defined as a single-valued mapping.

Theorem 3.1. *Let X be a complete* CAT(1) *space such that $d(x,y) < \pi/2$ for every $x, y \in X$. Let x^k be a point of X for every $k = 1, 2, \ldots, N$. Let $\alpha^k \in [0,1]$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^N \alpha^k = 1$. Then the set*

$$\operatorname*{argmax}_{y\in X} \sum_{k=1}^N \alpha^k \cos d(x^k, y)$$

consists of one point.

Proof. Let $D = \sup_{y\in X} \sum_{k=1}^{N} \alpha^k \cos d(x^k, y)$ and $\{y_n\}$ a sequence in X such that $\lim_{n\to\infty} \sum_{k=1}^{N} \alpha^k \cos d(x^k, y_n) = D$. For $m, n \in \mathbb{N}$, from Lemma 2.3, we have

$$\sum_{k=1}^{N} \alpha^k \cos d\left(x^k, \frac{1}{2}y_n \oplus \frac{1}{2}y_m\right) \cos \frac{d(y_n, y_m)}{2} \geq \sum_{k=1}^{N} \alpha^k \min\{\cos d(y_n, x^k), \cos d(y_m, x^k)\}.$$

Thus we get

$$\cos \frac{d(y_n, y_m)}{2} \geq \frac{\sum_{k=1}^{N} \alpha^k \min\{\cos d(y_n, x^k), \cos d(y_m, x^k)\}}{D}.$$

Hence we obtain $\{y_n\}$ is a Cauchy sequence. By the completeness of X, there exists $u = \lim_{n\to\infty} y_n$. From the continuity of the metric, we get $\sum_{k=1}^{N} \alpha^k \cos d(x^k, u) = \sup_{y\in X} \sum_{k=1}^{N} \alpha^k \cos d(x^k, y)$. Hence $\operatorname{argmax}_{y\in X} \sum_{k=1}^{N} \alpha^k \cos d(x^k, y)$ is nonempty. Let $u, v \in \operatorname{argmax}_{y\in X} \sum_{k=1}^{N} \cos d(x^k, y)$ and suppose $u \neq v$. By Lemma 2.1, we have

$$\begin{aligned}\sum_{k=1}^{N} \alpha^k \cos d(x^k, u) \sin d(u, v) &\geq \sum_{k=1}^{N} \alpha^k \cos d\left(x^k, \frac{1}{2}u \oplus \frac{1}{2}v\right) \sin d(u, v)\\ &\geq \sin \frac{d(u, v)}{2} \sum_{k=1}^{N} \alpha^k (\cos d(x^k, u) + \cos d(x^k, v)).\end{aligned}$$

Dividing by $\sin(d(u, v)/2)$, we get

$$2 \cos \frac{d(u, v)}{2} \sum_{k=1}^{N} \alpha^k \cos d(x^k, u) \geq \sum_{k=1}^{N} \alpha^k (\cos d(x^k, u) + \cos d(x^k, v)).$$

Similarly, we get

$$2 \cos \frac{d(u, v)}{2} \sum_{k=1}^{N} \alpha^k \cos d(x^k, v) \geq \sum_{k=1}^{N} \alpha^k (\cos d(x^k, u) + \cos d(x^k, v)).$$

Therefore, we obtain

$$2 \cos \frac{d(u, v)}{2} \sum_{k=1}^{N} \alpha^k (\cos d(x^k, u) + \cos d(x^k, v)) \geq 2 \sum_{k=1}^{N} \alpha^k (\cos d(x^k, u) + \cos d(x^k, v)).$$

Then we have

$$1 > \cos \frac{d(u, v)}{2} \geq 1,$$

which is a contradiction. Hence we get $u = v$. □

By Theorem 3.1, we know the set $\operatorname{argmax}_{y\in X} \sum_{k=1}^{N} \alpha^k \cos d(x^k, y)$ is a singleton. In what follows, a balanced mapping U from X to X for a sequence $\alpha^1, \alpha^2, \dots, \alpha^N \in [0, 1]$ and mappings $T^1, T^2, \dots, T^N$ is defined by

$$Ux = \operatorname*{argmax}_{y\in X} \sum_{k=1}^{N} \alpha^k \cos d(T^k x, y)$$

for every $x \in X$. We prove some basic properties of balanced mappings in this section.

Theorem 3.2. *Let X be a complete* CAT(1) *space such that $d(x,y) < \pi/2$ for every $x, y \in X$. Let T^k be a quasinonexpansive mapping from X to X for every $k = 1, 2, \dots, N$ such that $\bigcap_{k=1}^N F(T^k) \neq \emptyset$. Let $\alpha^k \in \,]0,1[$ for every $k = 1, 2, \dots, N$ such that $\sum_{k=1}^N \alpha^k = 1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then $F(U) = \bigcap_{k=1}^N F(T^k)$.*

Proof. Let $z \in \bigcap_{k=1}^N F(T^k)$. Then we have

$$\begin{aligned} Uz &= \operatorname*{argmax}_{y\in X} \sum_{k=1}^N \alpha^k \cos d(T^k z, y) \\ &= \operatorname*{argmax}_{y\in X} \sum_{k=1}^N \alpha^k \cos d(z, y) \\ &= \operatorname*{argmax}_{y\in X} \cos d(z, y) \\ &= z. \end{aligned}$$

Hence we get $z \in F(U)$. Let $z \in F(U), w \in \bigcap_{k=1}^N F(T^k)$ and $t \in \,]0,1[$. We may assume that $z \neq w$. From Lemma 2.1, we have

$$\begin{aligned} &\sum_{k=1}^N \alpha^k \cos d(T^k z, z) \sin d(z, w) \\ &\quad\geq \sum_{k=1}^N \alpha^k \cos d(T^k z, tz \oplus (1-t)w) \sin d(z, w) \\ &\quad\geq \sum_{k=1}^N \alpha^k \cos d(T^k z, z) \sin t d(z, w) + \sum_{k=1}^N \alpha^k \cos d(T^k z, w) \sin (1-t) d(z, w) \\ &\quad\geq \sum_{k=1}^N \alpha^k \cos d(T^k z, z) \sin t d(z, w) + \cos d(z, w) \sin (1-t) d(z, w). \end{aligned}$$

Hence we get

$$2 \sum_{k=1}^N \alpha^k \cos d(T^k z, z) (\sin d(z, w) - \sin t d(z, w)) \geq \cos d(z, w) \sin (1-t) d(z, w),$$

and it implies that

$$\begin{aligned} 2 \sum_{k=1}^N \alpha^k \cos d(T^k z, z) \sin \frac{(1-t) d(z, w)}{2} \cos \frac{(1+t) d(z, w)}{2} \\ \geq 2 \cos d(z, w) \sin \frac{(1-t) d(z, w)}{2} \cos \frac{(1-t) d(z, w)}{2}. \end{aligned}$$

Dividing by $2 \sin((1-t) d(z, w)/2) \cos d(z, w)$ and tending $t \to 1$, we get

$$\sum_{k=1}^N \alpha^k \cos d(T^k z, z) \geq 1.$$

Therefore we have $\cos d(T^k z, z) = 1$ for every $k = 1, 2, \dots, N$. Hence we get $z \in \bigcap_{k=1}^N F(T^k)$. □

Lemma 3.1. *Let X be a complete* CAT(1) *space such that $d(x,y) < \pi/2$ for every $x, y \in X$. Let T^k be a quasinonexpansive mapping from X to X for every*

$k = 1, 2, \ldots, N$ such that $\bigcap_{k=1}^{N} F(T^k) \neq \emptyset$. Let $\alpha^k \in [0,1]$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^{N} \alpha^k = 1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then we have

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \cos d(Ux, z) \geq \cos d(x, z)$$

for every $x \in X$ and $z \in \bigcap_{k=1}^{N} F(T^k)$.

Proof. Let $z \in \bigcap_{k=1}^{N} F(T^k)$ and $t \in]0,1[$. Then, from Lemma 3.2, we have $z \in F(U)$. We may assume that $Ux \neq z$ since if $Ux = z$, the inequality is obvious true. By Lemma 2.1, we get

$$\begin{aligned}
&\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \sin d(Ux, z) \\
&\quad \geq \sum_{k=1}^{N} \alpha^k \cos d(T^k x, tUx \oplus (1-t)z) \sin d(Ux, z) \\
&\quad \geq \sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \sin t d(Ux, z) + \sum_{k=1}^{N} \alpha^k \cos d(T^k x, z) \sin(1-t) d(Ux, z) \\
&\quad \geq \sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \sin t d(Ux, z) + \sum_{k=1}^{N} \alpha^k \cos d(x, z) \sin(1-t) d(Ux, z) \\
&\quad = \sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \sin t d(Ux, z) + \cos d(x, z) \sin(1-t) d(Ux, z).
\end{aligned}$$

Hence we obtain

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux)(\sin d(Ux, z) - \sin t d(Ux, z)) \geq \cos d(x, z) \sin(1-t) d(Ux, z),$$

and it implies that

$$\begin{aligned}
&2 \sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \sin \frac{(1-t)d(Ux, z)}{2} \cos \frac{(1+t)d(Ux, z)}{2} \\
&\qquad \geq 2 \cos d(x, z) \sin \frac{(1-t)d(Ux, z)}{2} \cos \frac{(1-t)d(Ux, z)}{2}.
\end{aligned}$$

Dividing by $2 \sin((1-t)d(Ux, z)/2)$ and tending $t \to 1$, we get

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \cos d(Ux, z) \geq \cos d(x, z)$$

for $x \in X$. ∎

Theorem 3.3. *Let X be a complete* CAT(1) *space such that $d(x, y) < \pi/2$ for every $x, y \in X$. Let T^k be a quasinonexpansive mapping from X to X for every $k = 1, 2, \ldots, N$ such that $\bigcap_{k=1}^{N} F(T^k) \neq \emptyset$. Let $\alpha^k \in [0,1]$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^{N} \alpha^k = 1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then U is a quasinonexpansive mapping.*

Proof. From Lemma 3.2, let $z \in F(U) = \bigcap_{k=1}^{N} F(T^k)$. By Lemma 3.1, we have

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \cos d(Ux, z) \geq \cos d(x, z)$$

for $x \in X$. Since $\cos d(T^k x, Ux) \leq 1$, we get

$$\begin{aligned}\cos d(Ux, z) &\geq \sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \cos d(Ux, z) \\ &\geq \cos d(x, z).\end{aligned}$$

Thus, we obtain

$$d(Ux, z) \leq d(x, z).$$

Hence U is a quasinonexpansive mapping. □

Theorem 3.4. *Let X be a complete* CAT(1) *space such that $d(x, y) < \pi/2$ for every $x, y \in X$. Let T^k be a quasinonexpansive and Δ-demiclosed mapping from X to X for every $k = 1, 2, \ldots, N$ such that $\bigcap_{k=1}^{N} F(T^k) \neq \emptyset$. Let $\alpha^k \in \,]0, 1[$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^{N} \alpha^k = 1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then U is a Δ-demiclosed mapping.*

Proof. From Lemma 3.2, let $z \in F(U) = \bigcap_{k=1}^{N} F(T^k)$. Let $\{x_n\} \subset X$ satisfying $d(Ux_n, x_n) \to 0$ and $\{x_n\}$ Δ-converges to $x_0 \in X$. By Lemma 3.1, we have

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k x_n, Ux_n) \cos d(Ux_n, z) \geq \cos d(x_n, z).$$

Then we get

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k x_n, Ux_n) \geq \frac{\cos d(x_n, z)}{\cos d(Ux_n, z)}.$$

Since $\lim_{n\to\infty}(\cos d(x_n, z)/\cos d(Ux_n, z)) = 1$, we obtain

$$\lim_{n\to\infty} \sum_{k=1}^{N} \alpha^k \cos d(T^k x_n, Ux_n) = 1.$$

Hence we get $\lim_{n\to\infty} d(T^k x_n, Ux_n) = 0$ for every $k = 1, 2, \ldots, N$. Then we have $\lim_{n\to\infty} d(T^k x_n, x_n) = 0$ for every $k = 1, 2, \ldots, N$. Since T^k is a Δ-demiclosed mapping for every $k = 1, 2, \ldots, N$, we obtain $x_0 \in F$. □

Theorem 3.5. *Let X be a complete* CAT(1) *space such that $d(x, y) < \pi/2$ for every $x, y \in X$. Let T^k be a strongly quasinonexpansive mapping from X to X for every $k = 1, 2, \ldots, N$ such that $\bigcap_{k=1}^{N} F(T^k) \neq \emptyset$. Let $\alpha^k \in \,]0, 1[$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^{N} \alpha^k = 1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then U is a strongly quasinonexpansive mapping.*

Proof. From Lemma 3.2, let $z \in F(U) = \bigcap_{k=1}^{N} F(T^k)$. Let $\{x_n\} \subset X$ satisfying $\limsup_{n\to\infty} d(x_n, z) < \pi/2$ and $\lim_{n\to\infty}(\cos d(x_n, z)/\cos d(Ux_n, z)) = 1$. By Lemma 3.1, we have

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k x_n, Ux_n) \cos d(Ux_n, z) \geq \cos d(x_n, z).$$

Then we get

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k x_n, Ux_n) \geq \frac{\cos d(x_n, z)}{\cos d(Ux_n, z)}.$$

Since $\lim_{n\to\infty}(\cos d(x_n, z)/\cos d(Ux_n, z)) = 1$, we obtain

$$\lim_{n\to\infty} \sum_{k=1}^{N} \alpha^k \cos d(T^k x_n, Ux_n) = 1.$$

Hence we get $\lim_{n\to\infty} d(T^k x_n, Ux_n) = 0$ for every $k = 1, 2, \ldots, N$. For any $k = 1, 2, \ldots, N$, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\cos d(x_n, z)}{\cos d(Ux_n, z)} &= \liminf_{n\to\infty} \frac{\cos d(x_n, z)}{\cos d(Ux_n, z)} \\
&\leq \liminf_{n\to\infty} \frac{\cos d(x_n, z)}{\cos(d(Ux_n, T^k x_n) + d(T^k x_n, z))} \\
&= \liminf_{n\to\infty} \frac{\cos d(x_n, z)}{\cos d(T^k x_n, z)} \\
&\leq \limsup_{n\to\infty} \frac{\cos d(x_n, z)}{\cos d(T^k x_n, z)} \\
&\leq \limsup_{n\to\infty} \frac{\cos d(x_n, z)}{\cos(d(T^k x_n, Ux_n) + d(Ux_n, z))} \\
&= \limsup_{n\to\infty} \frac{\cos d(x_n, z)}{\cos d(Ux_n, z)} \\
&= \lim_{n\to\infty} \frac{\cos d(x_n, z)}{\cos d(Ux_n, z)}.
\end{aligned}$$

Thus we obtain $\lim_{n\to\infty}(\cos d(x_n, z)/\cos d(T^k x_n, z)) = 1$. Since T^k is a strongly quasinonexpansive mapping for every $k = 1, 2, \ldots, N$, we get $\lim_{n\to\infty} d(T^k x_n, x_n) = 0$ for every $k = 1, 2, \ldots, N$. Since $\lim_{n\to\infty} d(T^k x_n, Ux_n) = 0$ and $\lim_{n\to\infty} d(T^k x_n, x_n) = 0$ for every $k = 1, 2, \ldots, N$, we obtain $\lim_{n\to\infty} d(Ux_n, x_n) = 0$. □

4. An iterative scheme for balanced mappings

In this section, we prove a convergence theorem of a Mann iterative sequence by using a balanced mapping in a complete CAT(1) space.

Lemma 4.1. *Let X be a complete* CAT(1) *space such that $d(x, y) < \pi/2$ for every $x, y \in X$. Let T^k be a mapping from X to X for every $k = 1, 2, \ldots, N$. Let $\alpha^k \in [0, 1]$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^{N} \alpha^k = 1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then we have*

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \geq \frac{\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Uy)}{\cos d(Ux, Uy)}$$

for every $x, y \in X$.

Proof. Let $t \in]0,1[$. We may assume $Ux \neq Uy$. By Lemma 2.1, we have

$$\begin{aligned}
&\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \sin d(Ux, Uy) \\
&\quad \geq \sum_{k=1}^{N} \alpha^k \cos d(T^k x, tUx \oplus (1-t)Uy) \sin d(Ux, Uy) \\
&\quad \geq \sum_{k=1}^{N} \alpha^k (\cos d(T^k x, Ux) \sin t d(Ux, Uy) + \cos d(T^k x, Uy) \sin(1-t) d(Ux, Uy)).
\end{aligned}$$

Then we get

$$\begin{aligned}
&2 \sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \cos \frac{(1+t)d(Ux,Uy)}{2} \sin \frac{(1-t)d(Ux,Uy)}{2} \\
&\quad \geq \sum_{k=1}^{N} \alpha^k \cos d(T^k x, Uy) \sin(1-t) d(Ux, Uy).
\end{aligned}$$

Dividing by $2\cos((1+t)d(Ux,Uy)/2)\sin((1-t)d(Ux,Uy)/2)$, we get

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \geq \sum_{k=1}^{N} \alpha^k \cos d(T^k x, Uy) \frac{\cos \dfrac{(1-t)d(Ux,Uy)}{2}}{\cos \dfrac{(1+t)d(Ux,Uy)}{2}}.$$

Tending $t \to 1$, we obtain the desired result. □

Theorem 4.1. *Let X be a complete* CAT(1) *space such that $d(x,y) < \pi/2$ for every $x, y \in X$. Let T^k be a quasinonexpansive and Δ-demiclosed mapping from X to X for every $k = 1, 2, \dots, N$ such that $\bigcap_{k=1}^{N} F(T^k) \neq \emptyset$. For a given real number $a \in]0, 1/2]$, let $\{\alpha_n^k\}, \{\delta_n\} \subset [a, 1-a]$ for every $k = 1, 2, \dots, N$ and $n \in \mathbb{N}$ such that $\sum_{k=1}^{N} \alpha_n^k = 1$. Let U_n be a balanced mapping for $\{\alpha_n^k\}$ and $\{T^k\}$. For a given point $x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by*

$$x_{n+1} = \delta_n x_n \oplus (1-\delta_n) U_n x_n$$

for every $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a point in $\bigcap_{k=1}^{N} F(T^k)$.

Proof. From Lemma 3.2, we know that $F(U_n) = \bigcap_{k=1}^{N} F(T^k)$ for every $n \in \mathbb{N}$. Let $z \in F = F(U_n) = \bigcap_{k=1}^{N} F(T^k)$. From Lemmas 2.2 and 3.3, we have

$$\begin{aligned}
\cos d(x_{n+1}, z) &= \cos d(\delta_n x_n \oplus (1-\delta_n) U_n x_n, z) \\
&\geq \delta_n \cos d(x_n, z) + (1-\delta_n) \cos d(U_n x_n, z) \\
&\geq \cos d(x_n, z).
\end{aligned}$$

Thus, we obtain $d(x_{n+1}, z) \leq d(x_n, z)$ for all $n \in \mathbb{N}$ and there exists

$$D = \lim_{n \to \infty} d(x_n, z) \leq d(x_1, z) < \frac{\pi}{2}.$$

Since $\{\delta_n\} \subset [a, 1-a]$, from Lemma 2.1, we get

$$\begin{aligned}
&\cos d(x_{n+1}, z) \sin d(x_n, U_n x_n) \\
&\quad = \cos d(\delta_n x_n \oplus (1-\delta_n) U_n x_n, z) \sin d(x_n, U_n x_n) \\
&\quad \geq \cos d(x_n, z) \sin \delta_n d(x_n, U_n x_n) + \cos d(U_n x_n, z) \sin(1-\delta_n) d(x_n, U_n x_n)
\end{aligned}$$

$$
\begin{aligned}
&\geq \cos d(x_n, z)(\sin \delta_n d(x_n, U_n x_n) + \sin(1-\delta_n) d(x_n, U_n x_n)) \\
&\geq 2\cos d(x_n, z) \sin a d(x_n, U_n x_n).
\end{aligned}
$$

Putting $E = \lim_{n\to\infty} d(x_n, U_n x_n)$ and tending $n \to \infty$, we get

$$\cos D \sin E \geq 2 \cos D \sin aE.$$

Using elementary calculation, we have $E = 0$, that is,

$$\lim_{n\to\infty} d(x_n, U_n x_n) = 0.$$

We show $\lim_{n\to\infty} d(x_n, T^k x_n) = 0$ for all $k = 1, 2, \dots, N$. Since $\{x_n\}$ is bounded, it follows that

$$
\begin{aligned}
D = \lim_{n\to\infty} d(x_n, z) &\leq \lim_{n\to\infty} (d(x_n, U_n x_n) + d(U_n x_n, z)) \\
&= \lim_{n\to\infty} d(U_n x_n, z) \\
&\leq \lim_{n\to\infty} d(x_n, z) = D.
\end{aligned}
$$

Thus we get $\lim_{n\to\infty} d(x_n, z) = \lim_{n\to\infty} d(U_n x_n, z) = D$. By Lemma 4.1, we have

$$
\begin{aligned}
\sum_{k=1}^{N} \alpha_n^k \cos d(T^k x_n, U_n x_n) &\geq \frac{\sum_{k=1}^{N} \alpha_n^k \cos d(T^k x_n, z)}{\cos d(U_n x_n, z)} \\
&\geq \frac{\sum_{k=1}^{N} \alpha_n^k \cos d(x_n, z)}{\cos d(U_n x_n, z)} \\
&\geq \frac{\cos d(x_n, z)}{\cos d(U_n x_n, z)}.
\end{aligned}
$$

Since $\alpha_n^k \leq 1 - a < 1$, we obtain $\lim_{n\to\infty} d(T^k x_n, U_n x_n) = 0$ for every $k = 1, 2, \dots, N$. Since $\lim_{n\to\infty} d(U_n x_n, x_n) = 0$, we also get $\lim_{n\to\infty} d(T^k x_n, x_n) = 0$ for every $k = 1, 2, \dots, N$. Let x_0 be an asymptotic center of $\{x_n\}$ and for every $\{x_{n_k}\} \subset \{x_n\}$, let y be an asymptotic center of $\{x_{n_k}\}$. There exists $\{x_{n_{k_l}}\} \subset \{x_{n_k}\}$ satisfying that $\{x_{n_{k_l}}\}$ Δ-converges to w. Since T^k is Δ-demiclosed and $\lim_{n\to\infty} d(T^k x_n, x_n) = 0$ for every $k = 1, 2, \dots, N$, we get $w \in F$. Since there exists $\lim_{n\to\infty} d(x_{n_k}, w)$, we have

$$
\begin{aligned}
\limsup_{k\to\infty} d(x_{n_k}, w) &= \lim_{k\to\infty} d(x_{n_k}, w) \\
&= \lim_{l\to\infty} d(x_{n_{k_l}}, w) \\
&\leq \limsup_{l\to\infty} d(x_{n_{k_l}}, y) \\
&\leq \limsup_{k\to\infty} d(x_{n_k}, y).
\end{aligned}
$$

Since y is an asymptotic center of $\{x_{n_k}\}$, we obtain $y = w$. Then we have $y \in F$. Hence we get

$$
\begin{aligned}
\limsup_{n\to\infty} d(x_n, y) &= \lim_{n\to\infty} d(x_n, y) \\
&= \lim_{k\to\infty} d(x_{n_k}, y) \\
&\leq \limsup_{k\to\infty} d(x_{n_k}, x_0) \\
&\leq \limsup_{n\to\infty} d(x_n, x_0).
\end{aligned}
$$

Since x_0 is an asymptotic center of $\{x_n\}$, we obtain $x_0 = y$. Therefore we obtain $\{x_n\}$ Δ-converges to $x_0 \in F$. $\square$

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