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COMMON FIXED POINT RESULTS FOR GENERALIZED CYCLIC CONTRACTION PAIRS INVOLVING CONTROL FUNCTIONS ON PARTIAL METRIC SPACES

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ABSTRACT. In this paper, our purpose is to establish some coincidence point and common fixed point results for a pair of self mappings satisfying some generalized cyclic contraction type conditions involving a control function with two variables in partial metric spaces. Moreover, we provide some examples to analyze and illustrate our main results.

KEYWORDS: partial metric, cyclic contraction, 0-completeness, common fixed point.
AMS Subject Classification: 54H25, 47H10.

1. Introduction

In 1994, Matthews [18] introduced the concept of partial metric spaces as a part of the study of denotational semantics of dataflow networks and proved the well known Banach Contraction Principle in this setting. Complete partial metric space is a useful framework to model several complex problems in theory of computation. The works of [5, 6, 8, 11, 9, 10, 19, 20, 21] are viable and have opened new avenues for applications in different fields of mathematics and applied sciences. It is interesting to note that in partial metric spaces, self distance of an arbitrary point need not be equal to zero. Matthews [18] introduced a class of open p-balls in partial metric spaces which generates a T_0 topology on X. This facilitated the initiation of open and closed sets, neighbourhoods and other allied notions in partial metric spaces. Recently, many authors studied fixed points of cyclic mappings in several spaces. In 2003, Kirk et al.[17] introduced the notion of cyclic mappings and proved some fixed point theorems for these mappings. Some results for cyclic contractions in partial metric spaces have been obtained in [4, 1, 7, 14, 15]. In 2013, Shatanawi et al.[24]

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proved some common fixed point theorem with the help of control functions, namely, altering distance functions due to Khan et al.[16]. After that, several generalized control functions were used to obtain fixed point results in various spaces. Motivated by the works in [12, 22, 25], we will prove some coincidence points and common fixed point results for a pair of self mappings satisfying some generalized cyclic contraction type conditions involving a control function with two variables in partial metric spaces. Our results extend and unify several existing results in the literature. Finally, we give some examples to justify the validity of our results.

2. Some Basic Concepts

In this section, we begin with some basic facts and properties of partial metric spaces.

Definition 2.1. [18] A partial metric on a nonempty set X is a function $p : X\times X \to \mathbb{R}^+$ such that for all $x,\, y,\, z \in X$:

(p_1) $p(x,x) = p(y,y) = p(x,y) \iff x = y$,
(p_2) $p(x,x) \leq p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X,p) is called a partial metric space.

It is clear that if $p(x,y) = 0$, then from (p_1) and (p_2), it follows that $x = y$. But if $x = y$, $p(x,y)$ may not be 0.

Example 2.2. [18] Let $X = [0,\infty)$ and $p(x,y) = max\,\{x,\, y\}$, for all $x,\, y \in X$. Then (X,p) is a partial metric space.

Example 2.3. [18] Let $X = \{[a,b] : a,\, b \in \mathbb{R},\, a \leq b\}$ and $p([a,b],[c,d]) = max\,\{b,\, d\} - min\,\{a,\, c\}$. Then (X,p) is a partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\epsilon) : x \in X,\, \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

Theorem 2.4. *If $U \in \tau_p$ and $x \in U$, then there exists $r > 0$ such that $B_p(x,r) \subseteq U$.*

Proof. Since U is an open set containing x, there exists an open p-ball, say $B_p(y,\epsilon)$ such that $x \in B_p(y,\epsilon) \subseteq U$. Then $p(x,y) < p(y,y) + \epsilon$. Let us choose $0 < r < p(y,y) - p(x,y) + \epsilon$ and consider the open p-ball $B_p(x,r)$. Then it is easy to verify that $B_p(x,r) \subseteq B_p(y,\epsilon) \subseteq U$. □

Remark 2.5. Let (X,p) be a partial metric space, (x_n) be a sequence in X and $x \in X$. Then (x_n) converges to x with respect to(w.r.t.) τ_p if and only if $\lim\limits_{n\to\infty} p(x_n,x) = p(x,x)$.

Let $x_n \to x$ w.r.t. τ_p and $\epsilon > 0$. Then there exists a natural number n_0 such that $x_n \in B_p(x,\epsilon)$ for all $n \geq n_0$. This gives that $p(x_n,x) - p(x,x) < \epsilon$ for all $n \geq n_0$. Since $p(x_n,x) - p(x,x) \geq 0$, it follows that $\mid p(x_n,x) - p(x,x) \mid < \epsilon$ for all $n \geq n_0$. This proves that $\lim\limits_{n\to\infty} p(x_n,x) = p(x,x)$.

Conversely, suppose that $\lim\limits_{n\to\infty} p(x_n,x) = p(x,x)$. We shall show that $x_n \to x$ w.r.t. τ_p. Let $U \in \tau_p$ and $x \in U$. Then there exists $\epsilon > 0$ such that $x \in B_p(x,\epsilon) \subseteq U$.

By hypotheses, it follows that

$$\lim_{n\to\infty}(p(x_n,x)-p(x,x))=0.$$

So, there exists $n_0\in\mathbb{N}$ such that $p(x_n,x)-p(x,x)<\epsilon$ for all $n\geq n_0$. This ensures that $x_n\in B_p(x,\epsilon)$ for all $n\geq n_0$ and hence $x_n\in U$ for all $n\geq n_0$. Therefore, (x_n) converges to x w.r.t. τ_p on X.

Definition 2.6. [18] Let (X,p) be a partial metric space and let (x_n) be a sequence in X. Then

(i) (x_n) converges to a point $x\in X$ if $\lim_{n\to\infty}p(x_n,x)=p(x,x)$. This will be denoted as $\lim_{n\to\infty}x_n=x$ or $x_n\to x(n\to\infty)$.
(ii) (x_n) is called a Cauchy sequence if $\lim_{n,m\to\infty}p(x_n,x_m)$ exists and is finite.
(iii) (X,p) is said to be complete if every Cauchy sequence (x_n) in X converges to a point $x\in X$ such that $p(x,x)=\lim_{n,m\to\infty}p(x_n,x_m)$.

Definition 2.7. [23] A sequence (x_n) in (X,p) is called 0-Cauchy if

$$\lim_{n,m\to\infty}p(x_n,x_m)=0.$$

The space (X,p) is said to be 0-complete if every 0-Cauchy sequence in X converges to a point $x\in X$ such that $p(x,x)=0$.

It is easy to verify that every closed subset of a 0-complete partial metric space is 0-complete.

Lemma 2.8. *Let (X,p) be a partial metric space.*

(a) *(see [3, 13]) If $p(x_n,z)\to p(z,z)=0$ as $n\to\infty$, then $p(x_n,y)\to p(z,y)$ as $n\to\infty$ for each $y\in X$.*
(b) *(see [23]) If (X,p) is complete, then it is 0-complete.*

The converse assertion of (b) may not hold, in general. The following example supports the above remark.

Example 2.9. [23] The space $X=[0,\infty)\cap\mathbb{Q}$ with the partial metric $p(x,y)=max\{x,y\}$ is 0-complete, but it is not complete. Moreover, the sequence (x_n) with $x_n=1$ for each $n\in\mathbb{N}$ is a Cauchy sequence in (X,p), but it is not a 0-Cauchy sequence.

Definition 2.10. [17] Let X be a nonempty set, $q\in\mathbb{N}$, and let $f:X\to X$ be a self-mapping. Then $X=\cup_{i=1}^{q}A_i$ is a cyclic representation of X with respect to f if

(a) $A_i,\ i=1,2,\cdots,q$ are nonempty subsets of X;
(b) $f(A_1)\subseteq A_2,\ f(A_2)\subseteq A_3,\ \cdots,\ f(A_{q-1})\subseteq A_q,\ f(A_q)\subseteq A_1$.

Definition 2.11. [2] Let T and S be self mappings of a set X. If $y=Tx=Sx$ for some x in X, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S.

Definition 2.12. [13] The mappings $T,S:X\to X$ are weakly compatible, if for every $x\in X$, the following holds:

$$T(Sx)=S(Tx)\ \textit{whenever}\ Sx=Tx.$$

Proposition 2.13. [2] *Let S and T be weakly compatible self maps of a nonempty set X. If S and T have a unique point of coincidence $y=Sx=Tx$, then y is the unique common fixed point of S and T.*

3. Main Results

In this section, we will prove some coincidence point and common fixed point theorems for a pair of self mappings defined on a 0-complete partial metric space and satisfying a generalized contraction type condition involving a control function of two variables. In 2013, Nashine et al.[22] introduced a class of generalized control functions as follows:

Let Φ denote the class of all functions $\varphi : [0,\infty)^2 \to [0,\infty)$ satisfying the following conditions:

(a) φ is lower semicontinuous;
(b) $\varphi(s,t) = 0$ if and only if $s = t = 0$.

We begin with the following theorem.

Theorem 3.1. *Let (X,p) be a 0-complete partial metric space, $q \in \mathbb{N}$ and A_1, A_2, $\cdots$, A_q be nonempty subsets of X. Suppose the mappings f, $g : X \to X$ are such that $g(A_1)$, $g(A_2)$, $\cdots$, $g(A_q)$ are closed subsets of (X,p) and satisfy the following conditions:*

$(C1)$ *$f(A_i) \subseteq g(A_{i+1})$ for $i = 1, 2, \cdots, q$, where $A_{q+1} = A_1$;*
$(C2)$ *there exists $\varphi \in \Phi$ such that*

$$p(fx,fy) \leq M(gx,gy) - \varphi(p(gx,gy), p(gx,fx))$$

for any $(gx,gy) \in g(A_i) \times g(A_{i+1})$, $i = 1, 2, \cdots, q$ with $A_{q+1} = A_1$, where $M(gx,gy) = max\,\{p(gx,gy), p(gx,fx), p(gy,fy), \frac{p(gx,fy)+p(fx,gy)}{2}\}$.

Then f and g have a unique point of coincidence u in $\cap_{i=1}^{q} g(A_i)$ with $p(u,u) = 0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $\cap_{i=1}^{q} g(A_i)$.

Proof. Let $Y = \cup_{i=1}^{q} A_i$ and $x_0 \in Y$ be arbitrary. Then there exists $i_0 \in \{1, 2, \cdots, q\}$ such that $x_0 \in A_{i_0}$. Since $f(A_{i_0}) \subseteq g(A_{i_0+1})$, there exists $x_1 \in A_{i_0+1}$ such that $gx_1 = fx_0$. Continuing this process, we can construct a sequence (x_n) such that $gx_n = fx_{n-1}$, $n = 1, 2, 3, \cdots$, where $x_n \in A_{i_0+n}$ and $A_{q+k} = A_k$.

If $p(gx_n, gx_{n+1}) = 0$ for some $n \in \mathbb{N}$, then $gx_n = gx_{n+1} = fx_n$ and hence gx_{n+1} is a point of coincidence of f and g.

Without loss of generality, we may assume that

$$p(gx_n, gx_{n+1}) > 0,\ \forall n \in \mathbb{N}.$$

Therefore,

$$\varphi(p(gx_n, gx_{n+1}), p(gx_n, gx_{n+1})) > 0,\ \forall n \in \mathbb{N}. \tag{3.1}$$

We note that for all $n \in \mathbb{N}$, there exists $i \in \{1, 2, \cdots, q\}$ such that $(x_n, x_{n+1}) \in A_i \times A_{i+1}$ and so, $(gx_n, gx_{n+1}) \in g(A_i) \times g(A_{i+1})$. By using condition $(C2)$, we obtain

$$\begin{aligned} p(gx_{n+1}, gx_{n+2}) &= p(fx_n, fx_{n+1}) \\ &\leq M(gx_n, gx_{n+1}) - \varphi(p(gx_n, gx_{n+1}), p(gx_n, fx_n)) \\ &= M(gx_n, gx_{n+1}) - \varphi(p(gx_n, gx_{n+1}), p(gx_n, gx_{n+1})), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} M(gx_n, gx_{n+1}) &= max \left\{ \begin{array}{l} p(gx_n, gx_{n+1}), p(gx_n, fx_n), p(gx_{n+1}, fx_{n+1}), \\ \frac{p(gx_n, fx_{n+1}) + p(fx_n, gx_{n+1})}{2} \end{array} \right\} \\ &= max \left\{ \begin{array}{l} p(gx_n, gx_{n+1}), p(gx_{n+1}, gx_{n+2}), \\ \frac{p(gx_n, gx_{n+2}) + p(gx_{n+1}, gx_{n+1})}{2} \end{array} \right\} \\ &\leq max \left\{ \begin{array}{l} p(gx_n, gx_{n+1}), p(gx_{n+1}, gx_{n+2}), \\ \frac{p(gx_n, gx_{n+1}) + p(gx_{n+1}, gx_{n+2})}{2} \end{array} \right\} \\ &= max\{p(gx_n, gx_{n+1}), p(gx_{n+1}, gx_{n+2})\}. \end{aligned}$$

Thus, we obtain from condition (3.2) that

$$\begin{aligned} p(gx_{n+1}, gx_{n+2}) &\leq max\{p(gx_n, gx_{n+1}), p(gx_{n+1}, gx_{n+2})\} \\ &\quad -\varphi(p(gx_n, gx_{n+1}), p(gx_n, gx_{n+1})). \end{aligned} \tag{3.3}$$

If $max\{p(gx_n, gx_{n+1}), p(gx_{n+1}, gx_{n+2})\} = p(gx_{n+1}, gx_{n+2})$, then by using condition (3.1), we get

$$\begin{aligned} p(gx_{n+1}, gx_{n+2}) &\leq p(gx_{n+1}, gx_{n+2}) - \varphi(p(gx_n, gx_{n+1}), p(gx_n, gx_{n+1})) \\ &< p(gx_{n+1}, gx_{n+2}), \end{aligned}$$

which is a contradiction.
Therefore, $max\{p(gx_n, gx_{n+1}), p(gx_{n+1}, gx_{n+2})\} = p(gx_n, gx_{n+1})$.

Thus, condition (3.3) reduces to

$$\begin{aligned} p(gx_{n+1}, gx_{n+2}) &\leq p(gx_n, gx_{n+1}) - \varphi(p(gx_n, gx_{n+1}), p(gx_n, gx_{n+1})) \\ &< p(gx_n, gx_{n+1}). \end{aligned} \tag{3.4}$$

This shows that $(p(gx_n, gx_{n+1}))$ is a nonincreasing sequence of positive real numbers. So, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p(gx_n, gx_{n+1}) = r. \tag{3.5}$$

Taking the upper limit as $n \to \infty$ in (3.4) and using condition (3.5) and lower semicontinuity of φ, we get

$$\begin{aligned} r &\leq r - \liminf_{n\to\infty} \varphi(p(gx_n, gx_{n+1}), p(gx_n, gx_{n+1})) \\ &\leq r - \varphi(r, r), \end{aligned}$$

which implies that $\varphi(r, r) = 0$ and hence $r = 0$.
Therefore,

$$\lim_{n\to\infty} p(gx_n, gx_{n+1}) = 0. \tag{3.6}$$

We now show that (gx_n) is 0-Cauchy in $g(Y)$.

If possible, suppose that (gx_n) is not a 0-Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find two subsequences (gx_{m_i}) and (gx_{n_i}) of (gx_n) such that n_i is the smallest positive integer satisfying

$$p(gx_{m_i}, gx_{n_i}) \geq \epsilon \ for \ n_i > m_i > i. \tag{3.7}$$

So, it must be the case that

$$p(gx_{m_i}, gx_{n_i-1}) < \epsilon. \tag{3.8}$$

Using conditions (3.7), (3.8) and (p_4), we obtain

$$\begin{aligned}\epsilon &\leq p(gx_{m_i}, gx_{n_i})\\ &\leq p(gx_{m_i}, gx_{n_i-1}) + p(gx_{n_i-1}, gx_{n_i}) - p(gx_{n_i-1}, gx_{n_i-1})\\ &< \epsilon + p(gx_{n_i-1}, gx_{n_i}).\end{aligned}$$

This gives that

$$\epsilon \leq p(gx_{m_i}, gx_{n_i}) < \epsilon + p(gx_{n_i-1}, gx_{n_i}).$$

Passing to the limit as $i \to \infty$ and using condition (3.6), we have

$$\lim_{i\to\infty} p(gx_{m_i}, gx_{n_i}) = \epsilon. \tag{3.9}$$

We observe that for all i, there exists $r_i \in \{1, 2, \cdots, q\}$ such that $n_i - m_i + r_i \equiv 1[q]$. Then $x_{m_i-r_i}$ (for large i, $m_i > r_i$) and x_{n_i} lie in different adjacently labelled sets A_j and A_{j+1} for certain $j \in \{1, 2, \cdots, q\}$ where $A_{q+1} = A_1$. So, $(gx_{m_i-r_i}, gx_{n_i}) \in g(A_j) \times g(A_{j+1})$.

By using condition $(C2)$, we get

$$\begin{aligned}p(gx_{m_i-r_i+1}, gx_{n_i+1}) &= p(fx_{m_i-r_i}, fx_{n_i})\\ &\leq M(gx_{m_i-r_i}, gx_{n_i})\\ &\quad -\varphi(p(gx_{m_i-r_i}, gx_{n_i}), p(gx_{m_i-r_i}, fx_{m_i-r_i})),\end{aligned} \tag{3.10}$$

where

$$M(gx_{m_i-r_i}, gx_{n_i}) = max\left\{\begin{array}{l} p(gx_{m_i-r_i}, gx_{n_i}), p(gx_{m_i-r_i}, fx_{m_i-r_i}),\\ \\ p(gx_{n_i}, fx_{n_i}), \frac{p(gx_{m_i-r_i}, fx_{n_i}) + p(fx_{m_i-r_i}, gx_{n_i})}{2}\end{array}\right\}. \tag{3.11}$$

We now compute that $\lim_{i\to\infty} p(gx_{m_i-r_i}, gx_{m_i}) = 0$. By repeated use of (p_4), we get

$$\begin{aligned}p(gx_{m_i-r_i}, gx_{m_i}) &\leq \sum_{l=0}^{r_i-1} p(gx_{m_i-r_i+l}, gx_{m_i-r_i+l+1})\\ &\leq \sum_{l=0}^{q-1} p(gx_{m_i-r_i+l}, gx_{m_i-r_i+l+1}).\end{aligned}$$

Taking the limit as $i \to \infty$ and using condition (3.6), it follows that

$$\lim_{i\to\infty} p(gx_{m_i-r_i}, gx_{m_i}) = 0. \tag{3.12}$$

Using (p_4), we have

$$\begin{aligned}p(gx_{m_i-r_i}, gx_{n_i}) &\leq p(gx_{m_i-r_i}, gx_{m_i}) + p(gx_{m_i}, gx_{n_i}) - p(gx_{m_i}, gx_{m_i})\\ &\leq p(gx_{m_i-r_i}, gx_{m_i}) + p(gx_{m_i}, gx_{n_i}).\end{aligned}$$

Taking the upper limit as $i \to \infty$ and using conditions (3.9) and (3.12), we get

$$\limsup_{i\to\infty} p(gx_{m_i-r_i}, gx_{n_i}) \leq \epsilon. \tag{3.13}$$

Again,

$$\begin{aligned}\epsilon &\leq p(gx_{m_i}, gx_{n_i})\\ &\leq p(gx_{n_i}, gx_{m_i-r_i}) + p(gx_{m_i-r_i}, gx_{m_i}) - p(gx_{m_i-r_i}, gx_{m_i-r_i})\\ &\leq p(gx_{n_i}, gx_{m_i-r_i}) + p(gx_{m_i-r_i}, gx_{m_i}).\end{aligned}$$

Passing to the upper limit as $i \to \infty$ and using conditions (3.12) and (3.13), we obtain

$$\epsilon \leq \limsup_{i\to\infty} p(gx_{m_i-r_i}, gx_{n_i}) \leq \epsilon.$$

Thus,

$$\limsup_{i\to\infty} p(gx_{m_i-r_i}, gx_{n_i}) = \epsilon.$$

By an argument similar to that used above, we can prove that

$$\liminf_{i\to\infty} p(gx_{m_i-r_i}, gx_{n_i}) = \epsilon.$$

Therefore,

$$\lim_{i\to\infty} p(gx_{m_i-r_i}, gx_{n_i}) = \epsilon. \tag{3.14}$$

By (p_4), we have

$$\begin{aligned} p(gx_{m_i-r_i}, gx_{n_i+1}) &\leq p(gx_{m_i-r_i}, gx_{n_i}) + p(gx_{n_i}, gx_{n_i+1}) - p(gx_{n_i}, gx_{n_i}) \\ &\leq p(gx_{m_i-r_i}, gx_{n_i}) + p(gx_{n_i}, gx_{n_i+1}). \end{aligned}$$

Passing to the upper limit as $i \to \infty$ and using conditions (3.6) and (3.14), we obtain

$$\limsup_{i\to\infty} p(gx_{m_i-r_i}, gx_{n_i+1}) \leq \epsilon. \tag{3.15}$$

Moreover,

$$\begin{aligned} \epsilon &\leq p(gx_{m_i}, gx_{n_i}) \\ &\leq p(gx_{n_i}, gx_{m_i-r_i}) + p(gx_{m_i-r_i}, gx_{m_i}) \\ &\leq p(gx_{n_i}, gx_{n_i+1}) + p(gx_{n_i+1}, gx_{m_i-r_i}) + p(gx_{m_i-r_i}, gx_{m_i}). \end{aligned}$$

Passing to the upper limit as $i \to \infty$ and using conditions (3.6), (3.12) and (3.15), we get

$$\epsilon \leq \limsup_{i\to\infty} p(gx_{n_i+1}, gx_{m_i-r_i}) \leq \epsilon.$$

Thus,

$$\limsup_{i\to\infty} p(gx_{n_i+1}, gx_{m_i-r_i}) = \epsilon.$$

Similarly, we can show that

$$\liminf_{i\to\infty} p(gx_{n_i+1}, gx_{m_i-r_i}) = \epsilon.$$

Therefore,

$$\lim_{i\to\infty} p(gx_{n_i+1}, gx_{m_i-r_i}) = \epsilon. \tag{3.16}$$

Furthermore, by (p_4), we have

$$p(gx_{n_i}, gx_{m_i-r_i+1}) \leq p(gx_{m_i-r_i}, gx_{m_i-r_i+1}) + p(gx_{m_i-r_i}, gx_{n_i}).$$

Passing to the upper limit as $i \to \infty$ and using conditions (3.6) and (3.14), we obtain

$$\limsup_{i\to\infty} p(gx_{n_i}, gx_{m_i-r_i+1}) \leq \epsilon. \tag{3.17}$$

Now,

$$\begin{aligned} \epsilon &\leq p(gx_{m_i}, gx_{n_i}) \\ &\leq p(gx_{n_i}, gx_{m_i-r_i}) + p(gx_{m_i-r_i}, gx_{m_i}) \\ &\leq p(gx_{n_i}, gx_{m_i-r_i+1}) + p(gx_{m_i-r_i+1}, gx_{m_i-r_i}) + p(gx_{m_i-r_i}, gx_{m_i}). \end{aligned}$$

Passing to the upper limit as $i \to \infty$ and using conditions (3.6), (3.12) and (3.17), we get

$$\epsilon \leq \limsup_{i\to\infty} p(gx_{n_i}, gx_{m_i-r_i+1}) \leq \epsilon.$$

Thus,

$$\limsup_{i\to\infty} p(gx_{n_i}, gx_{m_i-r_i+1}) = \epsilon.$$

By an argument similar to that used above, we can show that

$$\liminf_{i\to\infty} p(gx_{n_i}, gx_{m_i-r_i+1}) = \epsilon.$$

Therefore,

$$\lim_{i\to\infty} p(gx_{n_i}, gx_{m_i-r_i+1}) = \epsilon. \tag{3.18}$$

Similarly, we have

$$\lim_{i\to\infty} p(gx_{n_i+1}, gx_{m_i-r_i+1}) = \epsilon. \tag{3.19}$$

Taking the limit as $i \to \infty$ in (3.11) and using conditions (3.6), (3.14), (3.16), (3.18), we have

$$\lim_{i\to\infty} M(gx_{m_i-r_i}, gx_{n_i}) = \epsilon. \tag{3.20}$$

Passing to the upper limit as $i \to \infty$ in (3.10) and using conditions (3.19), (3.20) and lower semicontunuity of the function φ, we get

$$\begin{aligned} \epsilon &\leq \epsilon - \liminf_{i\to\infty} \varphi(p(gx_{m_i-r_i}, gx_{n_i}), p(gx_{m_i-r_i}, fx_{m_i-r_i})) \\ &\leq \epsilon - \varphi(\epsilon, 0), \end{aligned}$$

which implies that $\varphi(\epsilon, 0) = 0$, a contradiction, since $\epsilon > 0$. This proves that (gx_n) is a 0-Cauchy sequence in $g(Y)$. As $g(Y) = \cup_{i=1}^{q} g(A_i)$, it follows that $g(Y)$ is a closed subset of the 0-complete partial metric space (X, p) and hence $g(Y)$ is 0-complete. So, (gx_n) converges to some point $u \in g(Y)$ such that $p(u, u) = 0$. Therefore,

$$\lim_{n\to\infty} p(gx_n, u) = p(u, u) = 0. \tag{3.21}$$

We shall prove that $u \in \cap_{i=1}^{q} g(A_i)$.

As $x_0 \in A_{i_0}$, by $(C1)$, it follows that the sequence $(gx_{nq})_{n\geq 0} \subseteq g(A_{i_0})$. Since $g(A_{i_0})$ is closed, condition (3.21) ensures that $u \in g(A_{i_0})$. Again, by $(C1)$, we get $(gx_{nq+1})_{n\geq 0} \subseteq g(A_{i_0+1})$, where $A_{q+k} = A_k$. Proceeding as above, we obtain that $u \in g(A_{i_0+1})$. Continuing in this way, we get

$$u \in \cap_{i=1}^{q} g(A_i). \tag{3.22}$$

Now we shall show that u is a point of coincidence of f and g.

Indeed, since $u \in g(Y)$, there exists $t \in Y$ such that $u = gt$. Now, if $x_n \in A_i$ for some $i \in \{1, 2, \cdots, q\}$, then $(gt, gx_n) = (u, gx_n) \in g(A_{i-1}) \times g(A_i)$ where $A_0 = A_q$, because $u \in \cap_{i=1}^{q} g(A_i)$. By applying $(C2)$, we obtain that for all $n \in \mathbb{N}$,

$$\begin{aligned} p(ft, gx_{n+1}) &= p(ft, fx_n) \\ &\leq M(gt, gx_n) - \varphi(p(gt, gx_n), p(gt, ft)), \end{aligned} \tag{3.23}$$

where

$$M(gt, gx_n) = max\,\{p(gt, gx_n), (gt, ft), p(gx_n, fx_n), \frac{p(gt, fx_n) + p(ft, gx_n)}{2}\}.$$

By Lemma 2.8, we get

$$\lim_{n\to\infty} M(gt, gx_n) = max\{0, p(gt, ft), 0, \frac{p(ft, gt)}{2}\} = p(gt, ft).$$

Taking the upper limit as $n \to \infty$ in (3.23) and using Lemma 2.8 and lower semicontinuity of the function φ, it follows that

$$\begin{aligned} p(ft, gt) &\leq p(gt, ft) - \liminf_{n\to\infty} \varphi(p(gt, gx_n), p(gt, ft)) \\ &\leq p(gt, ft) - \varphi(0, p(gt, ft)), \end{aligned}$$

which implies that $\varphi(0, p(gt, ft)) = 0$ and hence $p(gt, ft) = 0$, that is, $gt = ft = u$.

Therefore, u is a point of coincidence of f and g such that $u \in \cap_{i=1}^{q} g(A_i)$ and $p(u, u) = 0$.

For uniqueness, we assume that there is another point of coincidence v of f and g such that $v \in \cap_{i=1}^{q} g(A_i)$ and $p(v, v) = 0$. By supposition, there exists $x \in X$ satisfying $v = gx = fx$. Taking $u \in g(A_i),\ v \in g(A_{i+1})$ and applying $(C2)$, we have

$$\begin{aligned} p(u, v) &= p(ft, fx) \\ &\leq max\{p(gt, gx), p(gt, ft), p(gx, fx), \frac{p(gt, fx) + p(ft, gx)}{2}\} \\ &\quad -\varphi(p(gt, gx), p(gt, ft)) \\ &= max\{p(u, v), p(u, u), p(v, v), \frac{p(u, v) + p(u, v)}{2}\} \\ &\quad -\varphi(p(u, v), p(u, u)) \\ &= p(u, v) - \varphi(p(u, v), 0). \end{aligned}$$

This gives that $\varphi(p(u, v), 0) = 0$ and hence $p(u, v) = 0$, that is, $u = v$. Thus, f and g have a unique point of coincidence $u \in \cap_{i=1}^{q} g(A_i)$ and $p(u, u) = 0$.

If f and g are weakly compatible, then by Proposition 2.13, f and g have a unique common fixed point in $\cap_{i=1}^{q} g(A_i)$. □

Corollary 3.2. *Let (X, p) be a 0-complete partial metric space and let $f, g : X \to X$ be self mappings. Suppose that $f(X) \subseteq g(X)$ and $g(X)$ is a closed subset of (X, p). If there exists $\varphi \in \Phi$ such that*

$$p(fx, fy) \leq M(gx, gy) - \varphi(p(gx, gy), p(gx, fx))$$

for all $x, y \in X$, then f and g have a unique point of coincidence u in $g(X)$ such that $p(u, u) = 0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof. The proof follows from Theorem 3.1 by taking $A_1 = A_2 = \cdots = A_q = X$. □

Corollary 3.3. *Let (X, p) be a 0-complete partial metric space and let $f : X \to X$ be a self mapping. Suppose there exists $\varphi \in \Phi$ such that*

$$p(fx, fy) \leq M(x, y) - \varphi(p(x, y), p(x, fx))$$

for all $x, y \in X$, where $M(x, y) = max\{p(x, y), p(x, fx), p(y, fy), \frac{p(x,fy)+p(y,fx)}{2}\}$. Then f has a unique fixed point u in X such that $p(u, u) = 0$.

Proof. The proof follows from Theorem 3.1 by taking $A_1 = A_2 = \cdots = A_q = X$ and $g = I$, the identity map on X. □

Corollary 3.4. *Let (X,p) be a 0-complete partial metric space, $q \in \mathbb{N}$ and A_1, A_2, $\cdots$, A_q be nonempty subsets of X. Suppose the mappings $f, g : X \to X$ are such that $g(A_1)$, $g(A_2)$, $\cdots$, $g(A_q)$ are closed subsets of (X,p) and satisfy the following conditions:*

- $(C1)$ *$f(A_i) \subseteq g(A_{i+1})$ for $i = 1, 2, \cdots, q$, where $A_{q+1} = A_1$;*
- $(C3)$ *there exists $r \in [0,1)$ such that*

$$p(fx,fy) \le r\, max\, \{p(gx,gy), p(gx,fx), p(gy,fy), \frac{p(gx,fy)+p(fx,gy)}{2}\}$$

for any $(gx,gy) \in g(A_i) \times g(A_{i+1})$, $i = 1, 2, \cdots, q$ with $A_{q+1} = A_1$.

Then f and g have a unique point of coincidence u in $\cap_{i=1}^{q} g(A_i)$ with $p(u,u) = 0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $\cap_{i=1}^{q} g(A_i)$.

Proof. From $(C3)$, we get

$$\begin{aligned} p(fx,fy) &\le r\, max\, \{p(gx,gy), p(gx,fx), p(gy,fy), \frac{p(gx,fy)+p(fx,gy)}{2}\} \\ &= M(gx,gy) - (1-r)\, M(gx,gy) \\ &\le M(gx,gy) - (1-r)\, max\, \{p(gx,gy), p(gx,fx)\} \\ &= M(gx,gy) - \varphi(p(gx,gy), p(gx,fx)), \end{aligned}$$

where $\varphi(s,t) = (1-r)\, max\, \{s, t\}$, $\forall s, t \in [0,\infty)$. Obviously, $\varphi \in \Phi$. The result now follows from Theorem 3.1 by considering $\varphi(s,t) = (1-r)\, max\, \{s, t\}$, $\forall s, t \in [0,\infty)$. □

Corollary 3.5. *Let (X,p) be a 0-complete partial metric space and $f : X \to X$ be a mapping. If there exists $r \in [0,1)$ such that*

$$p(fx,fy) \le r\, max\, \{p(x,y), p(x,fx), p(y,fy), \frac{p(x,fy)+p(fx,y)}{2}\} \tag{3.24}$$

for all $x, y \in X$, then f has a unique fixed point u in X with $p(u,u) = 0$.

Proof. Condition (3.24) gives that

$$\begin{aligned} p(fx,fy) &\le r\, max\, \{p(x,y), p(x,fx), p(y,fy), \frac{p(x,fy)+p(fx,y)}{2}\} \\ &= M(x,y) - (1-r)\, M(x,y) \\ &\le M(x,y) - (1-r)\, max\, \{p(x,y), p(x,fx)\} \\ &= M(x,y) - \varphi(p(x,y), p(x,fx)), \end{aligned}$$

where $\varphi(s,t) = (1-r)\, max\, \{s, t\}$, $\forall s, t \in [0,\infty)$. The result follows from Theorem 3.1 by taking $A_1 = A_2 = \cdots = A_q = X$, $g = I$ and $\varphi(s,t) = (1-r)\, max\, \{s, t\}$, $\forall s, t \in [0,\infty)$. □

Corollary 3.6. *Let (X,p) be a 0-complete partial metric space, $q \in \mathbb{N}$ and A_1, A_2, $\cdots$, A_q be nonempty subsets of X. Suppose the mappings $f, g : X \to X$ are such that $g(A_1)$, $g(A_2)$, $\cdots$, $g(A_q)$ are closed subsets of (X,p) and satisfy the following conditions:*

- $(C1)$ *$f(A_i) \subseteq g(A_{i+1})$ for $i = 1, 2, \cdots, q$, where $A_{q+1} = A_1$;*
- $(C4)$ *there exist $\alpha, \beta, \gamma, \delta \ge 0$ with $\alpha + \beta + \gamma + 2\delta < 1$ such that*

$$p(fx,fy) \le \alpha\, p(gx,gy) + \beta\, p(gx,fx) + \gamma\, p(gy,fy) + \delta\, (p(gx,fy) + p(fx,gy))$$

for any $(gx,gy) \in g(A_i) \times g(A_{i+1})$, $i = 1, 2, \cdots, q$ with $A_{q+1} = A_1$.

Then f and g have a unique point of coincidence u in $\cap_{i=1}^{q} g(A_i)$ with $p(u,u)=0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $\cap_{i=1}^{q} g(A_i)$.

Proof. From condition $(C4)$, we obtain

$$\begin{aligned} p(fx,fy) &\le \alpha\, p(gx,gy)+\beta\, p(gx,fx)+\gamma\, p(gy,fy)+\delta\,(p(gx,fy)+p(fx,gy)) \\ &\le (\alpha+\beta+\gamma+2\delta)\, M(gx,gy) \\ &= r\, M(gx,gy), \end{aligned}$$

where $r=(\alpha+\beta+\gamma+2\delta)\in[0,1)$. Thus, condition $(C3)$ holds true and Corollary 3.4 can be applied to obtain the desired result. □

Corollary 3.7. *Let (X,p) be a 0-complete partial metric space. Suppose the mapping $f:X\to X$ satisfies the following condition:*

$$p(fx,fy)\le M(x,y)-\frac{p(x,y)+p(x,fx)}{2+p(x,y)+p(x,fx)}$$

for all $x,y\in X$. Then f has a unique fixed point u in X such that $p(u,u)=0$.

Proof. The proof follows from Theorem 3.1 by taking $A_1=A_2=\cdots=A_q=X$, $g=I$ and $\varphi(s,t)=\frac{s+t}{2+s+t}$, $\forall s,\, t\in[0,\infty)$. □

Remark 3.8. Taking $g=I$ in Theorem 3.1, we obtain Theorem 13[22]. As a special case of Corollary 3.6, we obtain several important fixed point results in partial metric spaces including Matthews version of Banach contraction theorem [18].

Next we present our second main theorem.

Theorem 3.9. *Let (X,p) be a 0-complete partial metric space and let $f,\,T:X\to X$ be mappings. Suppose there exists $\varphi\in\Phi$ such that*

$$p(fx,Ty)\le N(x,y)-\varphi(p(x,y),\frac{p(x,fx)+p(y,Ty)}{2}) \tag{3.25}$$

for all $x,y\in X$, where $N(x,y)=max\left\{p(x,y),p(x,fx),p(y,Ty),\frac{p(x,Ty)+p(y,fx)}{2}\right\}$. Then f and T have a unique common fixed point u in X with $p(u,u)=0$.

Proof. We first prove that u is a fixed point of T if and only if u is a fixed point of f with $p(u,u)=0$.

Suppose that u is a fixed point of T, that is, $Tu=u$. Then, by using condition (3.25), we obtain

$$\begin{aligned} p(fu,u) &= p(fu,Tu) \\ &\le N(u,u)-\varphi(p(u,u),\frac{p(u,fu)+p(u,Tu)}{2}), \end{aligned}$$

where

$$\begin{aligned} N(u,u) &= max\left\{p(u,u),p(u,fu),p(u,Tu),\frac{p(u,Tu)+p(u,fu)}{2}\right\} \\ &= max\left\{p(u,u),p(u,fu),\frac{p(u,u)+p(u,fu)}{2}\right\} \\ &= max\,\{p(u,u),p(u,fu)\} \\ &= p(u,fu). \end{aligned}$$

Therefore,

$$p(fu,u)\leq p(u,fu)-\varphi(p(u,u),\frac{p(u,fu)+p(u,u)}{2}),$$

which implies that $\varphi(p(u,u),\frac{p(u,fu)+p(u,u)}{2})=0$. This gives that $\frac{p(u,fu)+p(u,u)}{2}=p(u,u)=0$, that is, $p(u,fu)=0$ and hence $fu=u$ with $p(u,u)=0$.

By an argument similar to that used above, we can show that if u is a fixed point of f, then u is also a fixed point of T with $p(u,u)=0$.

Let $x_0\in X$ be arbitrary. We can construct a sequence (x_n) in X such that

$$x_n=\begin{cases} fx_{n-1}, & if\ n\ is\ odd,\\ \\ Tx_{n-1}, & if\ n\ is\ even.\end{cases}$$

We assume that $x_n\neq x_{n-1}$ for every $n\in\mathbb{N}$. If $x_{2n}=x_{2n+1}$ for some $n\in\mathbb{N}\cup\{0\}$, then $x_{2n}=fx_{2n}$ and hence x_{2n} is a fixed point of f. By our previous discussion, it follows that x_{2n} is also a fixed point of T. So, x_{2n} becomes a common fixed point of f and T. The case $x_{2n+1}=x_{2n+2}$ for some $n\in\mathbb{N}\cup\{0\}$ can be treated similarly to achieve our goal. Therefore, $p(x_n,x_{n-1})>0$, $\forall n\in\mathbb{N}$ and hence

$$\varphi(p(x_n,x_{n-1}),\frac{p(x_n,x_{n-1})+p(x_{m+1},x_m)}{2})>0,\ \forall n,\ m\in\mathbb{N}. \tag{3.26}$$

We now show that $\lim\limits_{n\to\infty}p(x_n,x_{n+1})=0$.

By using condition (3.25), we obtain

$$\begin{aligned} p(x_{2n+1},x_{2n+2}) &= p(fx_{2n},Tx_{2n+1})\\ &\leq N(x_{2n},x_{2n+1})\\ &\quad -\varphi(p(x_{2n},x_{2n+1}),\frac{p(x_{2n},fx_{2n})+p(x_{2n+1},Tx_{2n+1})}{2}),\end{aligned}$$

where

$$\begin{aligned} N(x_{2n},x_{2n+1}) &= max\left\{\begin{array}{l} p(x_{2n},x_{2n+1}),p(x_{2n},fx_{2n}),p(x_{2n+1},Tx_{2n+1}),\\ \\ \frac{p(x_{2n},Tx_{2n+1})+p(x_{2n+1},fx_{2n})}{2}\end{array}\right\}\\ &= max\left\{\begin{array}{l} p(x_{2n},x_{2n+1}),p(x_{2n+1},x_{2n+2}),\\ \\ \frac{p(x_{2n},x_{2n+2})+p(x_{2n+1},x_{2n+1})}{2}\end{array}\right\}\\ &\leq max\left\{\begin{array}{l} p(x_{2n},x_{2n+1}),p(x_{2n+1},x_{2n+2}),\\ \\ \frac{p(x_{2n},x_{2n+1})+p(x_{2n+1},x_{2n+2})}{2}\end{array}\right\}\\ &= max\left\{p(x_{2n},x_{2n+1}),p(x_{2n+1},x_{2n+2})\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned} p(x_{2n+1},x_{2n+2}) &\leq max\left\{p(x_{2n},x_{2n+1}),p(x_{2n+1},x_{2n+2})\right\}\\ &\quad -\varphi(p(x_{2n},x_{2n+1}),\frac{p(x_{2n},x_{2n+1})+p(x_{2n+1},x_{2n+2})}{2}).\end{aligned} \tag{3.27}$$

If $max\,\{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\} = p(x_{2n+1}, x_{2n+2})$, then by using (3.26), we obtain from condition (3.27) that

$$\begin{aligned} p(x_{2n+1}, x_{2n+2}) &\leq p(x_{2n+1}, x_{2n+2}) \\ &\quad -\varphi(p(x_{2n}, x_{2n+1}), \frac{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})}{2}) \\ &< p(x_{2n+1}, x_{2n+2}), \end{aligned}$$

which is a contradiction. Therefore,

$$max\,\{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\} = p(x_{2n}, x_{2n+1}).$$

Thus, condition (3.27) becomes

$$\begin{aligned} p(x_{2n+1}, x_{2n+2}) &\leq p(x_{2n}, x_{2n+1}) \\ &\quad -\varphi(p(x_{2n}, x_{2n+1}), \frac{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})}{2}) \\ &< p(x_{2n}, x_{2n+1}). \end{aligned} \tag{3.28}$$

Similarly, we can show that

$$\begin{aligned} p(x_{2n}, x_{2n+1}) &\leq p(x_{2n-1}, x_{2n}) - \varphi(p(x_{2n-1}, x_{2n}), \frac{p(x_{2n+1}, x_{2n}) + p(x_{2n-1}, x_{2n})}{2}) \\ &< p(x_{2n-1}, x_{2n}). \end{aligned} \tag{3.29}$$

Combining conditions (3.28) and (3.29), we get

$$\begin{aligned} p(x_n, x_{n+1}) &\leq p(x_{n-1}, x_n) - \varphi(p(x_{n-1}, x_n), \frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{2}) \\ &< p(x_{n-1}, x_n), \ \forall n \in \mathbb{N}. \end{aligned} \tag{3.30}$$

Thus, $(p(x_n, x_{n+1}))$ is a nonincreasing sequence of positive numbers. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = r. \tag{3.31}$$

Taking the upper limit as $n \to \infty$ in (3.30) and using (3.31) and lower semicontinuity of φ, we obtain

$$\begin{aligned} r &\leq r - \liminf_{n\to\infty} \varphi(p(x_{n-1}, x_n), \frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{2}) \\ &\leq r - \varphi(r, r), \end{aligned}$$

which implies that $\varphi(r, r) = 0$ and hence $r = 0$. Therefore,

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{3.32}$$

We shall show that (x_n) is a 0-Cauchy sequence in X.

It is sufficient to show that (x_{2n}) is a 0-Cauchy sequence. If possible, suppose that (x_{2n}) is not a 0-Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find two subsequences (x_{2m_i}) and (x_{2n_i}) of (x_{2n}) such that n_i is the smallest positive integer for which

$$p(x_{2m_i}, x_{2n_i}) \geq \epsilon \ for \ n_i > m_i > i. \tag{3.33}$$

This implies that

$$p(x_{2m_i}, x_{2n_i-2}) < \epsilon. \tag{3.34}$$

By repeated use of (p_4) and by condition (3.34), we have

$$\begin{aligned} p(x_{2n_i+1}, x_{2m_i}) &\leq p(x_{2n_i+1}, x_{2n_i}) + p(x_{2n_i}, x_{2m_i}) - p(x_{2n_i}, x_{2n_i}) \\ &\leq p(x_{2n_i+1}, x_{2n_i}) + p(x_{2n_i}, x_{2n_i-1}) \\ &\quad + p(x_{2n_i-1}, x_{2n_i-2}) + p(x_{2n_i-2}, x_{2m_i}) \\ &< p(x_{2n_i+1}, x_{2n_i}) + p(x_{2n_i}, x_{2n_i-1}) \\ &\quad + p(x_{2n_i-1}, x_{2n_i-2}) + \epsilon. \end{aligned}$$

Passing to the upper limit as $i \to \infty$, we get

$$\limsup_{n\to\infty} p(x_{2n_i+1}, x_{2m_i}) \leq \epsilon.$$

From (3.33), we get

$$\epsilon \leq p(x_{2m_i}, x_{2n_i}) \leq p(x_{2m_i}, x_{2n_i+1}) + p(x_{2n_i+1}, x_{2n_i}).$$

Taking the upper limit as $i \to \infty$, we have

$$\epsilon \leq \limsup_{i\to\infty} p(x_{2m_i}, x_{2n_i+1}) \leq \epsilon.$$

Thus,

$$\limsup_{i\to\infty} p(x_{2m_i}, x_{2n_i+1}) = \epsilon.$$

Similarly, $\liminf_{i\to\infty} p(x_{2m_i}, x_{2n_i+1}) = \epsilon$. Therefore,

$$\lim_{i\to\infty} p(x_{2m_i}, x_{2n_i+1}) = \epsilon. \tag{3.35}$$

Again,

$$\begin{aligned} p(x_{2n_i}, x_{2m_i-1}) &\leq p(x_{2n_i}, x_{2n_i-1}) + p(x_{2n_i-1}, x_{2n_i-2}) \\ &\quad + p(x_{2n_i-2}, x_{2m_i}) + p(x_{2m_i}, x_{2m_i-1}) \\ &< \epsilon + p(x_{2n_i}, x_{2n_i-1}) + p(x_{2n_i-1}, x_{2n_i-2}) + p(x_{2m_i}, x_{2m_i-1}). \end{aligned}$$

Passing to the upper limit as $i \to \infty$, we obtain

$$\limsup_{i\to\infty} p(x_{2n_i}, x_{2m_i-1}) \leq \epsilon. \tag{3.36}$$

Also,

$$\epsilon \leq p(x_{2n_i}, x_{2m_i}) \leq p(x_{2n_i}, x_{2m_i-1}) + p(x_{2m_i-1}, x_{2m_i}).$$

Taking the upper limit as $i \to \infty$ and using conditions (3.32) and (3.36), we get

$$\epsilon \leq \limsup_{i\to\infty} p(x_{2n_i}, x_{2m_i-1}) \leq \epsilon.$$

Thus,

$$\limsup_{i\to\infty} p(x_{2n_i}, x_{2m_i-1}) = \epsilon.$$

Similarly, we can obtain

$$\liminf_{i\to\infty} p(x_{2n_i}, x_{2m_i-1}) = \epsilon.$$

Therefore,

$$\lim_{i\to\infty} p(x_{2n_i}, x_{2m_i-1}) = \epsilon. \tag{3.37}$$

By an argument similar to that used above, we can obtain

$$\lim_{i\to\infty} p(x_{2n_i}, x_{2m_i}) = \epsilon \tag{3.38}$$

and

$$\lim_{i\to\infty} p(x_{2n_i+1}, x_{2m_i-1}) = \epsilon. \tag{3.39}$$

By using condition (3.25), we have

$$\begin{aligned} p(x_{2n_i+1}, x_{2m_i}) &= p(fx_{2n_i}, Tx_{2m_i-1}) \\ &\leq N(x_{2n_i}, x_{2m_i-1}) \\ &\quad -\varphi(p(x_{2n_i}, x_{2m_i-1}), \frac{p(x_{2n_i}, fx_{2n_i}) + p(x_{2m_i-1}, x_{2m_i})}{2}), \end{aligned} \tag{3.40}$$

where

$$\begin{aligned} N(x_{2n_i}, x_{2m_i-1}) &= max \left\{ \begin{array}{l} p(x_{2n_i}, x_{2m_i-1}), p(x_{2n_i}, fx_{2n_i}), p(x_{2m_i-1}, Tx_{2m_i-1}), \\ \frac{p(x_{2n_i}, Tx_{2m_i-1}) + p(x_{2m_i-1}, fx_{2n_i})}{2} \end{array} \right\} \\ &= max \left\{ \begin{array}{l} p(x_{2n_i}, x_{2m_i-1}), p(x_{2n_i}, x_{2n_i+1}), p(x_{2m_i-1}, x_{2m_i}), \\ \frac{p(x_{2n_i}, x_{2m_i}) + p(x_{2m_i-1}, x_{2n_i+1})}{2} \end{array} \right\}. \end{aligned} \tag{3.41}$$

Taking the limit as $i \to \infty$ in (3.41) and using conditions (3.32), (3.37), (3.38), (3.39), we get

$$\lim_{i \to \infty} N(x_{2n_i}, x_{2m_i-1}) = max \{\epsilon, 0, 0, \frac{\epsilon + \epsilon}{2}\} = \epsilon. \tag{3.42}$$

Passing to the upper limit as $i \to \infty$ in (3.40) and using conditions (3.32), (3.35), (3.37), (3.42) and lower semicontinuity of φ, we get

$$\begin{aligned} \epsilon &= \limsup_{i \to \infty} p(x_{2n_i+1}, x_{2m_i}) \\ &\leq \limsup_{i \to \infty} N(x_{2n_i}, x_{2m_i-1}) \\ &\quad - \liminf_{i \to \infty} \varphi(p(x_{2n_i}, x_{2m_i-1}), \frac{p(x_{2n_i}, x_{2n_i+1}) + p(x_{2m_i-1}, x_{2m_i})}{2}) \\ &\leq \epsilon - \varphi(\epsilon, 0), \end{aligned}$$

which implies that $\varphi(\epsilon, 0) = 0$ and hence $\epsilon = 0$, a contradiction. Therefore, (x_n) is a 0-Cauchy sequence in X. Since (X, p) is 0-complete, there exists $u \in X$ such that $\lim_{n \to \infty} p(x_n, u) = p(u, u) = 0$. This ensures that $\lim_{n \to \infty} p(x_{2n}, u) = p(u, u) = 0$ and $\lim_{n \to \infty} p(x_{2n+1}, u) = p(u, u) = 0$. Moreover, by Lemma 2.8, $\lim_{n \to \infty} p(x_{2n}, Tu) = p(u, Tu)$ and $\lim_{n \to \infty} p(x_{2n+1}, Tu) = p(u, Tu)$.

By using condition (3.25), we obtain

$$\begin{aligned} p(x_{2n+1}, Tu) &= p(fx_{2n}, Tu) \\ &\leq N(x_{2n}, u)) - \varphi(p(x_{2n}, u), \frac{p(x_{2n}, fx_{2n}) + p(u, Tu)}{2}), \end{aligned} \tag{3.43}$$

where

$$\begin{aligned} N(x_{2n}, u) &= max \left\{ \begin{array}{l} p(x_{2n}, u), p(x_{2n}, fx_{2n}), p(u, Tu), \\ \frac{p(x_{2n}, Tu) + p(u, fx_{2n})}{2} \end{array} \right\} \\ &= max \left\{ \begin{array}{l} p(x_{2n}, u), p(x_{2n}, x_{2n+1}), p(u, Tu), \\ \frac{p(x_{2n}, Tu) + p(u, x_{2n+1})}{2} \end{array} \right\} \\ &\to p(u, Tu) \ as \ n \to \infty. \end{aligned}$$

Taking the upper limit as $n \to \infty$ in (3.43), we have

$$\begin{aligned} p(u,Tu) &\leq p(u,Tu) - \liminf_{i\to\infty} \varphi(p(x_{2n},u), \frac{p(x_{2n},x_{2n+1})+p(u,Tu)}{2}) \\ &\leq p(u,Tu) - \varphi(0, \frac{1}{2}\, p(u,Tu)), \end{aligned}$$

which gives that $\varphi(0, \frac{1}{2}\, p(u,Tu)) = 0$. This assures that $p(u,Tu) = 0$ and hence $Tu = u$. By our previous discussion, u is also a fixed point of f. Therefore, u is a common fixed point of f and T with $p(u,u) = 0$.

For uniqueness, let v be another common fixed point of f and T in X with $p(v,v) = 0$. By applying condition (3.25), we get

$$p(u,v) = p(fu,Tv) \leq N(u,v) - \varphi(p(u,v), \frac{p(u,fu)+p(v,Tv)}{2}), \tag{3.44}$$

where

$$\begin{aligned} N(u,v) &= max \left\{ p(u,v), p(u,fu), p(v,Tv) \}, \frac{p(u,Tv)+p(v,fu)}{2} \right\} \\ &= max\{p(u,v), 0, 0, p(u,v)\} \\ &= p(u,v). \end{aligned}$$

Thus, condition (3.44) becomes

$$p(u,v) \leq p(u,v) - \varphi(p(u,v), 0),$$

which implies that $\varphi(p(u,v), 0) = 0$ and hence $p(u,v) = 0$, that is, $u = v$. Therefore, f and T have a unique common fixed point in X. □

Corollary 3.10. *Let (X,p) be a 0-complete partial metric space and let the mappings $f, T : X \to X$ be such that*

$$p(fx,Ty) \leq r\, max \left\{ p(x,y), p(x,fx), p(y,Ty), \frac{p(x,Ty)+p(y,fx)}{2} \right\} \tag{3.45}$$

for all $x,y \in X$, where $r \in [0,1)$ is a constant. Then f and T have a unique common fixed point u in X with $p(u,u) = 0$.

Proof. From condition (3.45), we have

$$\begin{aligned} p(fx,Ty) &\leq r\, max\{p(x,y), p(x,fx), p(y,Ty), \frac{p(x,Ty)+p(y,fx)}{2}\} \\ &= N(x,y) - (1-r)\, N(x,y) \\ &\leq N(x,y) - (1-r)\, max\{p(x,y), \frac{p(x,fx)+p(y,Ty)}{2}\} \\ &= N(x,y) - \varphi(p(x,y), \frac{p(x,fx)+p(y,Ty)}{2}), \end{aligned}$$

where $\varphi(s,t) = (1-r)\, max\{s,t\}$, $\forall s,\, t \in [0,\infty)$. Obviously, $\varphi \in \Phi$. Now applying Theorem 3.9 we can obtain the desired result. □

Corollary 3.11. *Let (X,p) be a 0-complete partial metric space and let $f : X \to X$ be a mapping. Suppose there exists $\varphi \in \Phi$ such that*

$$p(fx,fy) \leq N^{'}(x,y) - \varphi(p(x,y), \frac{p(x,fx)+p(y,fy)}{2})$$

for all $x,y \in X$, where $N^{'}(x,y) = max \left\{ p(x,y), p(x,fx), p(y,fy), \frac{p(x,fy)+p(y,fx)}{2} \right\}$. Then f has a unique fixed point u in X with $p(u,u) = 0$.

Proof. The proof follows from Theorem 3.9 by considering $T = f$. □

Corollary 3.12. *Let (X, p) be a 0-complete partial metric space and let the mappings $f, T : X \to X$ be such that*

$$p(fx, Ty) \leq \alpha\, p(x, y) + \beta\, p(x, fx) + \gamma\, p(y, Ty) + \delta\, (p(x, Ty) + p(y, fx)) \qquad (3.46)$$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + \gamma + 2\delta < 1$. Then f and T have a unique common fixed point u in X with $p(u, u) = 0$.

Proof. From condition (3.46), we obtain

$$\begin{aligned} p(fx, Ty) &\leq \alpha\, p(x, y) + \beta\, p(x, fx) + \gamma\, p(y, Ty) + \delta\, (p(x, Ty) + p(y, fx)) \\ &\leq (\alpha + \beta + \gamma + 2\delta)\, N(x, y) \\ &= r\, N(x, y) \\ &= N(x, y) - (1 - r)\, N(x, y) \\ &\leq N(x, y) - (1 - r)\, max\{p(x, y), \frac{p(x, fx) + p(y, Ty)}{2}\} \\ &= N(x, y) - \varphi(p(x, y), \frac{p(x, fx) + p(y, Ty)}{2}), \end{aligned}$$

where $r = (\alpha + \beta + \gamma + 2\delta) \in [0, 1)$ and $\varphi(s, t) = (1 - r)\, max\{s, t\},\ \forall s, t \in [0, \infty)$. Now applying Theorem 3.9, we can obtain the desired result. □

Corollary 3.13. *Let (X, p) be a 0-complete partial metric space and let the mappings $f, T : X \to X$ be such that*

$$p(fx, Ty) \leq N(x, y) - \frac{p(x, y) + \frac{1}{2}\, (p(x, fx) + p(y, Ty))}{2 + p(x, y) + \frac{1}{2}\, (p(x, fx) + p(y, Ty))}$$

for all $x, y \in X$. Then f and T have a unique common fixed point u in X with $p(u, u) = 0$.

Proof. The proof follows from Theorem 3.9 by taking $\varphi : [0, \infty)^2 \to [0, \infty)$ as $\varphi(s, t) = \frac{s+t}{2+s+t}$. □

Remark 3.14. The results of this study are obtained under the weaker assumption that the underlying partial metric space is 0-complete. However, they also valid if the space is complete.

Finally, we give some examples to justify the validity of our main results.

Example 3.15. Let $X = \{[1 - 3^{-n}, 1] : n \in \mathbb{N}\} \cup \{[1, 1 + 3^{-n}] : n \in \mathbb{N}\} \cup \{\{1\}\}$, where $\{1\} = [1, 1]$. We define $p : X \times X \to \mathbb{R}^+$ by $p([a, b], [c, d]) = max\,\{b, d\} - min\,\{a, c\}$. Then (X, p) is a 0-complete partial metric space. Let $A_1 = \{[1 - 3^{-n}, 1] : n \in \mathbb{N}\} \cup \{\{1\}\}$ and $A_2 = \{[1, 1 + 3^{-n}] : n \in \mathbb{N}\} \cup \{\{1\}\}$. Obviously, $X = A_1 \cup A_2$. Define mappings $f, g : X \to X$ by

$$fx = \begin{cases} [1, 1 + 3^{-(n+2)}], & if\ \ x = [1 - 3^{-n}, 1], \\ [1 - 3^{-(n+2)}, 1], & if\ \ x = [1, 1 + 3^{-n}], \\ \{1\}, & if\ \ x = \{1\} \end{cases}$$

and

$$gx = \begin{cases} [1-3^{-(n+1)}, 1], \quad if \;\; x = [1-3^{-n}, 1], \\ \\ [1, 1+3^{-(n+1)}], \;\; if \;\; x = [1, 1+3^{-n}], \\ \\ \{1\}, \quad if \;\; x = \{1\}. \end{cases}$$

Then, $f(A_1) \subseteq g(A_2),\ f(A_2) \subseteq g(A_1)$ and $g(A_1),\ g(A_2)$ are closed subsets of (X, p). Thus, condition $(C1)$ holds true. We now verify condition $(C2)$ with the control function $\varphi : [0, \infty)^2 \to [0, \infty)$ given by $\varphi(s, t) = \frac{1}{2}\, max\, \{s,\, t\}$. We now consider the following cases:

Case-I: $x = [1-3^{-n}, 1] \in A_1,\ y = [1, 1+3^{-k}] \in A_2,\ n,\, k \in \mathbb{N}$ with $n < k$.

In this case, we have $3^{-k} < 3^{-n}$ and $3^{-k} \leq 3^{-(n+1)}$. Then,

$$p(fx, fy) = p([1, 1+3^{-(n+2)}], [1-3^{-(k+2)}, 1]) = \frac{1}{9}\,(3^{-n} + 3^{-k}) < \frac{2}{9}.\,3^{-n}.$$

$$\begin{aligned} p(gx, gy) &= p([1-3^{-(n+1)}, 1], [1, 1+3^{-(k+1)}]) = 3^{-(k+1)} + 3^{-(n+1)} \\ &= \frac{1}{3}.\,3^{-k} + 3^{-(n+1)} \leq (\frac{1}{3} + 1)\, 3^{-(n+1)} = \frac{4}{9}.\,3^{-n}. \end{aligned}$$

$$p(gx, fx) = p([1-3^{-(n+1)}, 1], [1, 1+3^{-(n+2)}]) = 3^{-(n+2)} + 3^{-(n+1)} = \frac{4}{9}.\,3^{-n}.$$

$$p(gy, fy) = p([1, 1+3^{-(k+1)}], [1-3^{-(k+2)}, 1]) = 3^{-(k+1)} + 3^{-(k+2)} < \frac{4}{9}.\,3^{-n}.$$

$$p(gx, fy) = p([1-3^{-(n+1)}, 1], [1-3^{-(k+2)}, 1]) = 3^{-(n+1)} = \frac{1}{3}.\,3^{-n}.$$

$$p(fx, gy) = p([1, 1+3^{-(n+2)}], [1, 1+3^{-(k+1)}]) = 3^{-(n+2)} = \frac{1}{9}.\,3^{-n}.$$

Now,

$$\frac{p(gx, fy) + p(fx, gy)}{2} = \frac{1}{2}(\frac{1}{3}.\,3^{-n} + \frac{1}{9}.\,3^{-n}) = \frac{2}{9}.\,3^{-n} < \frac{4}{9}.\,3^{-n}.$$

Thus, $M(gx, gy) = \frac{4}{9}.\,3^{-n}$ and

$$\varphi(p(gx, gy), p(gx, fx)) = \frac{1}{2}\, max\, \{p(gx, gy), p(gx, fx)\} = \frac{2}{9}.\,3^{-n}.$$

Therefore,

$$p(fx, fy) < \frac{2}{9}.\,3^{-n} = M(gx, gy) - \varphi(p(gx, gy), p(gx, fx)).$$

Case-II: $x = [1-3^{-n}, 1] \in A_1,\ y = [1, 1+3^{-k}] \in A_2,\ n,\, k \in \mathbb{N}$ with $n > k$.

In this case, we have $3^{-k} > 3^{-n}$ and $3^{-n} \leq 3^{-(k+1)}$. Then, $p(fx, fy) < \frac{2}{9}.\,3^{-k}$, $p(gx, gy) = \frac{1}{3}\,(3^{-k} + 3^{-n}) \leq \frac{4}{9}.\,3^{-k}$, $p(gx, fx) = \frac{4}{9}.\,3^{-n}$, $p(gy, fy) = \frac{4}{9}.\,3^{-k}$ and $p(gx, fy) = 3^{-(k+2)} = \frac{1}{9}.\,3^{-k}$, $p(fx, gy) = 3^{-(k+1)} = \frac{1}{3}.\,3^{-k}$. So, $\frac{p(gx,fy)+p(fx,gy)}{2} = \frac{2}{9}.\,3^{-k}$.

Moreover, we note that $p(gx, gy) = \frac{1}{3}\,(3^{-k} + 3^{-n}) > \frac{2}{3}.\,3^{-n} > \frac{4}{9}.\,3^{-n} = p(gx, fx)$.

Thus, $M(gx, gy) = \frac{4}{9}.\,3^{-k}$ and

$$\varphi(p(gx, gy), p(gx, fx)) = \frac{1}{2}\, max\, \{p(gx, gy), p(gx, fx)\} = \frac{1}{2}\, p(gx, gy) = \frac{1}{6}\,(3^{-k} + 3^{-n}).$$

Therefore,

$$\begin{aligned} M(gx,gy) - \varphi(p(gx,gy),p(gx,fx)) &= \frac{4}{9}.3^{-k} - \frac{1}{6}\left(3^{-k} + 3^{-n}\right) \\ &\geq \frac{4}{9}.3^{-k} - \frac{1}{6}.3^{-k} - \frac{1}{6}.3^{-(k+1)} \\ &= \frac{2}{9}.3^{-k} \\ &> p(fx,fy). \end{aligned}$$

Case-III: $x = [1 - 3^{-n}, 1] \in A_1,\ y = [1, 1 + 3^{-k}] \in A_2,\ n,\, k \in \mathbb{N}$ with $n = k$.

Then,
$p(fx,fy) = \frac{2}{9}.3^{-n}$, $p(gx,gy) = \frac{2}{3}.3^{-n}$, $p(gx,fx) = \frac{4}{9}.3^{-n}$, $p(gy,fy) = \frac{4}{9}.3^{-n}$ and $p(gx,fy) = \frac{1}{3}.3^{-n}$, $p(fx,gy) = \frac{1}{3}.3^{-n}$. So, $\frac{p(gx,fy)+p(fx,gy)}{2} = \frac{1}{3}.3^{-n}$.

Thus, $M(gx,gy) = \frac{4}{9}.3^{-n}$ and

$$\varphi(p(gx,gy),p(gx,fx)) = \frac{1}{2}\,max\,\{p(gx,gy),p(gx,fx)\} = \frac{2}{9}\,3^{-n}.$$

Therefore,

$$p(fx,fy) = \frac{2}{9}.3^{-n} = M(gx,gy) - \varphi(p(gx,gy),p(gx,fx)).$$

Case-IV: $x = [1 - 3^{-n}, 1] \in A_1$, $n \in \mathbb{N}$, $y = \{1\} \in A_2$.

Then,

$$p(fx,fy) = p([1, 1 + 3^{-(n+2)}], \{1\}) = 3^{-(n+2)} = \frac{1}{9}.3^{-n}.$$

$$p(gx,gy) = p([1 - 3^{-(n+1)}, 1], \{1\}) = 3^{-(n+1)} = \frac{1}{3}.3^{-n}.$$

$$p(gx,fx) = p([1 - 3^{-(n+1)}, 1], [1, 1 + 3^{-(n+2)}]) = 3^{-(n+2)} + 3^{-(n+1)} = \frac{4}{9}.3^{-n}.$$

$p(gy,fy) = p(\{1\},\{1\}) = 0$, $p(gx,fy) = p([1-3^{-(n+1)},1],\{1\}) = \frac{1}{3}.3^{-n}$, $p(fx,gy) = p([1, 1+3^{-(n+2)}],\{1\}) = \frac{1}{9}.3^{-n}$. Thus, $M(gx,gy) = \frac{4}{9}.3^{-n}$ and

$$\varphi(p(gx,gy),p(gx,fx)) = \frac{1}{2}\,max\,\{p(gx,gy),p(gx,fx)\} = \frac{2}{9}\,3^{-n}.$$

Therefore,

$$p(fx,fy) = \frac{1}{9}.3^{-n} < \frac{2}{9}.3^{-n} = M(gx,gy) - \varphi(p(gx,gy),p(gx,fx)).$$

Case-V: $x = \{1\} \in A_1$, $y = [1, 1 + 3^{-n}] \in A_2$, $n \in \mathbb{N}$.

In this case, we have

$$p(fx,fy) = \frac{1}{9}.3^{-n},\ M(gx,gy) = \frac{4}{9}.3^{-n},\ \varphi(p(gx,gy),p(gx,fx)) = \frac{1}{6}.3^{-n}.$$

Therefore,

$$p(fx,fy) = \frac{1}{9}.3^{-n} < \frac{5}{18}.3^{-n} = M(gx,gy) - \varphi(p(gx,gy),p(gx,fx)).$$

Case-VI: $x = y = \{1\}$ is trivial.

The other possibility is treated similarly. Moreover, f and g are weakly compatible. Thus, we have all the conditions of Theorem 3.1 and $\{1\}$ is the unique common fixed point of f and g in $g(A_1) \cap g(A_2)$ with $p(\{1\}, \{1\}) = 0$.

The following example supports our Theorem 3.9.

Example 3.16. Let $X = [0, 1]$ and define $p : X \times X \to \mathbb{R}^+$ by $p(x, y) = max\,\{x,\, y\}$ for all $x,\, y \in X$. Then (X, p) is a 0-complete partial metric space. Let $f,\, T : X \to X$ be defined by

$$fx = \frac{x^2}{1+x} \ and \ Tx = \frac{x^2}{2+x}.$$

Define $\varphi : [0, \infty)^2 \to [0, \infty)$ by $\varphi(s, t) = \frac{1}{2}\, max\,\{s,\, t\}$.

We now verify condition (3.25) for all $x,\, y \in X$.

Case-I: $x, y \in X$ with $y \le x$.
Then,

$$p(fx, Ty) = max\,\{\frac{x^2}{1+x},\, \frac{y^2}{2+y}\} \le max\,\{\frac{x^2}{1+x},\, \frac{y^2}{1+y}\} = \frac{x^2}{1+x} \le \frac{x}{2},$$

$$\begin{aligned} N(x,y) &= max\left\{p(x,y), p(x,fx), p(y,Ty), \frac{p(x,Ty)+p(y,fx)}{2}\right\} \\ &= max\left\{x,\, x,\, y,\, \frac{x + max\,\{y,\, \frac{x^2}{1+x}\}}{2}\right\} \\ &= x \end{aligned}$$

and

$$\varphi(p(x,y), \frac{p(x,fx)+p(y,Ty)}{2}) = \varphi(x, \frac{x+y}{2}) = \frac{1}{2}x.$$

Thus,

$$p(fx,Ty) \le \frac{x}{2} = N(x,y) - \varphi(p(x,y), \frac{p(x,fx)+p(y,Ty)}{2}).$$

Case-II: $x, y \in X$ with $x \le y$.
This case can be treated in a similar way to that of Case-I and we compute $p(fx, Ty) \le \frac{y}{2}$, $N(x, y) = y$, $\varphi(p(x, y), \frac{p(x,fx)+p(y,Ty)}{2}) = \frac{y}{2}$. Thus,

$$p(fx,Ty) \le \frac{y}{2} = N(x,y) - \varphi(p(x,y), \frac{p(x,fx)+p(y,Ty)}{2}).$$

Thus, we have all the conditions of Theorem 3.9 and 0 is the unique common fixed point of f and T in X with $p(0, 0) = 0$.

4. Conclusion

Matthews [18] exploited the idea of fixed points of contractive mappings in partial metric spaces. In recent investigations, the study of fixed point theory involving a control function takes a vital role in many aspects. In this paper, we used control functions to obtain some coincidence points and common fixed point results in partial metric spaces. Significance of this study lies in the fact that the results are obtained under the weaker assumption that the underlying partial metric space is 0-complete. However, they also valid if the space is complete.

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