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APPROXIMATION SOLVABILITY OF A PERTURBED MANN ITERATIVE ALGORITHM WITH ERRORS FOR A SYSTEM OF MIXED VARIATIONAL INCLUSIONS INVOLVING $\oplus$ OPERATION

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ABSTRACT. In this paper, we consider a new resolvent operator associated with XOR-NODSM mappings and give some of its fascinating properties supported by a well constructed example. As an application, we introduce and study a system of general mixed variational inclusions involving $\oplus$ operation in ordered Hilbert spaces. Further, we propose a perturbed Mann Iterative Algorithm with errors for approximating the solution of this class of problems. Our results can be complemented as the refinement and generalization of the corresponding results of recent works.

KEYWORDS: Variational inclusion, $\oplus$-operation, Resolvent operator, Algorithm, Convergence.
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1. Introduction

It is now a well known fact that variational inequality theory, introduced and studied by Stampacchia [27] and Fischera [12] in early 1960's, has been instrumental in the study of potential theory, elasticity, mathematical programming, network economics, transportation research and regional sciences.

Variational inclusions as the generalization of variational inequalities have been widely studied in recent years. An important aspect in the theory of variational inequalities is the existence of solution and development of efficient and implementable iterative algorithms. Among different methods for solving variational inclusion problems, resolvent operator technique has been widely used. The applications of the resolvent operator technique have been explored and improved

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recently, for instance, Fang and Huang [10] introduced a class of H-monotone operators and defined the associated class of resolvent operators which extended the classes of resolvent operators associated with η-subdifferential operators of Ding and Lou [8] and maximal monotone operators of Huang and Fang [14]. For more details, we refer [7, 8, 10, 11, 13, 16, 17] and the references therein.

In 2001, Huang and Fang [13] introduced the generalized m-accretive mappings and defined the resolvent operator for such class of mappings in Banach spaces. Since then a number of researchers investigated several classes of generalized m-accretive mappings such as H-accretive, $H(\cdot,\cdot)$-accretive, (H,η)-accretive and (A,η)-accretive mappings, see for example [4, 6–8, 11, 13, 16–18].

XOR operation, that is $\oplus$ operation, is a binary operation and behaves the same way as that of the ADD operation. This operation enjoys some nice properties such as commutativity, associativity and that every element under this operation is self-inverse. In Boolean algebra, it is the same as addition modulo(2). XOR is a logical operation that is true if and only if its arguments differ. XOR operator finds its applications in generating pseudo-random numbers, detecting errors in digital communications, etc. Until now several researchers have used XOR operation and its allied forms for solving some classes of variational inequalities and variational inclusions, see for example [1, 2, 20, 21, 23, 25].

Motivated and inspired by the above, in this paper, using the concept of XOR-NODSM mappings involving $\oplus$ operation and the new resolvent operator technique associated with XOR-NODSM mappings, we introduce and study a system of general mixed variational inclusions involving $\oplus$ operation in ordered positive Hilbet spaces and construct a new iterative algorithm with errors for this system of variational inclusions. Some properties of the associated resolvent operator have also been discussed by invoking $\oplus$ and $\odot$ operations supported by a well constructed example. Finally, we discuss the approximation solvability of the system considered. Our results improve and generalize the corresponding results of recent works, see for example [1, 3–11, 13–23, 25, 26].

2. Preliminaries

Let C be a cone with partial ordering " $\leq$ ". An ordered Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ is called positive if $0 \leq x$ and $0 \leq y$, then $0 \leq \langle x, y\rangle$ holds. Throughout the paper, $\mathcal{H}_p$ is assumed to be a real ordered positive Hilbert space. We denote the family of all nonempty (respectively, compact) subsets of $\mathcal{H}_p$ by $2^{\mathcal{H}_p}$ (respectively, $C^*(\mathcal{H}_p)$). The metric induced by the norm is denoted by d and the Hausdörff metric on $C^*(\mathcal{H}_p)$ by $\mathcal{D}(\cdot,\cdot)$.

Now, we recall some known definitions and results which are important to achieve the goal of this paper.

Definition 2.1 ([9]). A nonempty closed convex subset C of $\mathcal{H}_p$ is said to be a cone if:

(i) for any $x \in C$ and any $\lambda > 0, \lambda x \in C$;
(ii) $x \in C$ and $-x \in C$, then $x = 0$.

Definition 2.2 ([26]). A cone C is called a normal cone if and only if there exists a constant $\lambda_N > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \lambda_N \|y\|$, $\forall\ x, y \in \mathcal{H}_p$, where λ_N is called the normal constant of C.

Definition 2.3. For any $x, y \in \mathcal{H}_p, x \leq y$ if and only if $y - x \in C$.

The relation $\leq$ is a partial ordered relation in $\mathcal{H}_p$. The real Hilbert space $\mathcal{H}_p$ endowed with the ordered relation $\leq$ defined by C is called an ordered real Hilbert space.

Definition 2.4 ([26]). Let $x, y \in \mathcal{H}_p$, then x and y are said to be comparable to each other if either $x \leq y$ or $y \leq x$ holds and is denoted by $x \propto y$.

Definition 2.5 ([26]). For any $x, y \in \mathcal{H}_p$, $lub\{x, y\}$ denotes the least upper bound and $glb\{x, y\}$ denotes the greatest lower bound of the set $\{x, y\}$. Suppose $lub\{x, y\}$ and $glb\{x, y\}$ exist, then some binary operations are given below:

(i) $x \vee y = lub\{x, y\}$;
(ii) $x \wedge y = glb\{x, y\}$;
(iii) $x \oplus y = (x - y) \vee (y - x)$;
(iv) $x \odot y = (x - y) \wedge (y - x)$.

The operations $\vee, \wedge, \oplus$ and $\odot$ are called OR, AND, XOR and XNOR operations, respectively.

Lemma 2.6 ([9]). *If $x \propto y$, then $lub\{x, y\}$ and $glb\{x, y\}$ exist such that $(x - y) \propto (y - x)$ and $0 \leq (x - y) \vee (y - x)$.*

Lemma 2.7 ([9]). *For any natural number n, $x \propto y_n$ and $y_n \longrightarrow y^*$ as $n \longrightarrow \infty$, then $x \propto y^*$.*

Proposition 2.8 ([21, 22]). *Let $\oplus$ be an XOR operation and $\odot$ be an XNOR operation. Then the following relations hold for all $x, y, u, v, w \in \mathcal{H}_p$ and $\alpha, \beta, \lambda \in \mathbb{R}$:*

(i) $x \odot x = 0$, $x \odot y = y \odot x = -(x \oplus y) = -(y \oplus x)$;
(ii) $x \propto 0$, *then* $-x \oplus 0 \leq x \leq x \oplus 0$;
(iii) $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$;
(iv) $0 \leq x \oplus y$, *if* $x \propto y$;
(v) *if* $x \propto y$, *then* $x \oplus y = 0$ *if and only if* $x = y$;
(vi) $(x + y) \odot (u + v) \geq (x \odot u) + (y \odot v)$;
(vii) $(x + y) \odot (u + v) \geq (x \odot v) + (y \odot u)$;
(viii) *if x, y and w are comparable to each other, then* $(x \oplus y) \leq (x \oplus w) + (w \oplus y)$;
(ix) $\alpha x \oplus \beta x = |\alpha - \beta| x = (\alpha \oplus \beta) x$, *if* $x \propto 0$.

Proposition 2.9 ([9]). *Let C be a normal cone in $\mathcal{H}_p$ with constant λ_N, then for each $x, y \in \mathcal{H}_p$, the following relations hold:*

(i) $\|0 \oplus 0\| = \|0\| = 0$;
(ii) $\|x \vee y\| \leq \|x\| \vee \|y\| \leq \|x\| + \|y\|$;
(iii) $\|x \oplus y\| \leq \|x - y\| \leq \lambda_N \|x \oplus y\|$;
(iv) *if* $x \propto y$, *then* $\|x \oplus y\| = \|x - y\|$.

Definition 2.10 ([21]). Let $F : \mathcal{H}_p \longrightarrow \mathcal{H}_p$ be a single-valued mapping, then:

(i) F is said to be comparison mapping, if for each $x, y \in \mathcal{H}_p$, $x \propto y$ then $F(x) \propto F(y)$, $x \propto F(x)$ and $y \propto F(y)$;
(ii) F is said to be strongly comparison mapping, if F is a comparison mapping and $F(x) \propto F(y)$ if and only if $x \propto y$, $\forall\, x, y \in \mathcal{H}_p$.

Definition 2.11 ([20]). A single-valued mapping $F : \mathcal{H}_p \longrightarrow \mathcal{H}_p$ is said to be β-ordered compression mapping if F is a comparison mapping and

$$F(x) \oplus F(y) \leq k(x \oplus y), \text{ for } 0 < k < 1.$$

Definition 2.12 ([20]). Let $A, B : \mathcal{H}_p \longrightarrow \mathcal{H}_p$ and $H : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow \mathcal{H}_p$ be single-valued mappings. Then for all $x, y \in \mathcal{H}_p$:

(i) H is said to be t_1-ordered compression mapping in the first argument, if

$$H(x,\cdot)\oplus H(y,\cdot)\leq t_1(x\oplus y),\ 0<t_1<1;$$

(ii) H is said to be t_2-ordered compression mapping in the second argument, if

$$H(\cdot,x)\oplus H(\cdot,y)\leq t_2(x\oplus y),\ 0<t_2<1;$$

(iii) H is said to be k_1-ordered compression mapping with respect to A, if

$$H(A(x),\cdot)\oplus H(A(y),\cdot)\leq k_1(x\oplus y),\ 0<k_1<1;$$

(iv) H is said to be k_2-ordered compression mapping with respect to B, if

$$H(\cdot,B(x))\oplus H(\cdot,B(y))\leq k_2(x\oplus y),\ 0<k_2<1.$$

Definition 2.13 ([20]). A single-valued mapping $F:\mathcal{H}_p\longrightarrow\mathcal{H}_p$ is said to be Lipschitz-type-continuous if there exists a constant $\beta>0$ such that

$$\|F(x)\oplus F(y)\|\leq\beta\|x\oplus y\|,\ \forall\ x,y\in\mathcal{H}_p.$$

Definition 2.14 ([20]). A set-valued mapping $T:\mathcal{H}_p\longrightarrow C^*(\mathcal{H}_p)$ is said to be $\mathcal{D}$-Lipschitz-type-continuous if for all $x,y\in\mathcal{H}_p$, $x\propto y$, there exists a constant $\gamma>0$ such that

$$\mathcal{D}\big(T(x),T(y)\big)\leq\gamma\|x\oplus y\|.$$

Definition 2.15 ([21]). Let $M:\mathcal{H}_p\longrightarrow 2^{\mathcal{H}_p}$ be a set-valued mapping. Then:

(i) M is said to be a comparison mapping if for any $v_x\in M(x), x\propto v_x$, and if $x\propto y$, then for $v_x\in M(x)$ and $v_y\in M(y), v_x\propto v_y,\ \forall\ x,y\in\mathcal{H}_p$;

(ii) A comparison mapping M is said to be α-non-ordinary difference mapping if there exists a constant $\alpha>0$ such that

$$(v_x\oplus v_y)\oplus\alpha(x\oplus y)=0 \text{ holds},\ \forall\ x,y\in\mathcal{H}_p, v_x\in M(x), v_y\in M(y);$$

(iii) A comparison mapping M is said to be θ-ordered rectangular if there exists a constant $\theta>0$ such that

$$\langle v_x\odot v_y,-(x\oplus y)\rangle\geq\theta\|x\oplus y\|^2 \text{ holds},\ \forall\ x,y\in\mathcal{H}_p, v_x\in M(x), v_y\in M(y);$$

(iv) A comparison mapping M is said to be ρ-XOR-ordered strongly monotone compression mapping if for $x\propto y$, there exists a constant $\rho>0$ such that

$$\rho(v_x\oplus v_y)\geq x\oplus y,\ \forall\ x,y\in\mathcal{H}_p, v_x\in M(x), v_y\in M(y).$$

Definition 2.16 ([20]). Let $A,B:\mathcal{H}_p\longrightarrow\mathcal{H}_p$ and $H:\mathcal{H}_p\times\mathcal{H}_p\longrightarrow\mathcal{H}_p$ be single-valued mappings, then H is said to be:

(i) mixed comparison mapping with respect to A and B, if for each $x,y\in\mathcal{H}_p$, $x\propto y$, then $H(A(x),B(x))\propto H(A(y),B(y))$, $x\propto H(A(x),B(x))$ and $y\propto H(A(y),B(y))$;

(ii) mixed strongly comparison mapping with respect to A and B, if for each $x,y\in\mathcal{H}_p$, $H(A(x),B(x))\propto H(A(y),B(y))$ if and only if $x\propto y$.

Definition 2.17. Let $A,B:\mathcal{H}_p\longrightarrow\mathcal{H}_p$ and $H:\mathcal{H}_p\times\mathcal{H}_p\longrightarrow\mathcal{H}_p$ be single-valued mappings such that $H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B. Then, a set-valued comparison mapping $M:\mathcal{H}_p\times\mathcal{H}_p\longrightarrow 2^{\mathcal{H}_p}$ is said to be (α,ρ)-XOR-NODSM if:

(i) M is an α-non-ordinary difference mapping and ρ-XOR-ordered strongly monotone compression mapping;

(ii) $[H(A,B)\oplus\rho M(\cdot,\zeta)]\,(\mathcal{H}_p)=\mathcal{H}_p$, for some fixed $\zeta\in\mathcal{H}_p$.

Definition 2.18. Let $M : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow 2^{\mathcal{H}_p}$ be an (α, ρ)-XOR-NODSM mapping. Then, for fixed $\zeta \in \mathcal{H}_p$, the generalized resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)} : \mathcal{H}_p \longrightarrow \mathcal{H}_p$ is defined as:

$$\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) = \left[H(A,B) \oplus \rho M(\cdot,\zeta)\right]^{-1}(x),\ \forall\ x \in \mathcal{H}_p. \tag{2.1}$$

Now, we discuss some properties of the generalized resolvent operator.

Proposition 2.19. *Let $A, B : \mathcal{H}_p \longrightarrow \mathcal{H}_p, H : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow \mathcal{H}_p$ be single-valued mappings such that $H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B. Let $M : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow 2^{\mathcal{H}_p}$ is the set-valued θ-ordered rectangular mapping with $\rho\theta > |k_1 - k_2|$. Then the resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)} : \mathcal{H}_p \longrightarrow \mathcal{H}_p$ is single-valued.*

Proof. For any given $u \in \mathcal{H}_p$ and $\rho > 0$, let $x, y = \left[H(A,B) \oplus \rho M(\cdot,\zeta)\right]^{-1}(u)$. Then,

$$v_x = \frac{1}{\rho}\left[u \oplus H\big(A(x), B(x)\big)\right] \in M(x,\zeta) \text{ and } v_y = \frac{1}{\rho}\left[u \oplus H\big(A(y), B(y)\big)\right] \in M(y,\zeta).$$

Using (i) and (ii) of Proposition 2.8, we have

$$\begin{aligned}
v_x \odot v_y &= \frac{1}{\rho}\left[u \oplus H\big(A(x), B(x)\big)\right] \odot \frac{1}{\rho}\left[u \oplus H\big(A(y), B(y)\big)\right] \\
&= \frac{1}{\rho}\left\{\left[u \oplus H\big(A(x), B(x)\big)\right] \odot \left[u \oplus H\big(A(y), B(y)\big)\right]\right\} \\
&= -\frac{1}{\rho}\left\{\left[u \oplus H\big(A(x), B(x)\big)\right] \oplus \left[u \oplus H\big(A(y), B(y)\big)\right]\right\} \\
&= -\frac{1}{\rho}\left\{(u \oplus u) \oplus \left[H\big(A(x), B(x)\big) \oplus H\big(A(y), B(y)\big)\right]\right\} \\
&= -\frac{1}{\rho}\left\{0 \oplus \left[H\big(A(x), B(x)\big) \oplus H\big(A(y), B(y)\big)\right]\right\} \\
&\le -\frac{1}{\rho}\left[H\big(A(x), B(x)\big) \oplus H\big(A(y), B(y)\big)\right] \\
&\le -\frac{1}{\rho}\left\{\left[H\big(A(x), B(x)\big) \oplus H\big(A(x), B(y)\big)\right]\right. \\
&\quad \left. \oplus \left[H\big(A(x), B(y)\big) \oplus H\big(A(y), B(y)\big)\right]\right\}.
\end{aligned} \tag{2.2}$$

Since, M is θ-ordered rectangular mapping, $H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B and using (2.2), we have

$$\begin{aligned}
\theta\|x \oplus y\|^2 &\le \langle v_x \odot v_y, -(x \oplus y)\rangle \\
&\le \Big\langle -\frac{1}{\rho}\left\{\left[H\big(A(x), B(x)\big) \oplus H\big(A(x), B(y)\big)\right]\right. \\
&\quad \left. \oplus \left[H\big(A(x), B(y)\big) \oplus H\big(A(y), B(y)\big)\right]\right\}, -(x \oplus y)\Big\rangle \\
&\le \frac{1}{\rho}\left\{\big\langle H\big(A(x), B(x)\big) \oplus H\big(A(x), B(y)\big), x \oplus y\big\rangle\right. \\
&\quad \left. \oplus \big\langle H\big(A(x), B(y)\big) \oplus H\big(A(y), B(y)\big), x \oplus y\big\rangle\right\} \\
&\le \frac{1}{\rho}\langle k_1(x \oplus y), x \oplus y\rangle \oplus \langle k_2(x \oplus y), x \oplus y\rangle \\
&\le \frac{|k_1 - k_2|}{\rho}\|x \oplus y\|^2.
\end{aligned}$$

i.e.,

$$\left(\theta - \frac{|k_1 - k_2|}{\rho}\right) \|x \oplus y\|^2 \leq 0, \text{ for } \theta > \frac{|k_1 - k_2|}{\rho},$$

which shows that $\|x \oplus y\| = 0$, which implies $x \oplus y = 0$.

Therefore, $x = y$, that is the resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}$ is single-valued for $\rho\theta > |k_1 - k_2|$. □

Proposition 2.20. *Let $M : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow 2^{\mathcal{H}_p}$ be an (α, ρ)-XOR-NODSM set-valued mapping such that $H(\cdot, \cdot)$ is mixed strongly comparison mapping with respect to A and B. Then the generalized resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}$ is a comparison mapping.*

Proof. Since M is (α, ρ)-XOR-NODSM set-valued mapping, therefore M is α-non-ordinary difference as well as ρ-XOR-ordered strongly monotone compression mapping.

For any $x, y \in \mathcal{H}_p$, let $x \propto y$,

$$v^*_x = \frac{1}{\rho}\left[x \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right] \in M\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x), \zeta\right) \tag{2.3}$$

$$\text{and } v^*_y = \frac{1}{\rho}\left[y \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right] \in M\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y), \zeta\right). \tag{2.4}$$

Since M is ρ-XOR-ordered strongly monotone compression mapping, therefore in view of (2.3) and (2.4), we have

$$\begin{aligned}
(x \oplus y) &\leq \rho(v^*_x \oplus v^*_y) \\
&\leq \frac{\rho}{\rho}\Big\{\left[x \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right] \\
&\quad \oplus \left[y \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]\Big\} \\
&\leq (x \oplus y) \oplus \Big[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \\
&\quad \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\Big].
\end{aligned}$$

Thus,

$$0 \leq H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)$$

$$\begin{aligned}
0 \leq &\Big[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) - H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\Big] \\
&\vee \Big[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right) - H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\Big].
\end{aligned}$$

It follows that either

$$0 \leq \Big[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) - H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\Big]$$

or

$$0 \leq \Big[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right) - H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\Big]$$

This implies

$$H(A, B)\big(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\big) \propto H(A, B)\big(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\big).$$

Since $H(\cdot,\cdot)$ is mixed strongly comparison mapping with respect to A, B, it follows that, $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \propto \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)$, thereby showing that the resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}$ is a comparison mapping. $\square$

Proposition 2.21. *Let the mappings A, B, H, M be same as defined in Proposition 2.19, then the generalized resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)} : \mathcal{H}_p \longrightarrow \mathcal{H}_p$ is $\frac{1}{\rho\theta-(k_1+k_2)}$-Lipschitz-type-continuous for $\rho\theta > (k_1 + k_2)$, i.e.,*

$$\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\| \leq \frac{1}{\rho\theta - (k_1 + k_2)}\|x \oplus y\|, \ \forall\ x, y \in \mathcal{H}_p.$$

Proof. For any $x, y \in \mathcal{H}_p$, let $x \propto y$,

$$v^*_x = \frac{1}{\rho}\left[x \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right] \in M\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x), \zeta\right)$$

$$\text{and } v^*_y = \frac{1}{\rho}\left[y \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right] \in M\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y), \zeta\right).$$

$$\begin{aligned}
\text{Now,}\qquad v^*_x \oplus v^*_y &= \frac{1}{\rho}\Big\{\Big[x \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\Big] \\
&\quad \oplus \Big[y \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\Big]\Big\} \\
&= \frac{1}{\rho}\Big\{(x \oplus y) \oplus \Big[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \\
&\quad \oplus\ H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\Big]\Big\}. \qquad (2.5)
\end{aligned}$$

Since $M(\cdot,\zeta)$ is θ-ordered rectangular mapping and using (2.5), for any $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \in M(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x), \zeta)$ and $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y) \in M(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y), \zeta)$, we have

$$\begin{aligned}
&\theta\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|^2 \\
&\leq \left\langle v^*_x \odot v^*_x, -(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y))\right\rangle \\
&\leq \left\langle v^*_x \oplus v^*_x, \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\rangle \\
&= \frac{1}{\rho}\Big\langle(x \oplus y) \oplus \Big[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \\
&\quad \oplus\ H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\Big], \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\Big\rangle \\
&\leq \frac{1}{\rho}\Big\{\Big\|(x \oplus y) \oplus \Big[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \\
&\quad \oplus\ H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\Big]\Big\|\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|\Big\} \\
&\leq \frac{1}{\rho}\Big\{\Big\|(x \oplus y) - \Big[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \\
&\quad \oplus\ H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\Big]\Big\|\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|\Big\} \\
&\leq \frac{1}{\rho}\Big\{\Big[\|x \oplus y\| + \Big\|H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \\
&\quad \oplus\ H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\Big\|\Big]\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|\Big\} \\
&\leq \frac{1}{\rho}\Big\{\|x \oplus y\|\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|
\end{aligned}$$

$$+ \left\| H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right\| \left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\| \Big\}. \tag{2.6}$$

Since $H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B, we have

$$\begin{aligned}
&\left\| H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right\| \\
&= \Big\| \Big[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\Big] \\
&\quad \oplus \Big[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\Big]\Big\| \\
&\leq \Big\| \Big[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\Big] \\
&\quad - \Big[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\Big]\Big\| \\
&\leq \left\| H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right\| \\
&\quad + \left\| H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right\| \\
&\leq (k_1+k_2)\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|.
\end{aligned}$$

Thus, from (2.6), we have

$$\begin{aligned}
\theta\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|^2 \leq \frac{1}{\rho}\|x\oplus y\| \left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\| \\
+ \frac{k_1+k_2}{\rho}\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|^2.
\end{aligned}$$

This implies,

$$\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\| \leq \frac{1}{\rho\theta-(k_1+k_2)}\|x\oplus y\|, \ \forall\, x,y\in\mathcal{H}_p.$$

This completes the proof. ∎

In support of Propositions 2.19-2.21, we present the following example.

Example 2.22. Let $\mathcal{H}_p=[0,\infty)\times[0,\infty)$ with the usual inner product and norm and let $C=[0,1]\times[0,1]$ be a normal cone. Let $A,B:\mathcal{H}_p\longrightarrow\mathcal{H}_p$ and $H:\mathcal{H}_p\times\mathcal{H}_p\longrightarrow\mathcal{H}_p$ be defined by

$$A(x)=\left(\frac{x_1}{9}+3,\frac{x_2}{9}+6\right),\quad B(x)=\left(\frac{x_1}{3}+1,\frac{x_2}{3}+2\right)$$

$$\text{and}\qquad H\left(A(x),B(x)\right)=\frac{A(x)}{3}\oplus B(x),\ \forall\, x=(x_1,x_2)\in\mathcal{H}_p.$$

For $x=(x_1,x_2), y=(y_1,y_2)\in\mathcal{H}_p, x\propto y$, we have

$$\begin{aligned}
H\left(A(x),u\right)\oplus H\left(A(y),u\right) &= \left(\frac{A(x)}{3}\oplus u\right)\oplus\left(\frac{A(y)}{3}\oplus u\right) \\
&= \frac{1}{3}\left(A(x)\oplus A(y)\right) \\
&= \frac{1}{3}\left[\left(A(x)-A(y)\right)\vee\left(A(y)-A(x)\right)\right] \\
&= \frac{1}{3}\left[\left\{\left(\frac{x_1}{9}+3,\frac{x_2}{9}+6\right)-\left(\frac{y_1}{9}+3,\frac{y_2}{9}+6\right)\right\}\right.
\end{aligned}$$

$$\begin{aligned}
&\vee \left\{\left(\frac{y_1}{9}+3, \frac{y_2}{9}+6\right)-\left(\frac{x_1}{9}+3, \frac{x_2}{9}+6\right)\right\}\Big] \\
&= \frac{1}{3}\left[\left(\frac{x_1-y_1}{9}, \frac{x_2-y_2}{9}\right) \vee \left(\frac{y_1-x_1}{9}, \frac{y_2-x_2}{9}\right)\right] \\
&= \frac{1}{27}\left[(x-y)\vee(y-x)\right] \\
&= \frac{1}{27}(x\oplus y) \\
&\leq \frac{1}{24}(x\oplus y).
\end{aligned}$$

Hence, H is $\frac{1}{24}$-ordered compression mapping with respect to A. Similarly, we can show that H is $\frac{1}{2}$-ordered compression mapping with respect to B.

Suppose that the set-valued mapping $M : \mathcal{H}_p \longrightarrow 2^{\mathcal{H}_p}$ be defined by

$$M(x) = \big\{(3x_1, 3x_2)\big\},\ \forall\ x = (x_1, x_2) \in \mathcal{H}_p.$$

It can be easily verified that M is a comparison mapping, 1-XOR-ordered strongly monotone compression mapping and 3-non-ordinary difference mapping. Further, it is clear that for $\rho = 1$, $\big[H(A, B) + \rho M\big](\mathcal{H}_p) = \mathcal{H}_p$. Hence, M is an $(3, 1)$-XOR-NODSM strongly monotone compression mapping.

Let $\upsilon_x = (3x_1, 3x_2) \in M(x)$ and $\upsilon_y = (3y_1, 3y_2) \in M(y)$, then

$$\begin{aligned}
\big\langle \upsilon_x \odot \upsilon_y, -(x\oplus y)\big\rangle &= \big\langle \upsilon_x \oplus \upsilon_y, x\oplus y\big\rangle \\
&= \big\langle 3x \oplus 3y, x\oplus y\big\rangle \\
&= 3\big\langle x\oplus y, x\oplus y\big\rangle \\
&= 3\|x\oplus y\|^2 \\
&\geq 2\|x\oplus y\|^2.
\end{aligned}$$

i.e.,

$$\big\langle \upsilon_x \odot \upsilon_y, -(x\oplus y)\big\rangle\ \geq 2\ \|x\oplus y\|^2,\ \forall\ x, y \in \mathcal{H}_p.$$

Thus, M is 2-ordered rectangular comparison mapping.

The resolvent operator defined by (2.1) is given by

$$\mathcal{R}^{H(A,B)}_{\rho,M}(x) = \left(\frac{27x_1}{73}, \frac{27x_2}{73}\right),\ \forall\ x = (x_1, x_2) \in \mathcal{H}_p.$$

It is easy to verify that the resolvent operator defined above is comparison and single-valued mapping.

Further,

$$\begin{aligned}
\left\|\mathcal{R}^{H(A,B)}_{\rho,M}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M}(y)\right\| &= \left\|\frac{27x}{73} \oplus \frac{27y}{73}\right\| \\
&= \frac{27}{73}\|x\oplus y\| \\
&\leq \frac{24}{35}\|x\oplus y\|.
\end{aligned}$$

i.e.,

$$\left\|\mathcal{R}^{H(A,B)}_{\rho,M}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M}(y)\right\| \leq \frac{24}{35}\|x\oplus y\|,\ \forall\ x, y \in \mathcal{H}_p.$$

This shows that the resolvent operator is $\mathcal{R}^{H(A,B)}_{\rho,M}$ is $\frac{24}{35}$-Lipschitz-type-continuous.

3. Formulation of the Problem and Existence of Solution

Let $\mathcal{H}_p$ be a real ordered positive Hilbert space. For each $i = 1, 2$, let $A, B, g_i, p_i, G_i : \mathcal{H}_p \longrightarrow \mathcal{H}_p, F_i, H : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow \mathcal{H}_p$ be single-valued mappings and $S, T : \mathcal{H}_p \longrightarrow C^*(\mathcal{H}_p), M_i : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow 2^{\mathcal{H}_p}$ be set-valued mappings. Then, for fixed $\zeta, \zeta' \in \mathcal{H}_p$, we consider the following generalized system of mixed variational inclusion problem (in short, GSMVIP):

For any $\omega_1, \omega_2 \in \mathcal{H}_p$, find $x, y \in \mathcal{H}_p, u \in S(x), v \in T(x), u' \in S(y), v' \in T(y)$ such that

$$\begin{cases} \omega_1 \in F_1\big((g_1 - p_1)(u), G_1(v)\big) \oplus M_1\big((g_1 - p_1)(x), \zeta\big) \\ \omega_2 \in F_2\big(G_2(v'), (g_2 - p_2)(u')\big) \oplus M_2\big(\zeta', (g_2 - p_2)(y)\big). \end{cases} \tag{3.1}$$

For suitable choices of mappings and the underlying space $\mathcal{H}_p$, GSMVIP (3.1) encompasses several classes of variational inclusions including those involving involving XOR-operation as special cases, see for example [1, 2, 4, 21, 23, 25] and the related references cited therein.

Lemma 3.1. *The generalized system of mixed variational inclusion problem (3.1) admits a solution (x, y, u, v, u', v') where $x, y \in \mathcal{H}_p, u \in S(x), v \in T(x), u' \in S(y), v' \in T(y)$ if and only if it satisfies the following equations:*

$$\begin{aligned}(g_1 - p_1)(x) \\ = \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)}\big[\rho_1 F_1\big((g_1 - p_1)(u), G_1(v)\big) \oplus H(A, B)(g_1 - p_1)(x) \oplus \rho_1\omega_1\big]\end{aligned} \tag{3.2}$$

$$\begin{aligned}(g_2 - p_2)(y) \\ = \mathcal{R}^{H(A,B)}_{\rho_2, M_2(\zeta',\cdot)}\big[\rho_2 F_2\big(G_2(v'), (g_2 - p_2)(u')\big) \oplus H(A, B)(g_2 - p_2)(y) \oplus \rho_2\omega_2\big],\end{aligned} \tag{3.3}$$

where $\mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)} = \big[H(A, B) \oplus \rho_1 M_1(\cdot, \zeta)\big]^{-1}$, $\mathcal{R}^{H(A,B)}_{\rho_2, M_2(\zeta',\cdot)} = \big[H(A, B) \oplus \rho_2 M_2(\zeta', \cdot)\big]^{-1}$ *and* $\rho_1, \rho_2 > 0$.

Proof. Using the definition of the generalized resolvent operator and suppose

$$(g_1 - p_1)(x) = \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)}\big[\rho_1 F_1\big((g_1 - p_1)(u), G_1(v)\big) \oplus H(A, B)(g_1 - p_1)(x) \oplus \rho_1\omega_1\big].$$

Then,

$$\begin{aligned}(g_1 - p_1)(x) = \big[H(A, B) \oplus \rho_1 M_1(\cdot, \zeta)\big]^{-1}\big[&\rho_1 F_1\big((g_1 - p_1)(u), G_1(v)\big) \\ &\oplus H(A, B)(g_1 - p_1)(x) \oplus \rho_1\omega_1\big] \\ \Rightarrow H(A, B)(g_1 - p_1)(x) \oplus \rho_1 M_1\left((g_1 - p_1)(x), \zeta\right)& \\ \ni \rho_1 F_1\big((g_1 - p_1)(u), G_1(v)\big) &\oplus H(A, B)(g_1 - p_1)(x) \oplus \rho_1\omega_1.\end{aligned}$$

Which gives $\omega_1 \in F_1\big((g_1 - p_1)(u), G_1(v)\big) \oplus M_1\big((g_1 - p_1)(x), \zeta\big)$, the first inclusion of the GSMVIP (3.1). Similarly, we can get the second inclusion from equation (3.3). □

4. Iterative Algorithm and Convergence Analysis

Based on Lemma 3.1 and Nadler's Theorem [24], we establish the following iterative algorithm to approximate the solution of GSMVIP (3.1).

Iterative Algorithm 4.1. Step 1. For any $\omega_1, \omega_2 \in \mathcal{H}_p$ and $\rho_1, \rho_2 > 0$, choose $x_0, y_0 \in \mathcal{H}_p, u_0 \in S(x_0), v_0 \in T(x_0), u'_0 \in S(y_0)$ and $v'_0 \in T(y_0)$.
Step 2. Let

$$\begin{aligned}
&(g_1 - p_1)x_{n+1} \\
&= \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)}\Big\{\rho_1 F_1\big((g_1-p_1)(u_n), G_1(v_n)\big) \oplus H(A,B)(g_1-p_1)(x_n) \oplus \rho_1\omega_1 \oplus e_n\Big\}, \\
&(g_2 - p_2)y_{n+1} \\
&= \mathcal{R}^{H(A,B)}_{\rho_2, M_2(\zeta',\cdot)}\Big\{\rho_2 F_2\big(G_2(v'_n), (g_2-p_2)(u'_n)\big) \oplus H(A,B)(g_2-p_2)(y_n) \oplus \rho_2\omega_2 \oplus e'_n\Big\}.
\end{aligned}$$

Step 3. Choose $u_{n+1} \in S(x_{n+1}), v_{n+1} \in T(x_{n+1}), u'_{n+1} \in S(y_{n+1})$ and $v'_{n+1} \in T(y_{n+1})$ such that

$$\left\{\begin{array}{l}
\|u_{n+1} \oplus u_n\| \le \|u_{n+1} - u_n\| \le \big(1 + (1+n)^{-1}\big)\, \mathcal{D}\left(S(x_{n+1}), S(x_n)\right), \\
\|v_{n+1} \oplus v_n\| \le \|v_{n+1} - v_n\| \le \big(1 + (1+n)^{-1}\big)\, \mathcal{D}\left(T(x_{n+1}), T(x_n)\right), \\
\|u'_{n+1} \oplus u'_n\| \le \|u'_{n+1} - u'_n\| \le \big(1 + (1+n)^{-1}\big)\, \mathcal{D}\left(S(y_{n+1}), S(y_n)\right), \\
\|v'_{n+1} \oplus v'_n\| \le \|v'_{n+1} - v'_n\| \le \big(1 + (1+n)^{-1}\big)\, \mathcal{D}\left(T(y_{n+1}), T(y_n)\right).
\end{array}\right. \tag{4.1}$$

Step 4. Choose errors $\{e_n\}, \{e'_n\} \subset \mathcal{H}_p$ to take into account the possible inexact computations such that, for all $\nu_1, \nu_2 \in (0,1)$

$$\sum_{j=1}^{\infty} \|e_j \oplus e_{j-1}\|\nu_1^{-j} < \infty,\ \sum_{j=1}^{\infty} \|e'_j \oplus e'_{j-1}\|\nu_2^{-j} < \infty,\ \lim_{n\to\infty} e_n = 0,\ \lim_{n\to\infty} e'_n = 0.$$

Step 5. If $u_{n+1} \in S(x_{n+1}), v_{n+1} \in T(x_{n+1}), u'_{n+1} \in S(y_{n+1})$ and $v'_{n+1} \in T(y_{n+1})$ satisfy (4.1) to sufficient accuracy, stop; otherwise, set $n := n+1$ and return to Step 2.

Next, we prove the following theorem which ensures the convergence of iterative sequences generated by the Iterative Algorithm 4.1.

Theorem 4.2. *Let $C \subset \mathcal{H}_p$be a normal cone with constant λ_N. For $i = 1, 2$, let $A, B, g_i, p_i, G_i : \mathcal{H}_p \longrightarrow \mathcal{H}_p, F_i, H : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow \mathcal{H}_p$ be single-valued mappings such that:*

(i) *$H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B;*
(ii) *F_1 is τ_1-Lipschitz-type continuous with respect to $(g_1 - p_1)$ in the first argument and σ_1-Lipschitz-type-continuous with respect to G_1 in the second argument;*
(iii) *F_2 is τ_2-Lipschitz-type continuous with respect to $(g_2 - p_2)$ in the second argument and σ_2-Lipschitz-type-continuous with respect to G_2 in the first argument;*
(iv) *$(g_i - p_i)$ is r_i-Lipschitz-type-continuous and $(g_i - p_i \oplus I)$ is δ_i-Lipschitz-type-continuous, where I is the Identity mapping.*

Also, let $S, T : \mathcal{H}_p \longrightarrow C^*(\mathcal{H}_p)$ and $M_1, M_2 : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow 2^{\mathcal{H}_p}$ be set-valued mappings such that:

(i) *M_i is (α_i, ρ_i)-XOR-NODSM and θ_i-ordered rectangular mapping, repectively, for $i = 1, 2$;*
(ii) *S is γ_1-$\mathcal{D}$-Lipschitz-type-continuous and T is γ_2-$\mathcal{D}$-Lipschitz-type-continuous.*

If $x_{n+1} \propto x_n, y_{n+1} \propto y_n, (g_1 - p_1)(x_{n+1}) \propto (g_1 - p_1)(x_n), (g_2 - p_2)(y_{n+1}) \propto (g_2 - p_2)(y_n)$, for $n = 0, 1, 2, \cdots$ and the following conditions are satisfied:

$$0 < \varphi = \frac{\lambda_N|\rho_1|(\tau_1\gamma_1 + \sigma_1\gamma_2) + r_1\lambda_N(k_1 + k_2)}{(1-\delta_1)\left[\rho_1\theta_1 - (k_1 + k_2)\right]} < 1, \tag{4.2}$$

$$0 < \vartheta = \frac{\lambda_N|\rho_2|(\sigma_2\gamma_2 + \tau_2\gamma_1) + r_2\lambda_N(k_1 + k_2)}{(1-\delta_2)\left[\rho_2\theta_2 - (k_1 + k_2)\right]} < 1. \tag{4.3}$$

Then GSMVIP (3.1) has a solution (x, y, u, v, u', v'), where $x, y \in \mathcal{H}_p, u \in S(x), v \in T(x), u' \in S(y), v' \in T(y)$. Also, the sequences $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\}, \{u'_n\}, \{v'_n\}$ generated by the Iterative Algorithm 4.1 converge strongly to x, y, u, v, u', v', respectively.

Proof. By Algorithm 4.1 and Proposition 2.8, we have

$$\begin{aligned}
0 &\le (g_1 - p_1)x_{n+1} \oplus (g_1 - p_1)x_n \\
&= \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)}\Big\{\rho_1 F_1\big((g_1 - p_1)(u_n), G_1(v_n)\big) \oplus H(A,B)(g_1 - p_1)(x_n) \oplus \rho_1\omega_1 \oplus e_n\Big\} \\
&\quad \oplus \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)}\Big\{\rho_1 F_1\big((g_1 - p_1)(u_{n-1}), G_1(v_{n-1})\big) \oplus H(A,B)(g_1 - p_1)(x_{n-1}) \\
&\quad \oplus \rho_1\omega_1 \oplus e_{n-1}\Big\}.
\end{aligned}$$

Now, using Proposition 2.9 and Lipschitz-type-continuity of the generalized resolvent operator, we have

$$\begin{aligned}
&\big\|(g_1 - p_1)x_{n+1} \oplus (g_1 - p_1)x_n\big\| \\
&\le \lambda_N\Big\|\mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)}\Big\{\rho_1 F_1\big((g_1 - p_1)(u_n), G_1(v_n)\big) \oplus H(A,B)(g_1 - p_1)(x_n) \oplus \rho_1\omega_1 \oplus e_n\Big\} \\
&\quad \oplus \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)}\Big\{\rho_1 F_1\big((g_1 - p_1)(u_{n-1}), G_1(v_{n-1})\big) \oplus H(A,B)(g_1 - p_1)(x_{n-1}) \oplus \rho_1\omega_1 \\
&\quad \oplus e_{n-1}\Big\}\Big\| \\
&\le \frac{\lambda_N}{\rho_1\theta_1 - (k_1 + k_2)}\left\|\big[\rho_1 F_1\big((g_1 - p_1)(u_n), G_1(v_n)\big) \oplus H(A,B)(g_1 - p_1)(x_n) \oplus \rho_1\omega_1 \oplus e_n\big]\right. \\
&\quad \left.\oplus\big[\rho_1 F_1\big((g_1 - p_1)(u_{n-1}), G_1(v_{n-1})\big) \oplus H(A,B)(g_1 - p_1)(x_{n-1}) \oplus \rho_1\omega_1 \oplus e_{n-1}\big]\right\| \\
&\le \frac{\lambda_N|\rho_1|}{\rho_1\theta_1 - (k_1 + k_2)}\left\|F_1\big((g_1 - p_1)(u_n), G_1(v_n)\big) \oplus F_1\big((g_1 - p_1)(u_{n-1}), G_1(v_{n-1})\big)\right\| \\
&\quad + \frac{\lambda_N}{\rho_1\theta_1 - (k_1 + k_2)}\left\|H(A,B)(g_1 - p_1)(x_n) \oplus H(A,B)(g_1 - p_1)(x_{n-1})\right\| \\
&\quad + \frac{\lambda_N}{\rho_1\theta_1 - (k_1 + k_2)}\left\|e_n \oplus e_{n-1}\right\|.
\end{aligned} \tag{4.4}$$

Since, XOR operator is associative, $F_1 : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow \mathcal{H}_p$ is τ_1-Lipschitz-type-continuous with respect to $(g_1 - p_1)$ in first argument and σ_1-Lipschitz-type-continuous with respect to G_1 in second argument and S, T are γ_1, γ_2-$\mathcal{D}$-Lipschitz-type-continuous, respectively, therefore in view of Algorithm 4.1, we have

$$\begin{aligned}
&\big\|F_1\big((g_1 - p_1)(u_n), G_1(v_n)\big) \oplus F_1\big((g_1 - p_1)(u_{n-1}), G_1(v_{n-1})\big)\big\| \\
&\qquad \le \big\|F_1\big((g_1 - p_1)(u_n), G_1(v_n)\big) \oplus F_1\big((g_1 - p_1)(u_{n-1}), G_1(v_n)\big)\big\| \\
&\qquad\quad + \big\|F_1\big((g_1 - p_1)(u_{n-1}), G_1(v_n)\big) \oplus F_1\big((g_1 - p_1)(u_{n-1}), G_1(v_{n-1})\big)\big\| \\
&\qquad \le \tau_1\|u_n \oplus u_{n-1}\| + \sigma_1\|v_n \oplus v_{n-1}\| \\
&\qquad \le \tau_1\|u_n - u_{n-1}\| + \sigma_1\|v_n - v_{n-1}\|
\end{aligned}$$

$$\leq \tau_1\left(1+n^{-1}\right)\mathcal{D}\left(S(x_n),S(x_{n-1})\right)+\sigma_1\left(1+n^{-1}\right)\mathcal{D}\left(T(x_n),T(x_{n-1})\right)$$
$$\leq \tau_1\gamma_1\left(1+n^{-1}\right)\|x_n-x_{n-1}\|+\sigma_1\gamma_2\left(1+n^{-1}\right)\|x_n-x_{n-1}\|$$
$$=\left[(\tau_1\gamma_1+\sigma_1\gamma_2)\left(1+n^{-1}\right)\right]\|x_n-x_{n-1}\|. \tag{4.5}$$

Since $H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B and (g_1-p_1) is r_1-Lipschitz-type-continuous, we have

$$\|H(A,B)(g_1-p_1)(x_n)\oplus H(A,B)(g_1-p_1)(x_{n-1})\|$$
$$\leq \Big\|\big[H\big(A(g_1-p_1)(x_n),B(g_1-p_1)(x_n)\big)\oplus H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_n)\big)\big]$$
$$\oplus\big[H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_n)\big)\oplus H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_{n-1})\big)\big]\Big\|$$
$$\leq \Big\|\big[H\big(A(g_1-p_1)(x_n),B(g_1-p_1)(x_n)\big)\oplus H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_n)\big)\big]$$
$$-\big[H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_n)\big)\oplus H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_{n-1})\big)\big]\Big\|$$
$$\leq \Big\|H\big(A(g_1-p_1)(x_n),B(g_1-p_1)(x_n)\big)\oplus H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_n)\big)\Big\|$$
$$+\Big\|H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_n)\big)\oplus H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_{n-1})\big)\Big\|$$
$$\leq k_1\|(g_1-p_1)(x_n)\oplus(g_1-p_1)(x_{n-1})\|+k_2\|(g_1-p_1)(x_n)\oplus(g_1-p_1)(x_{n-1})\|$$
$$=(k_1+k_2)\|(g_1-p_1)(x_n)\oplus(g_1-p_1)(x_{n-1})\|$$
$$\leq r_1(k_1+k_2)\|x_n\oplus x_{n-1}\|$$
$$\leq r_1(k_1+k_2)\|x_n-x_{n-1}\|. \tag{4.6}$$

Using (4.5) and (4.6) in (4.4), we have

$$\Big\|(g_1-p_1)x_{n+1}\oplus(g_1-p_1)x_n\Big\|$$
$$\leq \frac{\lambda_N|\rho_1|}{\rho_1\theta_1-(k_1+k_2)}\left[(\tau_1\gamma_1+\sigma_1\gamma_2)\left(1+n^{-1}\right)\right]\|x_n-x_{n-1}\|$$
$$+\frac{\lambda_N}{\rho_1\theta_1-(k_1+k_2)}r_1(k_1+k_2)\|x_n-x_{n-1}\|+\frac{\lambda_N}{\rho_1\theta_1-(k_1+k_2)}\|e_n\oplus e_{n-1}\|$$
$$=\frac{\lambda_N|\rho_1|(\tau_1\gamma_1+\sigma_1\gamma_2)\left(1+n^{-1}\right)+r_1\lambda_N(k_1+k_2)}{\rho_1\theta_1-(k_1+k_2)}\|x_n-x_{n-1}\|$$
$$+\frac{\lambda_N}{\rho_1\theta_1-(k_1+k_2)}\|e_n\oplus e_{n-1}\|. \tag{4.7}$$

Since $(g_1-p_1\oplus I)$ is δ_1-Lipschitz-type-continuous mapping and in view of (4.7), we have

$$\|x_{n+1}\oplus x_n\|$$
$$=\|[(g_1-p_1)x_{n+1}\oplus(g_1-p_1)x_n]\oplus[(g_1-p_1)x_{n+1}\oplus x_{n+1}\oplus(g_1-p_1)x_n\oplus x_n]\|$$
$$\leq\|[(g_1-p_1)x_{n+1}\oplus(g_1-p_1)x_n]-[(g_1-p_1)x_{n+1}\oplus x_{n+1}\oplus(g_1-p_1)x_n\oplus x_n]\|$$
$$\leq\|(g_1-p_1)x_{n+1}\oplus(g_1-p_1)x_n\|+\|(g_1-p_1\oplus I)x_{n+1}\oplus(g_1-p_1\oplus I)x_n\|$$
$$\leq\frac{\lambda_N|\rho_1|(\tau_1\gamma_1+\sigma_1\gamma_2)\left(1+n^{-1}\right)+r_1\lambda_N(k_1+k_2)}{\rho_1\theta_1-(k_1+k_2)}\|x_n-x_{n-1}\|$$
$$+\frac{\lambda_N}{\rho_1\theta_1-(k_1+k_2)}\|e_n\oplus e_{n-1}\|+\delta_1\|x_{n+1}\oplus x_n\|.$$

This implies,

$$\|x_{n+1}\oplus x_n\|\leq\frac{\lambda_N|\rho_1|(\tau_1\gamma_1+\sigma_1\gamma_2)\left(1+n^{-1}\right)+r_1\lambda_N(k_1+k_2)}{(1-\delta_1)\left[\rho_1\theta_1-(k_1+k_2)\right]}\|x_n-x_{n-1}\|$$

$$+\frac{\lambda_N}{(1-\delta_1)\left[\rho_1\theta_1-(k_1+k_2)\right]}\left\|e_n\oplus e_{n-1}\right\|.$$

Since $x_{n+1}\propto x_n,\ n=0,1,2,\cdots$, we have

$$\|x_{n+1}-x_n\|\leq\varphi_n\|x_n-x_{n-1}\|+\eta\left\|e_n\oplus e_{n-1}\right\|, \tag{4.8}$$

where

$$\varphi_n=\frac{\lambda_N|\rho_1|(\tau_1\gamma_1+\sigma_1\gamma_2)\left(1+n^{-1}\right)+r_1\lambda_N(k_1+k_2)}{(1-\delta_1)\left[\rho_1\theta_1-(k_1+k_2)\right]},\eta=\frac{\lambda_N}{(1-\delta_1)\left[\rho_1\theta_1-(k_1+k_2)\right]}.$$

Let

$$\varphi=\frac{\lambda_N|\rho_1|(\tau_1\gamma_1+\sigma_1\gamma_2)+r_1\lambda_N(k_1+k_2)}{(1-\delta_1)\left[\rho_1\theta_1-(k_1+k_2)\right]}.$$

It is clear that $\varphi_n\longrightarrow\varphi$ as $n\longrightarrow\infty$. By (4.2), we know that $0<\varphi<1$ and hence there exists $n_0>0$ and $\varphi_0\in(0,1)$ such that $\varphi_n\leq\varphi_0$ for all $n\geq n_0$. Therefore, by (4.8), we have

$$\|x_{n+1}-x_n\|\leq\varphi_0\|x_n-x_{n-1}\|+\eta\left\|e_n\oplus e_{n-1}\right\|,\ \forall\ n\geq n_0. \tag{4.9}$$

(4.9) implies that

$$\|x_{n+1}-x_n\|\leq\varphi_0^{n-n_0}\|x_{n_0+1}-x_{n_0}\|+\eta\sum_{j=1}^{n-n_0}\varphi_0^{j-1}t_{n-(j-1)}, \tag{4.10}$$

where $t_n=\|e_n\oplus e_{n-1}\|$ for all $n\geq n_0$. Hence, for any $m\geq n>n_0$, we have

$$\begin{aligned}\|x_m-x_n\|&\leq\sum_{k=n}^{m-1}\|x_{k+1}-x_k\|\\&\leq\sum_{k=n}^{m-1}\varphi_0^{k-n_0}\|x_{n_0+1}-x_{n_0}\|+\eta\sum_{k=n}^{m-1}\varphi_0^{k}\left[\sum_{j=1}^{k-n_0}\frac{t_{k-(j-1)}}{\varphi_0^{k-(j-1)}}\right].\end{aligned}$$

Since $\sum_{j=1}^{\infty}\|e_j\oplus e_{j-1}\|\nu_1^{-j}<\infty,\ \forall\ \nu_1\in(0,1)$ and $0<\varphi_0<1$, it follows that $\|x_m-x_n\|\to 0$ as $n\longrightarrow\infty$, and so $\{x_n\}$ is a Cauchy sequence in $\mathcal{H}_p$. Thus, there exists $x\in\mathcal{H}_p$ such that $x_n\to u$, as $n\to\infty$. Similarly, we can show $\{y_n\}$ to be a Cauchy sequence in $\mathcal{H}_p$ and thus there exists $y\in\mathcal{H}_p$ such that $y_n\to y$, thanks to the completeness of $\mathcal{H}_p$.

By Algorithm 4.1 and the $\mathcal{D}$-Lipschitz continuity of S and T, we have

$$\left\{\begin{array}{l}\|u_{n+1}\oplus u_n\|\leq\|u_{n+1}-u_n\|\leq\left(1+(1+n)^{-1}\right)\gamma_1\|x_{n+1}-x_n\|,\\\|v_{n+1}\oplus v_n\|\leq\|v_{n+1}-v_n\|\leq\left(1+(1+n)^{-1}\right)\gamma_2\|x_{n+1}-x_n\|,\\\|u'_{n+1}\oplus u'_n\|\leq\|u'_{n+1}-u'_n\|\leq\left(1+(1+n)^{-1}\right)\gamma_1\|y_{n+1}-y_n\|,\\\|v'_{n+1}\oplus v'_n\|\leq\|v'_{n+1}-v'_n\|\leq\left(1+(1+n)^{-1}\right)\gamma_2\|y_{n+1}-y_n\|.\end{array}\right. \tag{4.11}$$

It follows that $\{u_n\},\{v_n\},\{u'_n\}$ and $\{v'_n\}$ are all Cauchy sequences. Thus, there exists $u,v,u',v'\in\mathcal{H}_p$ such that $u_n\to u,v_n\to v,u'_n\to u'$ and $v'_n\to v'$ as $n\to\infty$.

Now, we show that $u\in S(x),v\in T(x),u'\in S(y)$ and $v'\in T(y)$. Since $u_n\in S(x_n)$, we have

$$\begin{aligned}d(u,S(x))&\leq\|u\oplus u_n\|+d(u_n,S(x))\\&\leq\|u-u_n\|+\|u_n\oplus S(x)\|\\&\leq\|u-u_n\|+\mathcal{D}\left(S(x_n),S(x)\right)\\&\leq\|u-u_n\|+\gamma_1\|x_n-x\|\longrightarrow 0\text{ as }n\to\infty.\end{aligned}$$

Since $S(x)$ is closed, it follows that $u \in S(x)$. Similarly, we can show that $v \in T(x), u' \in S(y)$ and $v' \in T(y)$. Thus in view of Lemma 3.1 we conclude that (x, y, u, v, u', v'), such that $x, y \in \mathcal{H}_p, u \in S(x), v \in T(x), u' \in S(y), v' \in T(y)$, is a solution of GSMVIP (3.1). This completes the proof. □

5. Conclusion

The results presented in this paper generalizes many known results in the literature. The class of XOR-NODSM mappings involving ⊕ operation is much wider and more general than those of (A, η)-accretive operator, (H, η)-monotone operator as already discussed by many researchers. The resolvent operator associated with XOR-NODSM mappings can be further exploited to solve different classes of variational inclusions and related systems in the setting of Banach and semi-inner product spaces considered in, see for example [1–11, 13–23, 25, 26].

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