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GENERALIZED NONLOCAL BOUNDARY CONDITION FOR FRACTIONAL PANTOGRAPH DIFFERENTIAL EQUATION VIA HILFER FRACTIONAL DERIVATIVE

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ABSTRACT. Fractional calculus has been very popular due to the application in real world problems. This paper aimed to investigate the existence, uniqueness and Ulam stability for nonlinear fractional pantograph differential equations with generalized nonlocal boundary conditions involving Hilfer fractional derivative. The analysis were done through Banach and Kranoselskii's fixed point theorems. Finally, example are given to illustrate the theoretical results.

KEYWORDS: Pantograph differential equation, Ulam stability, Fixed point theorems, Nonlocal condition, Hilfer-fractional derivative.
AMS Subject Classification: 26A33, 34A12, 34A43, 34D20.

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1. Introduction

Fractional calculus is a generalized of ordinary differentiation and integration to arbitrary non-integer order. Fractional differential equations (FDE's) have picked up significance during the past decades due to its applicability in science and engineering. The primary concept of a fractional derivative was introduced in a letter written to Guillaume del' Hopital by Gottfried Wilhelm Leibniz in 1695 [28]. Because of the history effects associated with the dynamics of the models, non-integer order derivatives have been shown to be useful in modeling various phenomena. Non-integer order derivatives have been effectively utilized to describe physical processes in medicine, physics, image processing, optimization, electrodynamics, nanotechnology, biotechnology, engineering and many more fields, see [5, 17, 24, 16, 12, 44, 37] and the references cited therein.

Zhou et al. in [45], considered the presence of mild solutions for FDE's with Caputo fractional derivative. By applying the Laplace transform and probability density function, they gave a reasonable definition of mild solution. Utilizing the same strategy, Zhou et al. [46], gave a definition of mild solution for FDEs with Riemann Liouville fractional derivative. On the other hand, Hilfer proposed a generalized Rieman-Liouville fractional derivative, Hilfer fractional derivative, which includes Riemann-Liouville fractional derivative and Caputo fractional derivative. For case and details, see [33, 23, 32, 2, 1, 25, 21, 35, 36, 3, 30, 34, 6] and references therein.

An uncommon kind of delay differential equations is so called pantograph equations. It occurs in different fields of pure and applied mathematics, for examples, electrodynamics, control systems, number theory, probability, and quantum mechanics. Many researchers have studied the pantograph-type delay differential equation using analytical and numerical techniques [13, 18, 19, 26, 26, 38, 42, 41, 7, 8]. As of late, stability of FDE's has pulled in expanding interest due to it's applications in solving real life problems such as economics, biology and optimization. Different types of stability such as Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stability has been given much attention for FDE's which involves different types of operators, see [1, 14, 22, 27, 31, 39, 9, 21, 10, 20, 11]. For example, in [18], Balachandran et al. established the existence of solutions of abstract fractional pantograph equations with different types of initial conditions of the form:

$$\begin{cases} {}^C D^{\alpha}_{0+} z(t) = f(t, z(t), z(\gamma t)), \quad t \in J = [0, a], \quad 0 < \alpha < 1, \quad 0 < \gamma < 1, \\ z(0) = z_0, \end{cases} \tag{1.1}$$

where ${}^C D^{\alpha}_{0+}(\cdot)$ is the Caputo fractional derivative of order α and $f : J \times X \times X \to X$ is a continuous function. Vivek et al. [43] extended the results of [18] to differential equations involving Hilfer fractional derivative.

Vivek et al. [40], considered an implicit fractional differential equations with nonlocal condition described by:

$$\begin{cases} D^{\alpha,\beta}_{0^+} z(t) = f(t, z(t), D^{\alpha,\beta}_{0^+} z(t)), \qquad t \in [0, T], \\ I^{1-\gamma}_{0^+} z(0) = \sum_{i=1}^{m} c_i z(\eta_i), \qquad \alpha \le \gamma = \alpha + \beta - \alpha\beta, \quad \eta_i \in (0, T), \end{cases} \tag{1.2}$$

where $D^{\alpha,\beta}_{0^+}(\cdot)$ is the Hilfer fractional derivative of order $(0 < \alpha < 1)$ and type $0 \le \beta \le 1$, $I^{1-\gamma}_{0^+}(\cdot)$ is the Riemann-Liouville fractional integral of order $1 - \gamma$. The existence and uniqueness results were proved by Schaefer fixed point theorem and Banach's Contraction principle. Moreover, the authors addressed the stability analysis via Gronwall's lemma. Recently, Asawasamrit et al. [15] investigated the existence of solutions to nonlocal boundary value problems for fractional differential equations which involves Hilfer fractional derivative

$$\begin{cases} D_{a^+}^{\alpha,\beta} z(t) = f(t, z(t)), \qquad t \in [a,b], \quad 1 < \alpha < 2, \quad 0 \le \beta \le 1, \\ z(a) = 0, \quad z(b) = \sum_{i=1}^{m} c_i I_{a^+}^{\gamma_i} z(\eta_i), \qquad c_i \in \mathbb{R}, \quad \gamma_i > 0, \quad \eta_i \in [a,b], \end{cases} \tag{1.3}$$

where $D_{a^+}^{\alpha,\beta}(\cdot)$ is the Hilfer fractional derivative of order α and type β, $I_{a^+}^{\gamma_i}(\cdot)$ is the Riemann-Liouville fractional integral of order γ_i and $i = 1, \dots, m$. Using different types of fixed point theorems, the authors proved the existence and uniqueness results.

Motivated by the aforementioned discussions, this manuscript investigates the existence and uniqueness of the solutions of nonlinear fractional pantograph differential equations (NFPDE):

$$D_{0^+}^{\alpha,\beta} x(t) = f(t, x(t), x(\lambda t)),\ t \in J = [0,b],\ 1 < \alpha < 2,\ 0 \le \beta \le 1,\ 0 < \lambda < 1, \tag{1.4}$$

$$x(0) = 0,\ x(b) = \sum_{i=1}^{m} c_i x(\tau_i) + \sum_{j=1}^{k} d_j I_{0^+}^{\rho_j} x(\delta_j),\ \rho_j > 0,\ \tau_i, \delta_j \in [0,b], \tag{1.5}$$

where $D_{0^+}^{\alpha,\beta}(\cdot)$ is the Hilfer fractional derivative of order α and type β, $I_{0^+}^{\rho_j}(\cdot)$ is the Riemann-Liouville fractional integral of order $\rho_j > 0$. $\tau_i, d_j \in \mathbb{R}$, $i = 1, \dots, m$, $j = 1, \dots, k$ and $f : J \times \mathbb{R}^2 \to \mathbb{R}$, is a given continuous functions. Moreover, two different types of Ulam stability are investigated.

Remark 1.1. We note that the application of nonlocal condition: $\sum_{i=1}^{m} c_i x(\tau_i) + \sum_{j=1}^{k} d_j I_{0^+}^{\rho_j} x(\delta_j)$, in physical problems yields an excellent results than the initial condition $x(b) = x_b$ [4]. In addition,

- If $d_j = 0$, the generalized nonlocal condition reduces to multipoint nonlocal condition [15, 42, 40].
- If $c_i = 0$, the generalized nonlocal condition reduces to nonlocal Riemann-Liouville integral condition [21].
- If $\rho_j \to 1$ and $c_i = 0$, reduces to nonlocal integral condition.

The outline of the paper is as follows: In Section **??**, we give some prerequisite definitions and results concerning Hilfer fractional operator. In Section **??**, we derived the equivalence between the proposed problem and Volterra integral equation. The existence and uniqueness of the solution of NFPDE are investigated. Stability analysis in the frame of Ulam-Hyers and generalized Ulam-Hyers stable are proved. Example are given to demonstrate the theoretical results. Finally, conclusions part of the paper are given in Section 5

2. Preliminaries

In this section, we recall some preliminaries facts, lemmas and definitions with respect to fractional operators and Hilfer differential equation [29].

Let $J=[0,b]$ $(-\infty<0<b<\infty)$ be a finite interval of $\mathbb{R}$ and $C[0,b]$ be the space of continuous function on $[0,b]$. Let $X=\mathcal{C}([0,b],\mathbb{R})$ denotes the Banach space of all continuous from $[0,b]$ to $\mathbb{R}$ endowed with the norm defined by

$$\|x\| = \max_{t\in[0,b]} |x(t)|.$$

Definition 2.1. [29] The left-sided Riemann-Liouville fractional integral of order $\alpha\in\mathbb{R}^+$ of a function f is defined by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad t>0, \quad \alpha>0, \tag{2.1}$$

where $\Gamma(\cdot)$ denotes Gamma function.

Definition 2.2. [29] Let $\alpha\in\mathbb{R}^+,\ n\in\mathbb{N}$ and $f\in C([0,b],\mathbb{R})$. The operator

$${}^{RL}D_{0^+}^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds, & t>0,\ n-1<\alpha<n, \\ \frac{d^n}{dt^n} f(t), & \alpha=n, \end{cases} \tag{2.2}$$

is called left-sided Riemann-Liouville fractional derivative of order α of a function f.

Definition 2.3. [29] Suppose $\alpha\in\mathbb{R}^+,\ n\in\mathbb{N}$ and $f\in C^n[0,b]$. The Caputo fractional derivative of order $(n-1<\alpha<n)$ of a function f is given by

$${}^{C}\mathcal{D}_{0^+}^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_{0^+}^t (t-s)^{n-\alpha-1} \left(\frac{d}{dt} f\right)^n (s)ds, & t>0, \\ \frac{d^n}{dt^n} f(t), & \alpha=n, \end{cases} \tag{2.3}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.4. [29] Let $n-1<\alpha<n$ and $0\le\beta\le1$, with $n\in\mathbb{N}$. The left-sided Hilfer fractional derivative of order α and type β of a function f is defined by

$$\left(D_{0^+}^{\alpha,\beta} f\right)(t) = I_{0^+}^{\beta(n-\alpha)}\left[\mathcal{D}^n\left(I_{0^+}^{(1-\beta)(n-\alpha)} f\right)\right](t), \tag{2.4}$$

where $\mathcal{D}^n=\left(\frac{d}{dt}\right)^n$ and I is the Riemann-Liouville fractional integral defined in equation equations (2.1).

in particular, if $n=2$, Definition 2.4 is equivalent with

$$\left(D_{0^+}^{\alpha,\beta} f\right)(t) = I_{0^+}^{\beta(2-\alpha)}\left[\mathcal{D}^2\left(I_{0^+}^{(1-\beta)(2-\alpha)} f\right)\right](t). \tag{2.5}$$

Thus, throughout this manuscript, we discuss the case where $n=2$, $1<\alpha<2$, $0\le\beta\le1$ and $\gamma=\alpha+2\beta-\alpha\beta$.

Remark 2.5. It's worth to mention that:

- The derivative is considered as an interpolator between the Riemann-Liouville and Caputo fractional derivatives since

$$D_{0^+}^{\alpha,\beta} f(t) = \begin{cases} D_{0^+}^{\alpha} f(t), & \beta=0, \\ I_{0^+}^{n-\alpha}\mathcal{D}^n f(t), & \beta=1. \end{cases} \tag{2.6}$$

Next, we recall some properties of Hilfer derivative and integral operators.

Lemma 2.6. [29] *Let $\alpha,\beta\in\mathbb{C}$ such that $Re(\alpha)\ge0$ and $Re(\beta)>0$, then there exists,*

$$\left(I_{0^+}^{\alpha} s^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} t^{\beta+\alpha-1}$$

and

$$\left(D^{\alpha}_{0+}s^{\alpha-1}\right)(t) = 0, \quad 0 < \alpha < 1.$$

Lemma 2.7. [29] *Let $Re(\alpha) > 0$ and $Re(\beta) > 0$. If $f \in L^1(J)$, for each $t \in [0, b]$, then the following properties holds:*

$$\left(I^{\alpha}_{0+}I^{\beta}_{0+}f\right)(t) = \left(I^{\alpha+\beta}_{0+}f\right)(t)$$

and

$$\left(D^{\alpha}_{0+}I^{\alpha}_{0+}f\right)(t) = f(t).$$

Lemma 2.8. [29] *Let $Re(\alpha) > 0$, $n = -[-Re(\alpha)]$, $f \in L_1(0, b)$ and $(I^{\alpha}_{0+}f)(t) \in AC^n[0, b]$, then,*

$$(I^{\alpha}_{0+}D^{\alpha}_{0+}f)(t) = f(t) - \sum_{j=1}^{n} \frac{t^{\alpha-j}}{\Gamma(\alpha-j+1)}(I^{j-\alpha}_{0+}f)(0). \tag{2.7}$$

Furthermore, if $1 < \alpha < 2$, we get

$$(I^{\alpha}_{0+}D^{\alpha}_{0+}f)(t) = f(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}(I^{1-\alpha}_{0+}f)(0) + \frac{t^{\alpha-2}}{\Gamma(\alpha+1)}(I^{2-\alpha}_{0+}f)(0). \tag{2.8}$$

Theorem 2.1. ***(Krasnoselskii's fixed point theorem)*** *Let $\mathcal{B}$ be a nonempty bounded closed convex subset of a Banach space X. Let $T_1, T_2 : \mathcal{B} \to X$ be two continuous operators satisfying:*
(i) $T_1x + T_2y \in \mathcal{B}$ whenever $x, y \in \mathcal{B}$;
(ii) T_1 is compact and continuous;
(iii) T_2 is contraction mapping;
then, there exist $u \in \mathcal{B}$ such that $u = T_1u + T_2u$.

Theorem 2.2. ***(Contraction Mapping Principle)*** *Let X be a Banach space, $\mathcal{N} \subset X$ be closed and $T : \mathcal{N} \to \mathcal{N}$ a contraction mapping i.e*

$$\|Tx - Ty\| \leq k\|x - y\|, \text{ for all } x, y \in \mathcal{N} \text{ and } k \in (0, 1),$$

then $\mathcal{N}$ has a unique fixed point.

For shortness of notation, we take I^{α}_{0+} and I^{α}_{0+} as I^{α} and D^{α} respectively.

3. Main Results

This section presents the uniformity connecting NFPDE (1.4) – (1.5) and the Volterra integral equation. In addition, the existence and uniqueness of solutions of NFPDE (1.4) – (1.5) were prove using Banach and Kransnoselkii's fixed point theorems.

Lemma 3.1. *Let $1 < \alpha < 2$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + 2\beta - \alpha\beta$, and let $f : J \times \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f \in \mathcal{C}([J, \mathbb{R}])$ for any $x \in \mathcal{C}([J, \mathbb{R}])$. A function $x \in \mathcal{C}([J, \mathbb{R}])$ is a solution of problem (1.4) – (1.5) if and only if x satisfies the Volterra integral equation:*

$$\begin{aligned} x(t) =& \frac{t^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left(I^{\alpha}f(t, x(t), x(\lambda t))(b) - \sum_{i=1}^{m} c_i I^{\alpha} f(t, x(t), x(\lambda t))(\tau_i)\right. \\ &\left. + \sum_{j=1}^{k} d_j I^{\alpha+\rho_j} f(t, x(t), x(\lambda t))(\delta_j)\right) + I^{\alpha} f(t, x(t), x(\lambda t)), \end{aligned} \tag{3.1}$$

where

$$\Lambda = \frac{1}{\Gamma(\gamma)} \sum_{i=1}^{m} c_i \tau_i^{\gamma-1} + \sum_{j=1}^{k} \frac{d_j}{\Gamma(\gamma+\rho_j)} \delta_i^{\rho_j+\gamma-1} + \frac{b^{\gamma-1}}{\Gamma(\gamma)} \neq 0. \tag{3.2}$$

Proof. Suppose $x \in \mathcal{C}([J,\mathbb{R}])$ satisfies problem (1.4) − (1.5), then we show that x is also a satisfies the integral equation (3.1). Indeed, setting $(I^{2-\gamma,\rho}x)(0) = e_1$, $(I^{1-\gamma,\rho}x)(0) = e_2$, and applying definition 2.4 and Lemma 2.8, yields

$$x(t) = \frac{e_2}{\Gamma(\gamma)} t^{\gamma-1} + \frac{e_1}{\Gamma(\gamma)} t^{\gamma-2} + I^{\alpha} f(t, x(t), x(\lambda t)). \tag{3.3}$$

From the first boundary condition of equation (1.5), we can see that $e_1 = 0$, which implies

$$x(t) = \frac{e_2}{\Gamma(\gamma)} t^{\gamma-1} + I^{\alpha} f(t, x(t), x(\lambda t)). \tag{3.4}$$

Substituting $t = \tau_i$ and multiplying both sides by c_i in (3.4), give

$$c_i x(\tau_i) = \frac{c_i e_2}{\Gamma(\gamma)} \tau_i^{\gamma-1} + c_i I^{\alpha} f(t, x(t), x(\lambda t))(\tau_i), \tag{3.5}$$

which implies

$$\sum_{i=1}^{m} c_i x(\tau_i) = \frac{e_2}{\Gamma(\gamma)} \sum_{i=1}^{m} c_i \tau_i^{\gamma-1} + \sum_{i=1}^{m} c_i I^{\alpha} f(t, x(t), x(\lambda t))(\tau_i). \tag{3.6}$$

Now, putting $t = \delta_j$ and multiplying through by d_j in (3.4), we have

$$d_j x(\delta_j) = \frac{d_j e_2}{\Gamma(\gamma)} \delta_j^{\gamma-1} + d_j I^{\alpha} f(t, x(t), x(\lambda t))(\delta_j). \tag{3.7}$$

Applying I^{ρ_j} to both sides of (3.7) and using Lemma 2.6, we get

$$d_j I^{\rho_j} x(\delta_j) = \frac{d_j e_2}{\Gamma(\gamma+\rho_j)} \delta_j^{\gamma+\rho_j-1} + d_j I^{\alpha+\rho_j} f(t, x(t), x(\lambda t))(\delta_j). \tag{3.8}$$

Thus,

$$\sum_{j=1}^{m} d_j I^{\rho_j} x(\delta_j) = \sum_{j=1}^{m} \frac{d_j e_2}{\Gamma(\gamma+\rho_j)} \delta_j^{\gamma+\rho_j-1} + \sum_{j=1}^{m} d_j I^{\alpha+\rho_j} f(t, x(t), x(\lambda t))(\delta_j). \tag{3.9}$$

From the second boundary condition: $x(b) = \sum_{i=1}^{m} c_i x(\tau_i) + \sum_{j=1}^{k} d_j I^{\rho_j} x(\delta_j)$ and in view of equations (3.6) and (3.9), we obtain

$$\begin{aligned} \sum_{i=1}^{m} c_i x(\tau_i) + \sum_{j=1}^{k} d_j I^{\rho_j} x(\delta_j) =& \frac{e_2}{\Gamma(\gamma)} \sum_{i=1}^{m} c_i \tau_i^{\gamma-1} + \sum_{i=1}^{m} c_i I^{\alpha} f(t, x(t), x(\lambda t))(\tau_i) \\ &+ \sum_{j=1}^{m} \frac{d_j e_2}{\Gamma(\gamma+\delta_j)} \delta_j^{\gamma+\rho_j-1} + \sum_{j=1}^{m} d_j I^{\alpha+\rho_j} f(t, x(t), x(\lambda t))(\delta_j). \end{aligned} \tag{3.10}$$

It follows from (3.4), that

$$x(b) = \frac{e_2}{\Gamma(\gamma)} b^{\gamma-1} + I^{\alpha} f(t, x(t), x(\lambda t))(b). \tag{3.11}$$

In view of equations (3.10) and (3.11), we have

$$\frac{e_2}{\Gamma(\gamma)}b^{\gamma-1}+I^{\alpha}f(t,x(t),x(\lambda t))(b)=\frac{e_2}{\Gamma(\gamma)}\sum_{i=1}^{m}c_i\tau_i^{\gamma-1}+\sum_{i=1}^{m}c_iI^{\alpha}f(t,x(t),x(\lambda t))(\tau_i)$$
$$+\sum_{j=1}^{m}\frac{d_je_2}{\Gamma(\gamma+\rho_j)}\delta_j^{\gamma+\rho_j-1}+\sum_{j=1}^{m}d_jI^{\alpha+\rho_j}f(t,x(t),x(\lambda t))(\delta_j). \tag{3.12}$$

Hence,

$$e_2=\frac{1}{\Lambda}\left(I^{\alpha}f(t,x(t),x(\lambda t))(b)+\sum_{i=1}^{m}c_iI^{\alpha}f(t,x(t),x(\lambda t))(\tau_i)\right.$$
$$\left.+\sum_{j=1}^{m}d_jI^{\alpha+\delta_j}f(t,x(t),x(\lambda t))(\delta_j)\right). \tag{3.13}$$

Therefore, by substituting equation (3.13) in (3.4), the result follows. The converse follows directly. Hence the proof is completed. □

Let us denote

$$\phi=\frac{b^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)}+\frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)}\sum_{i=1}^{m}|c_i|\tau_i^{\alpha}+\frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{j=1}^{k}\frac{|d_j|\delta_j^{\alpha+\rho_j}}{\Gamma(\alpha+\rho_j+1)}+\frac{b^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.14}$$

3.1. **Existence result via Kransnoselskii's fixed point theorem.**

In this subsection, we investigate the existence of solution of problem (1.4) – (1.5) with helps of Kransnoselkii's fixed point theorem 2.1. Thus, followings hypotheses are needed.

(H_1) Let $f: J\times\mathbb{R}^2\to\mathbb{R}$ be a function such that $f\in C[0,b]$ for any $x\in C[0,b]$. For all $u,v,\bar{u},\bar{v}\in\mathbb{R}$ and $t\in J$ there exist a constants $K>0$ such that

$$|f(t,u,v)-f(t,\bar{u},\bar{v})|\le K(|u-\bar{u}|+|v-\bar{v}|).$$

(H_2) There exist $\theta\in\mathcal{C}([0,b],\mathbb{R})$ such that

$$|f(t,x(s),x(\gamma s))|\le\theta(t)$$

for each $t\in J$

(H_3) Suppose that

$$K\eta<\frac{1}{2}, \tag{3.15}$$

where

$$\eta=\frac{b^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)}+\frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)}\sum_{i=1}^{m}|c_i|\tau_i^{\alpha}+\frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{j=1}^{k}\frac{|d_j|\delta_j^{\alpha+\rho_j}}{\Gamma(\alpha+\rho_j+1)}. \tag{3.16}$$

Theorem 3.1. *Let $1<\alpha<2$, $0\le\beta\le1$ and $\gamma=\alpha+2\beta-\alpha\beta$. Suppose that the hypotheses $(H_1)-(H_3)$ are satisfied, then the problem (1.4) – (1.5) has at least one solution on J.*

Proof. Setting $\|\theta\| = \sup_{t\in J}|\theta(t)|$ and choosing $k \geq \phi\|\theta\|$ where ϕ is defined as in equation (3.14) and construct a closed convex set $x \in \mathcal{B}_k = \{x \in \mathcal{X} : \|x\| \leq k\}$. Define the operators T_1 and T_2 on $\mathcal{B}_k$ as follows

$$\begin{aligned}
T_1x(t) =& I^\alpha f(s,x(s),x(\lambda s))(t), \text{ for all } t \in [0,b].\\
T_2x(t) =& \frac{t^{\gamma-1}}{\Lambda\Gamma(\gamma)}\Bigg[I^\alpha f(s,x(s),x(\lambda s))(b) - \sum_{i=1}^{m} c_i I^\alpha f(s,x(s),x(\lambda s))(\tau_i)\\
&+ \sum_{j=1}^{k} d_j I^{\alpha+\rho_j} f(s,x(s),x(\lambda s))(\delta_j)\Bigg], \text{ for all } t \in [0,b].
\end{aligned}$$

We give the prove in the following steps.
Step 1. We show that $T_1x + T_2x \in \mathcal{B}_k$.
Thus, for any $x, y \in \mathcal{B}_k$, yields

$$\begin{aligned}
|(T_1x(t) + T_2y(t))| \leq& \sup_{t\in J}\Bigg\{I^\alpha|f(s,x(s),x(\lambda s))|(t) + \frac{t^{\gamma-1}}{|\Lambda\Gamma(\gamma)|}I^\alpha|f(s,y(s),y(\lambda s))|(b)\\
&+\frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{i=1}^{m}|c_i|I^\alpha|f(s,y(s),y(\lambda s))|(\tau_i)\\
&+\frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{j=1}^{k}|d_j|I^{\alpha+\rho_j}|f(s,y(s),y(\lambda s))|(\delta_j)\Bigg\}\\
\leq& \|\theta\|\Bigg(\frac{b^\alpha}{\Gamma(\alpha+1)} + \frac{b^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)}\sum_{i=1}^{m}|c_i|\tau_i^\alpha\\
&+ \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{j=1}^{k}\frac{|d_j|\delta_j^{\alpha+\rho_j}}{\Gamma(\alpha+\rho_j+1)}\Bigg)\\
\leq& \phi\|\theta\|\\
\leq& k < \infty.
\end{aligned} \tag{3.17}$$

Step 2. We show that, the operator T_2 is contractive.
Let $x, y \in \mathcal{C}([J,\mathbb{R}])$ and $t \in J$, then

$$\begin{aligned}
|(T_2x(t) + T_2y(t))| \leq& \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}I^\alpha|f(s,x(s),x(\lambda s)) - f(s,y(s),y(\lambda s))|(b)\\
&+ \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{i=1}^{m}|c_i|I^\alpha|f(s,x(s),x(\lambda s)) - f(s,y(s),y(\lambda s))|(\tau_i)\\
&+ \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{j=1}^{k}|d_j|I^{\alpha+\rho_j}|f(s,x(s),x(\lambda s)) - f(s,y(s),y(\lambda s))|(\delta_j)\\
\leq& 2K\Bigg(\frac{b^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)}\sum_{i=1}^{m}|c_i|\tau_i^\alpha
\end{aligned} \tag{3.18}$$

$$
\begin{aligned}
&+ \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{j=1}^{k} \frac{|d_j|\delta_j^{\alpha+\rho_j}}{\Gamma(\alpha+\rho_j+1)}\Bigg) \|x-y\| \\
&\leq 2K\eta\|x-y\|.
\end{aligned} \tag{3.19}
$$

Hence, it follows from (3.15), that T_2 is a contraction map.
Step 3. We show that the operator T_1 is continuous and compact.
Indeed, since f is continuous this implies that T_1 is also continuous and for any $x \in C[0,b]$, we get

$$
\|T_1x\| \leq \frac{b^\alpha}{\Gamma(\alpha+1)}\|\theta\|,
$$

which shows that the operator T_1 is uniformly bounded on $\mathcal{B}_k$. Finally, we shows that T_1 is compact.
Denoting $\sup_{(t,x)\in J\times\mathcal{B}_k} |f(t,x(t),x(\lambda t))| = f^* < \infty$. Thus, for any $0 < t_1 < t_2 < T$ gives

$$
\begin{aligned}
|(T_1x)(t_2)-(T_1x)(t_1)| \leq& \frac{1}{\Gamma(\alpha)}\int_0^{t_1} [(t_2-s)^{t-1}-(t_1-s)^{\alpha-1}]|f(s,x(s),x(\lambda s))|ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t_2-s)^{\alpha-1}|f(s,x(s),x(\lambda s))|ds \\
\leq& \frac{f^*}{\Gamma(\alpha+1)}\Big((t_2-t_1)^\alpha + t_2^\alpha - t_1^\alpha\Big) \\
\to& 0
\end{aligned} \tag{3.20}
$$

as $t_2 \to t_1$. As a consequence of Arzela-Ascoli theorem, implies that the operator T_1 is compact on $\mathcal{B}_\kappa$. Thus, by Theorem 2.1, problem (1.4) – (1.5) has at least one solution on J. □

3.2. Uniqueness result via Banach contraction principle.

Now, we prove the uniqueness of problem (1.4) – (1.5) by means of Banach contraction principle.

Theorem 3.2. *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta - \alpha\beta$. Suppose that assumption (H_1) holds such that $2K\phi < 1$, where ϕ is defined by (3.14). Then if there exist a solution of problem (1.4) – (1.5) is unique on J.*

Proof. Define the operator $T : X \to X$ by

$$
\begin{aligned}
(Tx)(t) =& \frac{t^{\gamma-1}}{\Lambda\Gamma(\gamma)}\Bigg[I^\alpha f(s,x(s),x(\lambda s))(b) - \sum_{i=1}^{m} c_i I^\alpha f(s,x(s),x(\lambda s))(\tau_i) \\
&+ \sum_{j=1}^{k} d_j I^{\alpha+\rho_j} f(s,x(s),x(\lambda s))(\delta_j)\Bigg] + I^\alpha f(s,x(s),x(\lambda s))(t),
\end{aligned} \tag{3.21}
$$

then, clearly the operator T is well defined. It enough to show that the operator T has a fixed point which is a solution of problem (1.4) – (1.5).
Let, $\mathcal{N} = \sup_{t\in J}|f(t,0,0)| < \infty$ and setting $\kappa \geq \frac{\mathcal{N}\phi}{1-2K\phi}$. It suffices to show that $T\mathcal{B}_\kappa \subset \mathcal{B}_\kappa$, where $x \in \mathcal{B}_\kappa = \{x \in C[0,b] : \|x\| \leq \kappa\}$.

Indeed, for any $x \in \mathcal{B}_\kappa$, we have

$$
\begin{aligned}
|(Tx)(t)| \le \sup_{t\in J}\Bigg\{ & \frac{t^{\gamma-1}}{|\Lambda\Gamma(\gamma)|} I^{\alpha}|f(s,x(s),x(\lambda s))|(b) + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |c_i| I^{\alpha} |f(s,x(s),x(\lambda s))|(\tau_i) \\
& + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{j=1}^{k} |d_j| I^{\alpha+\rho_j} |f(s,x(s),x(\lambda s))|(\delta_j) + I^{\alpha}|f(s,x(s),x(\lambda s))|(t) \Bigg\} \\
\le {} & \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^{\alpha}\big(|f(s,x(s),x(\lambda s)) - f(s,0,0)| + |f(s,0,0)|\big)(b)
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
& + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |c_i| I^{\alpha}\big(|f(s,x(s),x(\lambda s)) - f(s,0,0)| + |f(s,0,0)|\big)(\tau_i) \\
& + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{j=1}^{k} |d_j| |I^{\alpha+\rho_j}\big(|f(s,x(s),x(\lambda s)) - f(s,0,0)| + |f(s,0,0)|\big)(\delta_j) \\
& + I^{\alpha}\big(|f(s,x(s),x(\lambda s)) - f(s,0,0)| + |f(s,0,0)|\big)(t) \\
\le {} & \Big(2K\|x\| + \mathcal{N}\Big)\Bigg\{ \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^{\alpha}(b) + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |c_i| I^{\alpha}(\tau_i) \\
& + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{j=1}^{k} |d_j| |I^{\alpha+\rho_j}(\delta_j) + I^{\alpha}(t) \Bigg\} \\
\le {} & \Big(2K\|x\| + \mathcal{N}\Big)\Bigg\{ \frac{b^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} \sum_{i=1}^{m} |c_i| \tau_i^{\alpha} \\
& + \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{j=1}^{k} \frac{|d_j| \delta_j^{\alpha+\rho_j}}{\Gamma(\alpha+\rho_j+1)} + \frac{b^{\alpha}}{\Gamma(\alpha+1)} \Bigg\} \\
\le {} & \phi\big(2K\kappa + \mathcal{N}\big) \\
\le {} & \kappa.
\end{aligned} \tag{3.23}
$$

This shows that, $T\mathcal{B}_\kappa \subset \mathcal{B}_\kappa$.
Now, for any $x_1, x_2 \in \mathcal{X}$ and each $t \in J$, yields

$$\begin{aligned}
&|((Tx_1)(t) - (Tx_2)(t))| \\
&\leq \frac{b^{\gamma-1}}{|\Lambda\Gamma(\gamma)|} I^\alpha |f(s, x_1(s), x_1(\lambda s)) - f(s, x_2(s), x_2(\lambda s))|(b) \\
&+ \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |c_i| I^\alpha |f(s, x_1(s), x_1(\lambda s)) - f(s, x_2(s), x_2(\lambda s))|(\tau_i) \\
&+ \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{j=1}^{k} |d_j| I^{\alpha+\rho_j} |f(s, x_1(s), x_1(\lambda s)) - f(s, x_2(s), x_2(\lambda s))|(\delta_j) \\
&+ I^\alpha |f(s, x_1(s), x_1(\lambda s)) - f(s, x_2(s), x_2(\lambda s))|(t) \\
&\leq 2K \left(\frac{b^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} \sum_{i=1}^{m} |c_i| \tau_i^\alpha \right. \\
&\left. + \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{j=1}^{k} \frac{|d_j| \delta_j^{\alpha+\rho_j}}{\Gamma(\alpha+\rho_j+1)} + \frac{b^\alpha}{\Gamma(\alpha+1)} \right) \|x_1 - x_2\| \\
&\leq 2K\phi \|x_1 - x_2\|.
\end{aligned} \tag{3.24}$$

Therefore, it follows that the operator T is a contraction mapping. Thus, Theorem 2.2, guarantee the existence of a unique solution of problem (1.4) – (1.5) on J. □

4. Ulam-Hyers stability

In this section, the Ulam-Hyers and generalized Ulam-Hyers stability for NFPDE (1.4)–(1.5) are investigate. Thus, before we prove the theorem we need the following definitions, remark and lemma which are important in this section.

Definition 4.1. The NFPDE (1.4) – (1.5) is said to be Ulam-Hyers stable if there exists a real constant $\psi > 0$ such that for all $\epsilon > 0$ and for every solution $y \in C([0, b], \mathbb{R})$ of the inequality

$$|D^{\alpha,\beta} y(t) - f(t, y(t), y(\lambda t))| \leq \epsilon, \quad t \in J, \tag{4.1}$$

there exists a solution $x \in C([0, b], \mathbb{R})$ of the problem (1.4) – (1.5) with

$$|y(t) - x(t)| \leq \psi\epsilon, \quad t \in J. \tag{4.2}$$

Definition 4.2. The NFPDE (1.4) – (1.5) is said to be generalized Ulam-Hyers stable if there is $\nu_f \in (\mathbb{R}^+, \mathbb{R}^+)$ and $\nu_f(0) = 0$ such that for every solution $y \in C([0, b], \mathbb{R})$ of problem (1.4) – (1.5) there exists a solution $x \in C([0, b], \mathbb{R})$ of the problem (1.4) – (1.5) such that:

$$|y(t) - x(t)| \leq \nu_f(\epsilon), \quad t \in J, \tag{4.3}$$

holds.

Remark 4.3. A function $y \in C([0, b], \mathbb{R})$ is a solution of (1.4) – (1.5) if and only if there exists a function $h \in C([0, b], \mathbb{R})$ (which depends on y) such that

- $|g(t)| < \epsilon, \quad t \in J.$
- $D^{\alpha,\beta} y(t) = f(t, y(t), y(\lambda t)) + h(t) \quad t \in J.$

It follows from Remark 4.3, that

$$\begin{aligned} y(t) =& \frac{t^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[I^{\alpha}f(s,y(s),y(\lambda s))(b) - \sum_{i=1}^{m} c_i I^{\alpha} f(s,y(s),y(\lambda s))(\tau_i) \right. \\ &\left. + \sum_{j=1}^{k} d_j I^{\alpha+\rho_j} f(s,y(s),y(\lambda s))(\delta_j)\right] + I^{\alpha} f(s,y(s),y(\lambda s))(t) \\ &+ \frac{t^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[I^{\alpha}g(b) - \sum_{i=1}^{m} c_i I^{\alpha} g(\tau_i) + \sum_{j=1}^{k} d_j I^{\alpha+\rho_j} g(\delta_j)\right] + I^{\alpha} g(t), \end{aligned} \tag{4.4}$$

is the solution of the following equation:

$$D^{\alpha,\beta} y(t) = w(t,y(t),y(\lambda t)) + h(t), \quad t \in J. \tag{4.5}$$

Lemma 4.4. *Let $1 < \alpha < 2$ and $0 \leq \beta \leq 1$. If $y \in C([0,b],\mathbb{R})$ is a solution of problem (1.4) − (1.5), then y is a solution of the following integral inequality:*

$$|y(t) - B_y - I^{\alpha} f(s,y(s),y(\lambda s))(t)| \leq \phi\epsilon, \tag{4.6}$$

where

$$\begin{aligned} B_y =& \frac{t^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[I^{\alpha}f(s,y(s),y(\lambda s))(b) - \sum_{i=1}^{m} c_i I^{\alpha} f(s,y(s),y(\lambda s))(\tau_i)\right. \\ &\left. + \sum_{j=1}^{k} d_j I^{\alpha+\rho_j} f(s,y(s),y(\lambda s))(\delta_j)\right]. \end{aligned}$$

Proof. Indeed, from Remark 4.3 and equation (4.4), that

$$\begin{aligned} |y(t) - B_y - I^{\alpha} f(s,y(s),y(\lambda s))(t)| &= \left|\frac{t^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[I^{\alpha}g(b) - \sum_{i=1}^{m} c_i I^{\alpha} g(\tau_i)\right.\right. \\ &\quad \left.\left. + \sum_{j=1}^{k} d_j I^{\alpha+\rho_j} g(\delta_j)\right] + I^{\alpha} g(t)\right| \\ &\leq I^{\alpha}|g(t)| + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^{\alpha}|g(b)| + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} c_i I^{\alpha}|g(\tau_i)| \\ &\quad + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{j=1}^{k} d_j I^{\alpha+\rho_j}|g(\delta_j)| \\ &\leq \epsilon\left[\frac{b^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} \sum_{i=1}^{m} |c_i|\tau_i^{\alpha}\right. \\ &\quad \left. + \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{j=1}^{k} \frac{|d_j|\delta_j^{\alpha+\rho_j}}{\Gamma(\alpha+\rho_j+1)} + \frac{b^{\alpha}}{\Gamma(\alpha+1)}\right] \\ &= \phi\epsilon. \end{aligned} \tag{4.7}$$

□

Theorem 4.1. *Suppose that the assumption* (H_1) *holds with* $K\phi < \frac{1}{2}$*, then the NFPDE* (1.4) − (1.5) *is Ulam-Hyers stable on* J *and accordingly generalized Ulam-Hyers stable.*

Proof. Let $y \in C([0,b],\mathbb{R})$ be the solution of the inequality (4.1) and $x \in C([0,b],\mathbb{R})$ be the unique solution of problem (1.4) − (1.5). Thus,

$$\begin{aligned}
&|y(t)-x(t)| = \\
&\left| y(t) - \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^{\alpha} f(s,x(s),x(\lambda s))(b) + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |c_i| I^{\alpha} f(s,x(s),x(\lambda s))(\tau_i) \right. \\
&\left. - \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{j=1}^{k} |d_j| I^{\alpha+\rho_j} f(s,x(s),x(\lambda s))(\delta_j) - I^{\alpha} f(s,x(s),x(\lambda s))(t) \right| \\
&\leq \left| y(t) - B_y - I^{\alpha} f(s,y(s),y(\lambda s))(t) \right| \\
&\quad + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |c_i| I^{\alpha} \big| f(s,y(s),y(\lambda s)) - f(s,x(s),x(\lambda s)) \big| (\tau_i) \\
&\quad + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{j=1}^{k} |d_j| I^{\alpha+\rho_j} \big| f(s,y(s),y(\lambda s)) - f(s,x(s),x(\lambda s)) \big| (\delta_j) \\
&\quad + I^{\alpha} \big| f(s,y(s),y(\lambda s)) - f(s,x(s),x(\lambda s)) \big| (t) \\
&\leq \epsilon\phi + 2K\phi |y(t)-x(t)|,
\end{aligned} \tag{4.8}$$

which implies that

$$|y(t)-x(t)| \leq \frac{\epsilon\phi}{1-2K\phi}. \tag{4.9}$$

Therefore,

$$|y(t)-x(t)| \leq \psi\epsilon, \tag{4.10}$$

where

$$\psi = \frac{\phi}{1-2K\phi},$$

such that $K\phi < \frac{1}{2}$. Hence, we conclude that the NFPDE(1.4) − (1.5) is Ulam-Hyers stable. Moreover, setting $\nu_f(\epsilon) = \psi\epsilon$ such that $\nu_f(0) = 0$, the NFPDE (1.4) − (1.5) is generalized Ulam-Hyers stable. □

Example 4.5. Consider NFPDE of the form:

$$\begin{cases} D^{\frac{6}{5},\frac{1}{5}} x(t) = \frac{1}{10^{t+3}\left(1+|x(t)|+|x(\frac{1}{6}t)|\right)}, & t \in J = [0,1], \\ x(0) = 0, \quad x(1) = \frac{1}{3}x(\frac{1}{3}) - \frac{1}{2}x(\frac{1}{2}) + \frac{1}{4} I^{\frac{1}{4}} x(\frac{1}{4}). \end{cases} \tag{4.11}$$

By comparing (1.4) − (1.5) with (4.11), we obtain the followings:
$\alpha = \frac{6}{5}$, $\beta = \frac{1}{5}$, $\gamma = \frac{1}{35}$, $\lambda = \frac{1}{6}$, $b = 1$, $c_1 = \frac{1}{3}$, $c_2 = \frac{-1}{2}$, $\tau_1 = \frac{1}{3}$, $\tau_2 = \frac{1}{2}$, $d_1 = \frac{1}{4}$, $\rho_1 = \frac{1}{4}$ and $f : J \times \mathbb{R}^2 \to \mathbb{R}$ is a function defined by

$$f(t,u,v) = \frac{1}{10^{t+3}\left(1+|u|+|v|\right)}, \quad t \in J, \quad u,v \in \mathbb{R}.$$

Clearly, the function f is continuous and for all $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in J$,

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leq \frac{1}{10^3}\left(|u-\bar{u}| + |v-\bar{v}|\right).$$

Thus, assumption (H_1) is satisfies with $K = \frac{1}{10^3}$. Hence, by simple calculation, we obtain $|\Lambda| \approx 0.7578$ and $\phi \approx 2.3206$.
So,

$$2K\phi = \frac{2}{10^3} \times 2.3206 < 1.$$

Thus, it follows from Theorem 3.14 that problem (1.4) − (1.5) has a unique solution on J, since all the assumptions are satisfied.
In addition, $K\phi = \frac{1}{10^3} \times 2.3206 < \frac{1}{2}$. Thus, by Theorem 4.1, problem (1.4) − (1.5) is both Ulam-Hyers and generalized Ulam-Hyers stable on J.

5. Conclusions

We investigate the existence and uniqueness of solutions for problem (1.4) − (1.5) by employed the techniques of Banach and Kransnoselkii's fixed point theorems. We also establish the uniformity between generalized problem (1.4) − (1.5) and the Volterra integral equation. Ulam-Hyers and generalized Ulam-Hyers stability of solutions to (1.4) − (1.5) using the classical calculus approach are established. Finally, as an application example were given to illustrate the main results.

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