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APPROXIMATING FIXED POINTS OF THE NEW SP*-ITERATION FOR GENERALIZED $\alpha-$NONEXPANSIVE MAPPINGS IN $CAT(0)$ SPACES

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ABSTRACT. In this paper, we study the convergence of the newly defined SP*-iteration to fixed point for the generalized $\alpha-$nonexpansive mappings in $CAT(0)$ spaces. Our results improve and extend some recently results in the literature of fixed point theory in $CAT(0)$ spaces.

KEYWORDS:Fixed point, iteration process, $\Delta-$convergence, $CAT(0)$ space, generalized $\alpha-$nonexpansive mappings.
AMS Subject Classification: 47H09; 47H10.

1. Introduction and Preliminaries

The existence of a fixed point is very important in several areas of mathematics and other sciences. The numerous numbers of researchers attracted in these direction and developed iterative process has been investigated to approximate fixed point for not only nonexpansive mapping, but also for some wider class of nonexpansive mappings. This is an active area of research, several well known scientists in the world paid and still pay attention to the qualitative study of iteration methods. The well-known Banach contraction theorem use Picard iteration process[25] for approximation of fixed point. Some of the well-known iterative processes are those of Mann [21], Ishikawa [13], Noor [23] , SP-İteration [26], Picard Normal S-iteration [14] and so on.

It is well-known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a $CAT(0)$ space. Other examples include Pre-Hilbert spaces, any convex subset of a Euclidian space $\mathbb{R}^n$ with the induced metric, the complex Hilbert ball with a hyperbolic metric and many others. For discussion of these spaces and of the fundamental role they play in geometry see

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Bridson and Haefliger [4]. Burago et al. [5] contains a somewhat more elementary treatment, and Gromov [12] a deeper study. Fixed point theory in $CAT(0)$ space has been first studied by Kirk (see [15],[16]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete $CAT(0)$ space always has a fixed point. On the other hand, we know that not every Banach space is a $CAT(0)$ space. Since then, the fixed point theory in CAT(0) spaces has been rapidly developed and many papers have appeared(see [6],[7],[8],[9],[10],[15],[16],[17], [18]).

Recently, Kirk and Panyanak [18] used the concept of $\Delta-$convergence introduced by Lim [20] to prove on the $CAT(0)$ space analogs of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [6] obtained $\Delta-$convergence theorems for the Picard, Mann and Ishikawa iteration processes for nonexpansive mappings in the $CAT(0)$ space. In addition, the convergence results for generalized nonexpansive mappings are obtained by using different iteration processes in $CAT(0)$ spaces (see [2], [27]).

In the sequel, we need the following definitions and useful lemmas to prove our main results of this paper.

Lemma 1.1. [6] *Let X be a $CAT(0)$ space.*

(i) For $x, y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that $d(x,z) = td(x,y)$ and $d(y,z) = (1-t)d(x,y)$ (I). We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (I).

(ii) For $x, y \in X$ and $t \in [0,1]$, we have $d((1-t)x \oplus ty, z) \le (1-t)d(x,z) + td(y,z)$.

Let $\{x_n\}$ be a bounded sequence in a closed convex subset K of a $CAT(0)$ space X. For $x \in X$, set $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by $r(K, \{x_n\}) = \inf_n \{r(x, \{x_n\}) : x \in K\}$ and the asymptotic center of x_n relative to K is the set $A(K, \{x_n\}) = \{x \in K : r(x, \{x_n\}) = r(K, \{x_n\})\}$. It is known that, in a $CAT(0)$ space, $A(K, \{x_n\})$ consists of exactly one point; please, see [9], Proposition 7.

We now recall the definition of Δ-convergence and weak convergence in $CAT(0)$ space.

Definition 1.2. ([20],[18]) A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to $\Delta-$converge to $x \in X$ if x is the unique asymptotic center of u_n for every subsequence $\{x_n\}$ of $\{x_n\}$.In this case we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x is the $\Delta-$limit of $\{x_n\}$.

Lemma 1.3. *([18]) Given $\{x_n\} \in X$ such that $\{x_n\}$, $\Delta-$converges to x and given $y \in X$ with $y \neq x$, then $\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y)$.*

In a Banach space the above condition is known as the Opial property.

Let (X, d) be a metric space and K a nonempty subset of X. Let $T : K \to K$ be a mapping. A point $x \in K$ is called a fixed point of T if $Tx = x$ and we denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$. X is a complete $CAT(0)$ space, K is a nonempty convex subset of X and $T : K \to K$ is a mapping. T is called nonexpansive if for each $x, y \in K$, $d(Tx, Ty) \le d(x, y)$.

Lemma 1.4. *([18]) Every bounded sequence in a complete $CAT(0)$ space always has a $\Delta-$convergent subsequence.*

Lemma 1.5. *([8]) Let K be closed convex subset of a complete $CAT(0)$ space and $\{x_n\}$ be a bounded sequence in K. Then asymptotic center of $\{x_n\}$ is in K.*

Lemma 1.6. [19] *Suppose that X is a complete $CAT(0)$ space and $x \in X$. Let T be a mapping on K. $0 < k \leq t_n \leq m < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$ and $\limsup_{n\to\infty} d(t_n x_n \oplus (1-t_n)y_n, x) = r$ hold for $r \geq 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

A number of extensions and generalizations of nonexpansive mappings have been considered by many mathematicians, see [[1],[11] [24], [29]], in recent years. In 2008, Suzuki [29] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called (C) *condition*. Let K be a nonempty convex subset of a Banach space X, a mapping $T : K \to K$ is satisfy condition (C) if for all $x, y \in K$, $\frac{1}{2}d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq d(x, y)$. Suzuki [29] showed that the mapping satisfying *condition* (C) is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. In 2011, Aoyama and Kohsaka [1] introduced the class of α-nonexpansive mappings in the setting of Banach spaces and obtained some fixed point results for such mappings.

In 2017, Pant and Shukla [24] introduced a new type of nonexpansive mappings called generalized α-nonexpansive mappings and obtain a number of existence and convergence theorems.This new class of nonlinear mappings properly contains nonexpansive, Suzuki-type generalized nonexpansive mappings and partially extends firmly nonexpansive and α-nonexpansive mappings.

In what follows, we give the following definition and lemma to be used in main results.

Definition 1.7. [24]. A mapping $T : K \to K$ is called a generalized α-nonexpansive mapping if there exists an $\alpha \in [0, 1)$ and for each $x, y \in K$,

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq \alpha d(Tx, y) + \alpha d(Ty, x) + (1-2\alpha)d(x, y) \tag{1.1}$$

The next simple examples can show these facts. We see that T is a generalized α-nonexpansive mappings but does not satisfy condition (C).

Example 1.8. Let $K = [0, 6]$ be a subset of $\mathbb{R}$ endowed with the usual norm. Define a mapping $T : K \to K$ by $Tx = \begin{cases} 0, & x \neq 6 \\ 3, & x = 6 \end{cases}$

For $x \in (3, 4]$ and $y = 6$, $\frac{1}{2}d(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) = 3 > 6 - x = d(x, y)$. Thus T does not satisfy Suzuki's condition (C). However, T is a generalized $\alpha-$nonexpansive mapping with $\alpha \geq \frac{1}{3}$.

Example 1.9. Let $M = \{(0,0)(3,0),(0,6),(6,0),(6,7),(7,6)\}$ is a subset of $\mathbb{R}^2$. Define a norm $\|.\|$ by $\|(x_1, x_2)\| = |x_1| + |x_2|$. Then $(M, \|.\|)$ is a Banach space.

Define a mapping $T : M \to M$ by $T(0,0) = (0,0)$, $T(3,0) = (0,0)$, $T(0,6) = (0,0)$, $T(6,0) = (3,0)$, $T(6,7) = (6,0)$, $T(7,6) = (0,6)$.

We note that for $\alpha \geq \frac{1}{4}$,

$$\|Tx - Ty\| \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1-2\alpha)\|x - y\|,$$

if $(x, y) \neq ((6,7),(7,6))$. In case of $x = (6,7)$ and $y = (7,6)$, we have

$$\frac{1}{2}\|x - Tx\| = \frac{1}{2}\|y - Ty\| = \frac{7}{2} > 2 = \|x - y\|.$$

Thus, T is a generalized $\alpha-$nonexpansive mapping. However, for $x = (6,7)$ and $y = (7,6)$,

$$\|Tx - Ty\|^2 = 144 > 90\alpha + 4$$

$$= 49\alpha + 49\alpha + (1-2\alpha).4$$
$$= \alpha\|Tx-y\|^2 + \alpha\|Ty-x\|^2 + (1-2\alpha)\|x-y\|^2.$$

Thus, T is not an $\alpha-$nonexpansive mapping for any $\alpha < 1$. Further, for $x = (6,0)$ and $y = (7,6)$

$$\frac{1}{2}\|x-Tx\| = \frac{3}{2} < 7 = \|x-y\| \ \ but, \|Tx-Ty\| = 9 > 7 = \|x-y\|.$$

Thus, T does not satisfy Suzuki's condition (C).

Now we give the following well-known facts about generalized α-nonexpansive mapping, which can be found in [24].

Lemma 1.10. (1) *If T is a generalized α -nonexpansive mapping and has a fixed point, then T is a quasi-nonexpansive mapping.*

(2) *If T is a generalized α -nonexpansive mapping, then $F(T)$ is closed. Moreover if X is strictly convex and K is convex, then $F(T)$ is also convex.*

(3) *If T is a generalized α -nonexpansive mapping, then for each $x, y \in K$, for each $x, y \in K$,*

$$d(x,Ty) \leq (\frac{3+\alpha}{1-\alpha})d(Tx,x) + d(x,y).$$

(4) *If X has Opial property, T is a generalized α -nonexpansive mapping, $\{x_n\}$ converges weakly to a point z and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $z \in F(T)$.*

2. The new iteration process

Let be X be a real Banach space and K be a nonempty subset of X, and $T : K \to K$ be a mapping. We have $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are real sequences in $[0,1]$. Recently, Phuengrattana and Suantai ([26]) defined the SP-iteration as follows:

$$\begin{cases} z_n = (1-c_n)x_n + c_nTx_n, \\ y_n = (1-b_n)z_n + b_nTz_n, \\ x_{n+1} = (1-a_n)y_n + a_nTy_n, \forall n \in \mathbb{N}, \end{cases} \tag{2.1}$$

where $x_1 \in K$. They showed that the Mann, Ishikawa, Noor and SP-iterations are equivalent and the SP-iteration converges better than the others for the class of continuous and nondecreasing functions. In 2014, Kadioglu and Yildirim [14] introduced Picard Normal S-iteration process and they established that the rate of convergence of the Picard Normal S-iteration process is faster than other fixed point iteration process that was in existence then. The Picard Normal S-iteration[14] as follows:

$$\begin{cases} z_n = (1-b_n)x_n + b_nTx_n, \\ y_n = (1-a_n)z_n + a_nTz_n, \\ x_{n+1} = Ty_n, \forall n \in \mathbb{N}, \end{cases} \tag{2.2}$$

where $x_1 \in K$. Let's note here that some fixed points properties and demiclosedness principle for generalized α-nonexpansive mappings in the frame work of uniformly convex hyperbolic spaces are studied(see [22]). They further established strong and Δ-convergence theorems of Picard Normal S-iteration scheme generated by (2.2) for the generalized α-nonexpansive mappings in the frame work of uniformly convex hyperbolic spaces.

Motivated by above, in this paper, we introduce a new iteration called as SP*-iteration scheme:for arbitrary $x_1 \in K$ construct a sequence $\{x_n\}$ by

$$\begin{cases} z_n = T((1-c_n)x_n + c_nTx_n), \\ y_n = T((1-b_n)z_n + b_nTz_n), \\ x_{n+1} = T((1-a_n)y_n + a_nTy_n), \forall n \in \mathbb{N}, \end{cases} \tag{2.3}$$

First we give a useful definition that is used to determine the faster iteration which converge to the same point. The following definition about the rate of convergence is given by [3].

Definition 2.1. [3] Let α_n and β_n be two sequences of positive numbers that converge to a, respectively b. Assume that there exists

$$\ell = \lim_{n\to\infty} \frac{|\alpha_n - a|}{|\beta_n - b|}.$$

(1) If $\ell = 0$, then it can be said that α_n converges to a faster than β_n converges to b.
(2) If $0 < \ell < \infty$, then it can be said that α_n and β_n have the same rate of convergence.

Theorem 2.2. *Let K be a nonempty closed convex subset of a norm space X. A mapping $T : K \to K$ is contraction with contraction factor $\theta \in (0, 1)$ and $p \in F(T)$. Let $\{u_n\}$ defined by the iteration (2.2) and $\{x_n\}$ defined by the iteration (2.3), where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are real sequences in $[\varsigma, 1-\varsigma]$ for all $n \in \mathbb{N}$ and for some ς in $[0, 1]$. Then $\{x_n\}$ converges faster than $\{u_n\}$. That is, our new iteration defined by (2.3) faster than (2.2).*

Proof. As proved in [14],

$$\|u_{n+1} - p\| \le \theta^n[(1-\varsigma(1-\theta))^2]^n\|u_1 - p\|,$$

for all $n \in \mathbb{N}$. Let

$$k_n = \theta^n[(1-\varsigma(1-\theta))^2]^n\|u_1 - p\|.$$

It follows from (2.3), we have,

$$\begin{aligned} \|z_n - p\| &= \|T((1-c_n)x_n + c_nTx_n) - p\| \\ &\le \theta[\|(1-c_n)(x_n - p) + c_n(Tx_n - p)\|] \\ &\le \theta[(1-c_n)\|x_n - p\| + c_n\theta\|x_n - p\|] \\ &= \theta[1 - c_n(1-\theta)]\|x_n - p\|. \end{aligned}$$

Similarly, using (2.3), we get

$$\begin{aligned} \|y_n - p\| &= \|T((1-b_n)z_n + b_nTz_n) - p\| \\ &\le \theta[\|(1-b_n)(z_n - p) + b_n(Tz_n - p)\|] \\ &\le \theta[(1-b_n)\|z_n - p\| + b_n\theta\|z_n - p\|] \\ &\le \theta^2[1 - b_n(1-\theta)][1 - c_n(1-\theta)]\|x_n - p\|. \end{aligned}$$

Again using (2.3), we get

$$\begin{aligned} \|x_{n+1} - p\| &= \|T((1-a_n)y_n + a_nTy_n) - p\| \\ &\le \theta[\|(1-a_n)(y_n - p) + a_n(Ty_n - p)\|] \\ &\le \theta[(1-a_n)\|y_n - p\| + a_n\theta\|y_n - p\|] \\ &\le \theta^3[(1 - a_n(1-\theta)][1 - b_n(1-\theta)][1 - c_n(1-\theta)]\|x_n - p\|. \end{aligned}$$

Hence, we get

$$\|x_{n+1}-p\| \leq \theta^{3n}[(1-\varsigma(1-\theta))^3]^n\|x_1-p\|.$$

Let

$$m_n = \theta^{3n}[(1-\varsigma(1-\theta))^3]^n\|x_1-p\|.$$

Then, we get

$$\begin{aligned} \frac{m_n}{k_n} &= \frac{\theta^{3n}[(1-\varsigma(1-\theta))^3]^n\|x_1-p\|}{\theta^n[(1-\varsigma(1-\theta))^2]^n\|u_1-p\|} \\ &\rightarrow 0 \quad as \quad n\rightarrow\infty. \end{aligned}$$

Consequently $\{x_n\}$ converges faster than $\{u_n\}$. □

In order to show numerically that our new iteration (SP*-iteration) process (2.3) have a good speed of convergence comparatively to (2.1) and (2.2) , we consider the following example.

Example 2.3. Let us define a function $T : [0, 10) \rightarrow [0, 10)$ by $T(x) = \sqrt{2x+3}$. Then clearly T is a contraction map. Let $\{a_n\} = 0.75, \{b_n\} = 0.75, \{c_n\} = 0.75$ $\forall n \in \mathbb{N}$. 3 is the fixed point of T. The iterative values for initial value $x_1 = 4$ are given in Table 1. The efficiency of new iteration process is clear. We can see that our new iteration process (2.3) have a good speed of convergence comparatively to (2.1) and (2.2) iteration processes.

Table 1. Sequences generated by SP-iteration, Picard Normal S-iteration and SP*-iteration processes for mapping T of Example 2.1.

	SP-iteration	Picard Normal S-iteration	New iteration(SP*-iteration)
x_1	4	4	4
x_2	3,119517509335370	3,079156197588850	3,004353238197250
x_3	3,014854024750290	3,006567594269410	3,000020148061540
x_4	3,001855414595090	3,000547099976860	3,000000093277940
x_5	3,000231905910340	3,000045590279090	3,000000000431840
x_6	3,000028987912010	3,000003799180300	3,000000000002000
x_7	3,000003623483900	3,000000316598290	3,000000000000010
x_8	3,000000452935410	3,000000026383190	3.000000000000000
x_9	3,000000056616920	3,000000002198600	3.000000000000000
x_{10}	3,000000007077120	3,000000000183220	3.000000000000000
x_{11}	3,000000000884640	3,000000000015270	3.000000000000000
x_{12}	3,000000000110580	3,000000000001270	3.000000000000000
x_{13}	3,000000000013820	3,000000000000110	3.000000000000000
x_{14}	3,000000000001730	3,000000000000010	3.000000000000000
x_{15}	3,000000000000220	3.000000000000000	3.000000000000000
x_{16}	3,000000000000030	3.000000000000000	3.000000000000000
x_{17}	3,000000000000000	3.000000000000000	3.000000000000000

In this paper, we apply SP*-iteration (2.3) in a $CAT(0)$ space for generalized nonexpansive mappings as follows

$$\begin{cases} z_n = T((1-c_n)x_n \oplus c_nTx_n), \\ y_n = T((1-b_n)z_n \oplus b_nTz_n), \\ x_{n+1} = T((1-a_n)y_n \oplus a_nTy_n)\forall n\in\mathbb{N}, \end{cases} \tag{2.4}$$

where K is a nonempty closed convex subset of a $CAT(0)$ space, $x_1 \in K$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\} \in [0,1]$.

Inspired and motivated by these facts, in this paper, we consider generalized α-nonexpansive mappings. Further we prove some convergence theorems of a new iterative process generated by (2.4) to fixed point for generalized α-nonexpansive mappings in $CAT(0)$ spaces.

3. Convergence of SP*-iteration process for generalized $\alpha-$nonexpansive mappings

Lemma 3.1. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space X , T be a generalized $\alpha-$nonexpansive mapping with $F(T) \neq \emptyset$. For arbitrary chosen $x_1 \in K$, let $\{x_n\}$ be a sequence generated by (2.4) with $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ real sequences in $[0,1]$, then $\lim_{n\to\infty} d(x_n,p)$ exits for any $p \in F(T)$.*

Proof. For any $p \in F(T)$, and $x \in K$, since for T a generalized $\alpha-$nonexpansive mapping, $\frac{1}{2}d(p,Tp) = 0 \leq d(p,x)$ implies that

$$\begin{aligned} d(Tp,Tx) &\leq \alpha d(Tp,x) + \alpha d(Tx,p) + (1-2\alpha)d(p,x) \\ &\leq \alpha d(Tp,x) + \alpha d(Tp,Tx) + (1-2\alpha)d(p,x) \end{aligned}$$

$$\begin{aligned} (1-\alpha)d(Tp,Tx) &\leq \alpha d(Tp,x) + (1-2\alpha)d(p,x) \\ &= (1-\alpha)d(p,x) \end{aligned}$$

Thus, $d(Tp,Tx) \leq d(p,x)$. Then we show that T is a quasi-nonexpansive mapping. Now, using (2.4) and Lemma 1.10(1), we have,

$$\begin{aligned} d(z_n,p) &= d(T((1-c_n)x_n \oplus c_nTx_n),p) \\ &\leq d((1-c_n)x_n \oplus c_nTx_n,p) \\ &\leq (1-c_n)d(x_n,p) + c_nd(x_n,p) \\ &= d(x_n,p). \end{aligned} \tag{3.1}$$

Using (2.4), (3.1) and Lemma 1.10(1), we get

$$\begin{aligned} d(y_n,p) &= d(T((1-b_n)z_n \oplus b_nTz_n),p) \\ &\leq d((1-b_n)z_n \oplus b_nTz_n,p) \\ &\leq (1-b_n)d(z_n,p) + b_nd(z_n,p) \\ &= d(z_n,p) \\ &\leq d(x_n,p). \end{aligned}$$

By using (2.4) , (3.2) and Lemma 1.10(1), we get

$$\begin{aligned} d(x_{n+1},p) &= d(T((1-a_n)y_n \oplus a_nTy_n),p) \\ &\leq d((1-a_n)y_n \oplus a_nTy_n,p) \\ &\leq (1-a_n)d(y_n,p) + a_nd(Ty_n,p) \\ &\leq (1-a_n)d(x_n,p) + a_nd(y_n,p) \\ &\leq (1-a_n)d(x_n,p) + a_nd(x_n,p) \\ &= d(x_n,p) \end{aligned} \tag{3.2}$$

This implies that $\{d(x_n - p)\}$ is bounded and non-increasing for all $p \in F(T)$. It follows that $\lim_{n\to\infty} d(x_n, p)$ exits. □

Theorem 3.2. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space X, T be a generalized $\alpha-$nonexpansive mapping with $F(T) \neq \emptyset$. For arbitrary chosen $x_1 \in K$, let $\{x_n\}$ be a sequence in K defined by (2.4) with $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ real sequences in $[0,1]$. Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. By Lemma 3.1, $\lim_{n\to\infty} d(x_n, p)$ exits and $\{x_n\}$ is bounded. Put $\lim_{n\to\infty} d(x_n, p) = r$. From (3.1) and (3.2), we have

$$\limsup_{n\to\infty} d(z_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = r$$

and

$$\limsup_{n\to\infty} d(y_n, p) \leq \limsup_{n\to\infty} d(z_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = r.$$

Next,

$$\limsup_{n\to\infty} d(Tx_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = r.$$

On the other hand,

$$\begin{aligned} d(x_{n+1} - p) &= d(T((1-a_n)y_n \oplus a_n Ty_n, p)) \\ &\leq (1-a_n)d(y_n, p) + a_n d(Ty_n, p) \\ &\leq (1-a_n)d(y_n, p) + a_n d(y_n, p) \\ &= d(y_n, p). \end{aligned}$$

So we can get $d(x_{n+1}, p) \leq d(y_n, p)$. Therefore $r \leq \liminf_{n\to\infty} d(y_n, p)$. Thus we have $r = \lim_{n\to\infty} d(y_n, p)$. Now

$$\begin{aligned} r = \lim_{n\to\infty} d(y_n, p) &\leq \lim_{n\to\infty} d(z_n, p) \\ &= \lim_{n\to\infty} d(T((1-c_n)x_n \oplus c_n Tx_n), p) \\ &\leq \lim_{n\to\infty} d((1-c_n)x_n \oplus c_n Tx_n, p) \\ &\leq \lim_{n\to\infty} (1-c_n)d(x_n, p) + c_n d(Tx_n, p) \\ &\leq \lim_{n\to\infty} (1-c_n)d(x_n, p) + c_n d(x_n, p) \\ &= \lim_{n\to\infty} d(x_n, p) = r. \end{aligned}$$

Hence, we get

$$\lim_{n\to\infty} d((1-c_n)x_n \oplus c_n Tx_n, p) = r.$$

Thus by Lemma 1.6, we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Conversely, suppose that $\{x_n\}$ is bounded $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Let $p \in A(K, \{x_n\})$. By Lemma 1.10(3), we have,

$$
\begin{aligned}
r(Tp,\{x_n\}) = \limsup_{n\to\infty} d(x_n,Tp) &\leq \limsup_{n\to\infty} \frac{3+\alpha}{1-\alpha} d(Tx_n,x_n) + d(x_n,p) \\
&\leq \limsup_{n\to\infty} d(x_n,p) = r(p,\{x_n\})
\end{aligned}
$$

This implies that for $Tp = p \in A(K,\{x_n\})$. Since X is complete $CAT(0)$ then $A(K,\{x_n\})$ is singleton, hence $Tp = p$. This completes the proof. □

Now , we prove the $\Delta-$convergence theorem of a iterative process generated by (2.4) in $CAT(0)$ spaces.

Theorem 3.3. *Let X, K, T and $\{x_n\}$ be as in Theorem 3.2 with $F(T) \neq \emptyset$. Then x_n, $\Delta-$converges to a fixed point of T.*

Proof. Theorem 3.2 guarantees that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n,x_n) = 0$. Let $W_\Delta(x_n) = \bigcup A(\{u_n\})$; where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$: We claim that $W_\Delta(x_n) \subseteq F(T)$. Let $u \in W_\Delta(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = u$. By Lemma 1.4 and Lemma 1.5, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v \in K$. Since $\lim_{n\to\infty} d(v_n,Tv_n) = 0$ and T is a generalized $\alpha-$nonexpansive mapping, then, we have $d(v_n,Tv) \leq \frac{3+\alpha}{1-\alpha} d(Tv_n,v_n) + d(v_n,v)$ By taking lim sup and using Opial property, we obtain $v \in F(T)$. Now, we claim that $u = v$. Assume on contrary, that $u \neq v$. By Lemma 3.1, $\lim_{n\to\infty} d(x_n,v)$ exists and by the uniqueness of asymptotic centers, then we have

$$
\begin{aligned}
\lim_{n\to\infty} d(v_n,v) &< \lim_{n\to\infty} d(v_n,u) \\
&\leq \lim_{n\to\infty} d(u_n,u) \\
&< \lim_{n\to\infty} d(u_n,v) \\
&= \lim_{n\to\infty} d(x_n,v) \\
&= \lim_{n\to\infty} d(v_n,v),
\end{aligned}
$$

which is contraction. Thus $u = v \in F(T)$ and $W_\Delta(x_n) \subseteq F(T)$. To show that $\{x_n\}$, Δ-converges to a fixed point of T, we show that $W_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 1.4 and Lemma 1.5, there exists a subsequence $\{v_n\}$ of u_n such that $\Delta - \lim_{n\to\infty} v_n = v \in K$. Let $A(\{u_n\}) = u$ and $A(\{x_n\}) = x$. We have already seen that $u = v$ and $v \in F(T)$. Finally, we claim that $x = v$. If not, then existence $\lim_{n\to\infty} d(x_n,v)$ and uniqueness of asymptotic centers imply that

$$
\begin{aligned}
\lim_{n\to\infty} d(v_n,v) &< \lim_{n\to\infty} d(v_n,x) \\
&\leq \lim_{n\to\infty} d(x_n,x) \\
&< \lim_{n\to\infty} d(x_n,v) \\
&= \lim_{n\to\infty} d(v_n,v).
\end{aligned}
$$

This is a contradiction and hence $x = v \in F(T)$. Therefore, $W_\Delta(x_n) = x$. □

In the next result, we prove the strong convergence theorem as follows.

Theorem 3.4. *Let T be a generalized $\alpha-$nonexpansive mapping on a compact convex subset K of a complete $CAT(0)$ space X. $\{x_n\}$ be as in Theorem 3.2 with $F(T) \neq \emptyset$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Teorem 3.2, we have $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$. Since K is compact, by Lemma 1.4, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $p \in K$ such that $\{x_{n_k}\}$ converges p. By Lemma 1.10, we have $d(x_{n_k}, Tp) \leq \frac{3+\alpha}{1-\alpha} d(Tx_{n_k}, x_{n_k}) + d(x_{n_k}, p)$ for all $k \geq 0$. Then $\{x_{n_k}\}$ converges Tp. This implies $Tp = p$. Since T is quasinonexpansive , we have $d(x_{n+1}, p) \leq d(x_n, p)$for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ converges strongly to p. □

Finally, we briefly discuss the strong convergence theorem using condition (I) introduced by Senter and Dotson[28] in $CAT(0)$ space X as follows.

Theorem 3.5. *Let X, K, T and $\{x_n\}$ be as in Theorem 3.2 with $F(T) \neq \emptyset$. Also if, for T satisfies condition (A), then $\{x_n\}$ defined by (2.4) converges strongly to a fixed point of T.*

Proof. By Lemma 3.1, we have $\lim\limits_{n\to\infty} d(x_n, p)$ exists and so $\lim\limits_{n\to\infty} d(x_n, F(T))$. Also by Theorem 3.2, $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$.

It follows from condition (A) that $\lim\limits_{n\to\infty} f(d(x_n, F(T)) \leq \lim\limits_{n\to\infty} d(x_n, Tx_n)$. That is, $\lim\limits_{n\to\infty} f(d(x_n, F(T)) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, we have $\lim\limits_{n\to\infty} d(x_n, F(T)) = 0$. Thus , we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_k\} \subset F(T)$ such that $d(x_{n_k}, y_k) < \frac{1}{2^k}$ for all $k \in \mathbb{N}$. We can easily show that $\{y_k\}$ is a Cauchy sequence in $F(T)$ and so it converges to a point p. Since $F(T)$ is closed, therefore $p \in F(T)$ and $\{x_{n_k}\}$ converges strongly to p. Since $\lim\limits_{n\to\infty} d(x_n, p)$ exists, we have that $x_n \to p$. The proof is completed. □

4. Conclusions

We study the convergence of the newly defined SP*-iteration process (2.4) to fixed point for the generalized α-nonexpansive mappings in nonlinear $CAT(0)$ spaces. These results presented in this paper extend and generalize some works for $CAT(0)$ space in the literature.

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