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FIXED POINT THEOREMS IN PARTIAL b-METRIC-LIKE SPACES

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ABSTRACT. In this paper, we introduce an interesting extension of the b-metric space called b-metric-like space. We investigate some contraction mapping in partial b-metric-like space and prove the existence of fixed point of this mapping in partial b-metric-like space under some conditions.

KEYWORDS: Partial b-metric-like space, Partial b-metric space, Cauchy sequence, Fixed point.
AMS Subject Classification: :46C05, 47H09, 47H10,

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1. Introduction

In 1920, Banach [4] introduced a Banach Contraction Principle.

Theorem 1.1. *Let (X, d) be a complete metric space and let T be a contraction on X, there exists $r \in [0, 1)$ such that*

$$d\left(Tx, Ty\right) \leq rd\left(x, y\right), \forall x, y \in X.$$

Then T has a unique fixed point.

In recent years, many scholars have proposed a series of new concepts of contraction mapping and new fixed point theorems [5, 6, 7, 8, 9, 10].

In 1993, Bakhtin [5] introduced the concept of b-metric space which is a generalization of metric space. He proved the famous Banach Contraction Principle in the b-metric space, also see [6]. In 1994, S.G. Matthews [7] introduced the concept of partial metric space and proved the Banach Contraction Principle in the partial metric space.

In 2013, the notion of b-metric-like spaces were introduced by Alghamdi [8] and some fixed point theorems were studied in such spaces.

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Definition 1.2. Let X be a nonempty set, $s \geq 1$ be a given real number and let $b : X \times X \longrightarrow [0, \infty)$ be a mapping such that for all $x,\ y,\ z \in X$, the following conditions hold:

$(Pb1)$ $x = y$ if and only if $b(x,x) \ = b(x,y) \ = b(y,y)$;
$(Pb2)$ $b(x,x) \leq b(x,y)$;
$(Pb3)$ $b(x,y) = b(y,x)$;
$(Pb4)$ $b(x,y) \leq s[b(x,z) + b(z,y)] \ - b(z,z)$.

Then the pair (X, b) is called a partial b-metric space. The number s is called the coefficient of (X, b, s).

In 2014, S. Satish [9] introduced the concept of partial b-metric space and the fixed point theorem of Banach Contraction Principle and Kannan type mapping was proved in partial b-metric space. In 2018, J. Zhou, D. Zheng and G. Zhang [10] proved some fixed point theorem for C-contractive mapping and Meir-Keeler mapping in partial b-metric space which generalized and extended the result of S.K. Chatterjea [6] and S. Satish [9], respectively.

In this paper, we introduce a new definition for a partial b-metric-like space and the fixed point theorem for C-contractive mapping and Meir-Keeler mapping was proved in partial b-metric-like space. The new results can be viewed as some unified forms of the previous results. That is, some fixed point theorem in partial b-metric space considered and studied by J. Zhou, D. Zheng and G. Zhang.

2. Preliminaries

The following concepts and results are needed for the results.

Definition 2.1. Let X be a nonempty set, $s \geq 1$ be a given real number and let $P_L : X \times X \longrightarrow [0, \infty)$ be a mapping such that for all $x,\ y,\ z \in X$, the following conditions hold:

(Pb_L1) if $P_L(x,x) = P_L(x,y) = P_L(y,y)$, then $x = y$;
(Pb_L2) $P_L(x,x) \leq P_L(x,y)$;
(Pb_L3) $P_L(x,y) = P_L(y,x)$;
(Pb_L4) $P_L(x,y) \leq s[P_L(x,z) + P_L(z,y)] - P_L(z,z)$.

Then the pair (X, P_L) is called a partial b-metric-like space. The number s is called the coefficient of (X, P_L, s).

Remark 2.2. It is clear that every partial b-metric space is a partial b-metric-like space with the zero self-distance. However, the converse of this fact need not hold.

Remark 2.3. In a partial b-metric space (X, b, s), if $x,\ y \in X$ and $b(x,y) = 0$, then $x = y$ but the converse may not be true.

Remark 2.4. ([10]) It is clear that every partial metric space is a partial b-metric space with coefficient $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. However, the converse of this fact need not hold.

Definition 2.5. Let (X, P_L, s) be a partial b-metric-like space. Let x_n be any sequence in X and $x \in X$. Then

(i) the x_n sequence is said to be convergent and converges to x if $\lim\limits_{n \longrightarrow \infty} P_L(x_n, x)$ exists and is finite.
(ii) the x_n sequence is said to be Cauchy sequence in (X, P_L, s) if $\lim\limits_{n,m \longrightarrow \infty} P_L(x_n, x_m)$ exists and is finite.

(iii) (X, P_L, s) is said to be a complete partial b-metric-like space if for every Cauchy sequence x_n in X there exists $x \in X$ such that

$$\lim_{n,m\longrightarrow\infty} P_L(x_n, x_m) = \lim_{n\longrightarrow\infty} P_L(x_n, x) = P_L(x, x).$$

3. Main Results

In this section, we shall prove the existence of fixed point in partial b-metric-like space under some conditions.

Theorem 3.1. *Let (X, P_L, s) be a complete partial b-metric-like space with coefficient $s \geq 1$ and $f : X \longrightarrow X$ be a mapping satisfying the following condition: for $x, y \in X$*

$$P_L(fx, fy) \leq \lambda \left[P_L(x, fy) + P_L(y, fx)\right] \tag{3.1}$$

where $\lambda \in [0, \frac{1}{2s})$. Then f has unique fixed point $z \in X$ and $P_L(z, z) = 0$.

Proof. First we prove the existence of fixed point. Let $x_n = f^n x_0$ and $P_{L_n} = P_L(x_n, x_{n+1})$, where x_0 is arbitrary point of X.

If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point of f. Therefore, we can suppose $x_{n+1} \neq x_n,\ P_{L_n} > 0$ for each $n \in \mathbb{N}$, from (3.1) and definition of partial b-metric-like space. Consider

$$\begin{aligned}
P_{L_n} =& P_L(x_n, x_{n+1})\\
=& P_L(fx_{n-1}, fx_n)\\
\leq& \lambda \left[P_L(x_{n-1}, fx_n) + P_L(x_n, fx_{n-1})\right]\\
=& \lambda \left[P_L(x_{n-1}, x_{n+1}) + P_L(x_n, x_n)\right]\\
\leq& \lambda \left[s\left[P_L(x_{n-1}, x_n) + P_L(x_n, x_{n+1})\right] - P_L(x_n, x_n) + P_L(x_n, x_n)\right]\\
=& \lambda \left[s\left[P_L(x_{n-1}, x_n) + P_L(x_n, x_{n+1})\right]\right]\\
=& \lambda s \left[P_{L_{n-1}} + P_{L_n}\right].
\end{aligned}$$

Let $\mu = \lambda s$ and $\lambda \in [0, \frac{1}{2s})$. Then $P_{L_n} \leq \mu[P_{L_{n+1}} + P_{L_n}]$, where $\mu \in [0, \frac{1}{2})$. Therefore, $P_{L_n} \leq \alpha P_{L_{n-1}}$ where $\alpha = \frac{\mu}{1-\mu} < 1$. On repeating this process we obtain

$$P_{L_n} \leq \alpha^n b_0.$$

Hence $\lim_{n\longrightarrow\infty} P_{L_n} = 0$. Next, we shall show that x_n is a Cauchy sequence in X. Let $P_{L_n} = P_L(x_n, x_m)$, from (3.1) and (Pb_L4) that for $n, m \in \mathbb{N}$ with $n < m$,

$$\begin{aligned}
P_L(x_n, x_m) =& P_L(f^n x_0,\ f^m x_0)\\
\leq& \lambda \left[P_L(x_{n+1}, fx_{m-1}) + P_L(x_{m-1}, fx_{n-1})\right]\\
=& \lambda \left[P_L(x_{n-1}, x_m) + P_L(x_{m-1}, x_n)\right]\\
\leq& \lambda\ \left[s\left[P_L(x_{n-1}, x_n) + P_L(x_n, x_m)\right] - P_L(x_n, x_n)\right.\\
&\left. + s\left[P_L(x_{m-1}, x_m) + P_L(x_n, x_m)\right] - P_L(x_m, x_m)\right]\\
=& \lambda s P_L(x_{n-1}, x_n) + 2\lambda s P_L(x_n, x_m) + \lambda s P_L(x_{m-1}, x_m).
\end{aligned}$$

Since (3.1) and (Pb_L4), we have $P_L(x_n, x_m) \leq \beta[P_{L_{n-1}} + P_{L_{m-1}}]$ where $\beta = \frac{\lambda s}{1-2\lambda s}$. Therefore, $\{x_n\}$ is a Cauchy sequence in X and $\lim_{n,m\longrightarrow\infty} P_L(x_n, x_m) = 0$.

By the completeness of X, there exists $z \in X$ such that

$$\lim_{n\longrightarrow\infty} P_L(x_n, z) = \lim_{n,m\longrightarrow\infty} P_L(x_n, x_m) = P_L(z, z) = 0.$$

Now, we shall prove that z is a fixed point of f. Let $d_n = P_L(fx_n, fu)$ and $P_{L_n} = (z, fx_n)$ for each $n \in \mathbb{N}$. Consider, for $n \in \mathbb{N}$,

$$\begin{aligned} d_n =& P_L(fx_n, fz) \\ \leq& \lambda\left[P_L(x_n, fz) + P_L(fx_n, z)\right] \\ =& \lambda\left[P_L(fx_{n-1}, fz) + P_L(fx_n, z)\right] \\ =& \lambda(d_{n-1} + \ P_{L_n}). \end{aligned}$$

We take upper limit on both sides to the above inequality,

$$\limsup_{n\longrightarrow\infty} d_n \leq \limsup_{n\longrightarrow\infty} d_{n-1} + \limsup_{n\longrightarrow\infty} P_{L_n}.$$

Since $\lim\limits_{n\longrightarrow\infty} P_{L_n} = 0$, we have $\limsup\limits_{n\longrightarrow\infty} d_n = 0$. Thus $\lim\limits_{n\longrightarrow\infty} d_n = 0$.
That is,

$$\begin{aligned} d_n =& P_L(fx_n, fz) \\ \leq& \lambda\left[P_L(x_n, fz) + P_L(fx_n, z)\right] \\ =& \lambda\left[P_L(fx_{n-1}, fz) + P_L(fx_n, z)\right] \\ =& \lambda(d_{n-1} + \ P_{L_n}). \end{aligned}$$

We take limit on both sides to the inequality, then $P_L(z, fz) \leq 0$. Hence $fz = z$. Therefore z is a fixed point of f.

Next, we prove unique fixed point. Let $z, v \in X$ be two distinct fixed points of f, that is, $z = fz \neq fv = v$. Then, we have $P_L(z, z) = P_L(v, v)$. Since (3.1), we have

$$\begin{aligned} P_L(z, v) =& P_L(fz, fv) \\ \leq& \lambda[P_L(z, fv) + P_L(v, fz)] \\ =& \lambda[P_L(z, v) + P_L(v, z)] \\ =& 2\lambda P_L(z, v) \\ <& \frac{1}{s} P_L(z, v), \end{aligned}$$

a contradiction. Thus, we have $z = v$.

Next, we prove that $z \in X$ is the fixed point of f, that is $fz = z$. From (3.1), we obtain

$$\begin{aligned} P_L(z, z) =& P_L(fz, fz) \\ \leq& \lambda[P_L(z, fz) + P_L(z, fz)] \\ =& \lambda[P_L(z, z) + P_L(z, z)] \\ =& 2\lambda P_L(z, z) \\ <& \frac{1}{s} P_L(z, z) \\ <& P_L(z, z). \end{aligned}$$

It is a contradiction. Hence, we have $P_L(z, z) = 0$. □

If (X, P_L, s) is a partial b-metric space and $P_L = b$, then Theorem 3.1 reduces to the following result.

Corollary 3.2. *([10]) Let (X, b, s) be a complete partial b-metric space with coefficient $s \geq 1$ and $f : X \longrightarrow X$ be a mapping satisfying the following condition:*

$$b(fx, fy) \leq \lambda[b(x, fy) + b(y, fx)] \quad x, y \in X,$$

where $\lambda \in [0, \frac{1}{2s})$. *Then* f *has unique fixed point* $z \in X$ *and* $b(z,z) = 0$.

If (X, b, s) is a b-metric space in Corollary 3.2, then we have the following corollary.

Corollary 3.3. *([9]) Let* (X, b, s) *be a complete* b*-metric space with coefficient* $s \geq 1$ *and* $f : X \longrightarrow X$ *be a mapping satisfying the following condition:*

$$b(fx, fy) \leq \lambda[b(x, fy) + b(y, fx)] \quad x, y \in X,$$

where $\lambda \in [0, \frac{1}{2s})$. *Then* f *has unique fixed point* $z \in X$ *and* $b(z,z) = 0$.

Theorem 3.4. *Let* (X, P_L, s) *be a complete partial* b*-metric-like space with coefficient* $s > 1$ *and* $f : X \longrightarrow X$ *be a mapping satisfying the following condition:*
for each $\varepsilon > 0$ *there exist* $\delta > 0$ *such that*

$$\varepsilon \leq P_L(x, z) < \varepsilon + \delta \Rightarrow sP_L(fx, fz) < \varepsilon. \tag{3.2}$$

Then f *has a unique fixed point* $z \in X$ *and* $P_L(z, z) = 0$.

Proof. By (3.2), for all $x, y \in X$ and $x \neq y$,

$$sP_L(fx, fy) < \ P_L(x, y). \tag{3.3}$$

Let $x_0 \in X$. We can choose sequence $\{x_n\}$ in X such that

$$x_{n+1} = fx_n = f^2x_{n-1} = \cdots = f^{n+1}x_0,$$

for $n = 0, 1, 2, 3 \cdots$.
If $x_{n+1} = x_n$ for all $n \in \mathbb{N}$, then f have a fixed point.

Let $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. By the inequality (3.3) with $x = x_{n-1}$ and $y = x_n$, we obtain

$$sP_L(x_n, x_{n+1}) < P_L(x_{n-1}, x_n).$$

For $s > 1$, $\{P_L(x_n, x_{n+1})\}$ is a decrease sequence, it is easy to see that

$$\lim_{n \longrightarrow \infty} P_L(x_n, x_{n+1}) = 0.$$

Next, we will show that $\{x_n\}$ is a Cauchy sequence in X. We can choose an N sufficiently large such that when $n > N$,

$$P_L(x_n, x_{n+1}) \ \frac{\varepsilon - \frac{\varepsilon}{s}}{s + s^2}.$$

Let $K(x_N, \varepsilon) = \{y \in X : P_L(y, x_N) \leq \varepsilon\}$.
If $x_m \in K(x_N, \varepsilon)$ with $m > N$, then $x_m \neq x_N$. Making use of the inequality $P_L(x, y) \leq s\,[P_L(x, z) + P_L(z, y)] - P_L(z, z)$, we obtain that

$$\begin{aligned}
P_L(f^2x_m, x_N) \leq & s\left[P_L(f^2x_m, f^2x_N) + P_L(f^2x_N, x_N)\right] - P_L(f^2x_N, f^2x_N) \\
\leq & s\left[P_L(f^2x_m, f^2x_N) + P_L(f^2x_N, x_N)\right] \\
\leq & s\left[\frac{1}{s}P_L(x_{m+1}, x_{N+1}) + P_L(f^2x_N, x_N)\right] \\
\leq & P_L(x_{m+1}, x_{N+1}) + s^2\left[P_L(x_{N+2}, x_{N+1}) + P_L(x_{N+1}, x_N)\right] - sP_L(x_N, x_N) \\
\leq & \frac{1}{s}P_L(x_m, x_N) + \left(s + s^2\right)P_L(x_{N+1}, x_N) \\
\leq & \frac{\varepsilon}{s} + \left(s + s^2\right)\left(\frac{\varepsilon - \frac{\varepsilon}{s}}{s + s^2}\right) \\
= & \varepsilon.
\end{aligned}$$

Therefore $f^2x_m \in K(x_N, \varepsilon)$. That is f^2 maps $K(x_N, \varepsilon)$ into itself.
Since $x_{N+1} \in K(x_N, \varepsilon)$, we have $x_{N+3}, x_{N+5} \in K(x_N, \varepsilon)$. By

$$P_L(x, y) \leq s\left[P_L(x, z) + P_L(z, y)\right] - P_L(z, z),$$

we get

$$\begin{aligned} P_L(x_{N+2}, x_N) \leq & s\left[P_L(x_{N+2}, x_{N+1}) + P_L(x_{N+1}, x_N)\right] - P_L(x_{N+1}, x_{N+1}) \\ \leq & sP_L(x_{N+1}, x_N) + s\left[\frac{1}{s}P_L(x_N, x_{N+1})\right] \\ \leq & s\left(\frac{\varepsilon - \frac{\varepsilon}{s}}{s + s^2} + \frac{\varepsilon - \frac{\varepsilon}{s}}{s + s^2}\right) \\ < & \varepsilon. \end{aligned}$$

Hence $x_{N+2} \in K(x_N, \varepsilon)$. Similarly, $x_{N+4}, x_{N+6} \in K(x_N, \varepsilon)$.
This implies that $\{x_n : n \geq N\} \subset K(x_N, \varepsilon)$. Since $x_n,\ x_m \in K(x_N, \varepsilon)$, for $n > m > N$, we get

$$\begin{aligned} P_L(x_n, x_m) \leq & s\left[P_L(x_n, x_N) + P_L(x_N, x_m)\right] - P_L(x_N, x_N) \\ \leq & 2s\varepsilon. \end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X and $\lim_{n,m \longrightarrow \infty} P_L(x_n, x_m) = 0$.
By completeness of X there exists $z \in X$, such that

$$\lim_{n \longrightarrow \infty} P_L(x_n, z) = \lim_{n,m \longrightarrow \infty} P_L(x_n, x_m) = P_L(z, z) = 0. \tag{3.4}$$

To show that, z is a fixed point of f. We must prove that $fz = z$.
By (3.4) and $P_L(x, y) \leq s\left[P_L(x, z) + P_L(z, y)\right] - P_L(z, z)$, We have

$$\begin{aligned} P_L(fz, z) \leq & s\left[P_L(z, fx_n) + P_L(fx_n, fz)\right] - P_L(fx_n, fx_n) \\ \leq & s\left[P_L(z, x_{n+1}) + \frac{1}{s}P_L(x_n, z)\right] \\ = & sP_L(z, x_{n+1}) + P_L(x_n, z). \end{aligned}$$

Passing to limit as $n \longrightarrow \infty$, we obtain

$$P_L(fz, z) \leq 0.$$

Hence $fz = z$, so z is a fixed point of f.
We want to show that $P_L(z, z) = 0$. Suppose that $P_L(z, z) > 0$. From (3.4), we can get

$$P_L(z, z) = P_L(fz, fz) \leq \frac{1}{s}P_L(z, z) < P_L(z, z),$$

a contradiction. Therefore $P_L(z, z) = 0$.

Next, we prove unique fixed point. Let $z, v \in X$ be two distinct fixed points of f, that is, $z = fz \neq fv = v$. Since (3.3), we have

$$\begin{aligned} P_L(z, v) = & P_L(fz, fv) \\ < & \frac{1}{s}P_L(z, v) \\ < & P_L(z, v), \end{aligned}$$

a contradiction. Thus $z = v$.
Therefore z is a fixed point of f and it is unique fixed point of f. ☐

If (X, P_L, s) is a partial b-metric space and $P_L = b$, then Theorem 3.4 reduces to the following result.

Corollary 3.5. *Let (X, b, s) be a complete partial b-metric space with coefficient $s > 1$and $f : X \longrightarrow X$ be a mapping satisfying the following condition:*
for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq b(x, z) < \varepsilon + \delta \Rightarrow sb(fx, fz) < \varepsilon.$$

Then f has a unique fixed point $z \in X$ and $b(z, z) = 0$.

If (X, b, s) is a b-metric space in Corollary 3.5, then we have the following corollary.

Corollary 3.6. *Let (X, b, s) be a complete b-metric space with coefficient $s > 1$ and $f : X \longrightarrow X$ be a mapping satisfying the following condition:*
for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq b(x, z) < \varepsilon + \delta \Rightarrow sb(fx, fz) < \varepsilon.$$

Then f has a unique fixed point $z \in X$ and $b(z, z) = 0$.

4. Conclusion

We have introduced a new extension of the concept of partial b-metric space, called a partial b-metric-like space. Furthermore, we proved some fixed point results for these C-contractive mappings. One can easily extend these results to some fixed point theorem in partial b-metric space (see [10]).

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