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BEST PROXIMITY PAIRS IN CONE METRIC SPACES

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ABSTRACT. In this paper we introduce cone metric space and give some contraction about existence best proximity pairs and best proximity points. Also, we prove some fixed point theorems on cone metric space.

KEYWORDS : Cone metric space; Fixed point; Normal cone; Cone-cyclic contraction.
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1. INTRODUCTION

Let E be a normed linear space. the subset P of E is called a cone if
(i) P is closed, non-empty and $P \neq \{0\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
(iii) $P \cap -P = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leqslant$ with respect to P by $x \leqslant y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leqslant y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $int\ P$ denotes the interior of P. The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in E$, $0 \leqslant x \leqslant y$ implies $\|x\| \leq M\|y\|$.

The least positive number satisfying the above is called the normal constant of P.

Definition 1.1. Let X be a non-empty set, $(E, \|.\|)$ a normed space that ordered by a normal cone P with constant normal $M = 1$, $intP \neq \emptyset$ and $\leqslant$ is a partial ordering with respect to P and the mapping $d : X \times X \to E$ satisfies
(i)$0 \leqslant d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$
(iii) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example 1.2. (i) Suppose $E = R^2$, $P = \{(x, y) \in E : x,\ y \geq 0\}$, $X = R$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$ where $\alpha \geq 0$ is a cone metric

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space.
(ii) Suppose $E = C_R[0,1]$, $P = \{\varphi \in E : \varphi \geq 0\}, X = R$ and $d : X \times X \to E$ defined by $d(x,y) = |x-y|\varphi$ where $\varphi : [0,1] \to R$ such that $\varphi(t) = e^t$. It easy to see that d is a cone metric space.

Definition 1.3. (X,d) a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is N such that for all $n > N$, $d(x_n, x) \ll c$. Then $\{x_n\}$ is said to be convergent to x. We denote this by

$$\lim_{n\to\infty} x_n = x.$$

Also sequence $\{x_n\}$ is said to be bounded if there is $M \gg 0$ such that for all $n \in \mathbb{N}$ we have

$$d(0, x_n) \ll M.$$

Definition 1.4. Let (X,d) be a cone metric space. If every Cauchy sequence be convergence, then X is called a complete cone metric space.

Definition 1.5. Let (X,d) be a cone metric space. If there is a pair $(x_0, y_0) \in A \times B$ such that

$$\|d(x_0, y_0)\| = \inf_{(a,b)\in A\times B} \|d(a,b)\|$$

the pair (x_0, y_0) is called a best proximity pair for A and B and said that the pair (A,B) has best proximity pair in X. Put $Prox(A,B)$ the set of all best proximity pairs for the pair (A,B).

Fixed point theory is an important tool for solving equations $T(x) = x$. However, if T does not have fixed points, then one often tries to find an element x which is in some sense closest to $T(x)$. A classical result in this direction is a best approximation theorem due to Ky Fan [4].
a best proximity pair evolves as a generalization of the best approximation considered by Sahney and Singh [6], Singer [7] and Xu [8], of exploring some the sufficient conditions for the non-empty of the set $Prox(A,B)$.

In this paper we consider sufficient conditions that ensure the existence of an element $x \in X$ for two subsets A, B in cone metric space X such that $\|d(x,Tx)\| = d(A,B)$ for $T : A\cup B \to A \cup B$, where

$$d(A,B) = \inf_{(a,b)\in A\times B} \|d(a,b)\|.$$

It is clear that if $d(A,B) = 0$, then T has fixed point. In continue we consider some fixed point theorems on cone metric space. It is notable we use of results in [1-3].

2. MAIN RESULTS

In this section at first we give a new definition and use for present new results.

Definition 2.1. Let A and B be nonempty subsets of a cone metric space (X,d) and P a cone of normed space E. A map $T : A\cup B \to A\cup B$ is a cone-cyclic contraction map if there exists a continuous map $\varphi : P \to [0,1)$ such that

i) $d(Tx,Ty) \leqslant \varphi(d(x,y))d(x,y) + (1-\varphi(d(x,y)))dist(A,B)$
ii) $\lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = 0$, where, $x_{n+1} = T^{n+1}x_0, x_0 \in A\cup B$
iii) $T(A) \subset B$ and $T(B) \subset A$.
where $dist(A,B) = inf\{d(a,b) : a \in A, b \in B\}$.

Proposition 2.2. *Let A and B be nonempty subsets of a cone metric space X, P a normal cone with normal constant K, $T : A \cup B \to A \cup B$ a cone-cyclic contraction map and $x_{n+1} = T^{n+1}x_0$ for $x_0 \in A \cup B$. Then $d(x_n, Tx_n) \to dist(A, B)$.*

Proof. Choose $x_0 \in A \cup B$. Set $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_n = T^{n+1}x_0, \cdots$ and $\varphi_n := \varphi(d(x_{n-1}, x_n))$. By definition of cone-cyclic contraction map we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant \varphi_n d(x_{n-1}, x_n) + (1 - \varphi_n)dist(A, B) \\ &\leqslant \varphi_n\varphi_{n-1}d(x_{n-2}, x_{n-1}) + (1 - \varphi_n\varphi_{n-1})dist(A, B) \\ &\leqslant \\ &\vdots \\ &\leqslant \varphi_n\varphi_{n-1} \dots \varphi_1 d(x_0, x_1) + (1 - \varphi_n\varphi_{n-1} \dots \varphi_1)dist(A, B). \end{aligned}$$

Therefore

$$d(x_n, x_{n+1}) - (1 - \varphi_n\varphi_{n-1} \dots \varphi_1)dist(A, B) \leqslant \varphi_n\varphi_{n-1} \dots \varphi_1 d(x_0, x_1)$$

and so since P is normal we obtain,

$$\|d(x_n, x_{n+1}) - (1 - \varphi_n\varphi_{n-1} \dots \varphi_1)dist(A, B)\| \leqslant K\varphi_n\varphi_{n-1} \dots \varphi_1\|d(x_0, x_1)\|.$$

Thus $d(x_n, x_{n+1}) \to dist(A, B)$. □

Theorem 2.3. *Let A and B be nonempty subsets of a complete cone metric space X, P be a normal cone with normal constant K. Let $T : A \cup B \to A \cup B$ is a cone-cyclic contraction map, let $x_0 \in A$ and define $x_{n+1} = Tx_n$. Suppose $\{x_{2n}\}$ has a convergent subsequence in A. Then there exists x in A such that $d(x, Tx) = dist(A, B)$.*

Proof. Suppose $\{x_{2n_k}\}$ is a subsequence of $\{x_{2n}\}$ converging to some $x \in A$. Now

$$dist(A, B) \leqslant d(x, x_{2n_k-1}) \leqslant d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}).$$

Thus, we have $d(x, x_{2n_k-1})$ converges to $dist(A, B)$. Since

$$dist(A, B) \leqslant d(x_{2n_k}, Tx) \leqslant d(x_{2n_k-1}, x),$$

$d(x, Tx) = dist(A, B)$. □

Proposition 2.4. *Let A and B be nonempty subsets of a cone metric space X, $T : A \cup B \to A \cup B$ a cone-cyclic contraction map, $x_0 \in A \cup B$ and $x_{n+1} = Tx_n,\ n = 0, 1, 2, \cdots$. Then the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded.*

Proof. Suppose $x_0 \in A$ then since by Proposition 2.2 $d(x_{2n}, x_{2n+1})$ converges to $dist(A, B)$. It is enough to prove that $\{x_{2n+1}\}$ is bounded. Suppose $\{x_{2n+1}\}$ is not bounded, then there exists $n_0 \in \mathbb{N}$ such that

$$d(x_2, x_{2n_0+1}) \gg M \text{ and } d(x_2, x_{2n_0-1}) \ll M,$$

where $M \gg max\{\frac{2d(x_0,Tx_0)}{1/k^2-1} + dist(A, B), d(Tx_2, Tx_0)\}$. By the cone-cyclic contraction property of T,

$$\begin{aligned} \frac{M - dist(A, B)}{k^2} + dist(A, B) &\ll d(x_0, x_{2n_0-1}) \\ &\leqslant d(x_0, x_2) + d(x_2, x_{2n_0-1}) \\ &\leqslant 2d(x_0, Tx_0) + M. \end{aligned}$$

Thus $M \ll \frac{2d(x_0,Tx_0)}{1/k^2-1} + dist(A, B)$, hence

$$\frac{2d(x_0, Tx_0)}{1/k^2 - 1} + dist(A, B) - M \in intP, \frac{2d(x_0, Tx_0)}{1/k^2 - 1} + dist(A, B) - M \in int(-P),$$

which is a contradiction, since $intP \cap int(-P) = \emptyset$. The proof when x_0 in B is similar. □

The Proposition 2.4 leads us to an existence result when one of the sets is boundedly compact. We remember that a set A is boundedly compact if every bounded sequence in A has a subsequence convergence.

Corollary 2.5. *Let A and B be nonempty closed subsets of a complete cone metric space X and $T : A \cup B \to A \cup B$ a cone-cyclic contraction map. If either A or B is boundedly compact, then there exists $x \in A \cup B$ with $d(x, Tx) = dist(A, B)$.*

Proof. It follows directly from Theorem 2.3 and Proposition 2.4 □

Proposition 2.6. *Let A and B be nonempty compact subsets of a cone metric space X, P a normal cone with normal constant $K = 1$. Then Prox(A,B) is nonempty.*

Proof. Define $f : A \times B \to [0, \infty)$ by $f(a, b) = \|d(a, b)\|$. Now choose $0 \ll c$ such that $\|c\| < \frac{\epsilon}{2}$. Suppose $z \in B(a, c)$ and $w \in B(b, c)$. Since

$$d(a, b) \leqslant d(a, z) + d(z, b),$$

$$d(z, b) \leqslant d(z, w) + d(w, b),$$

hence

$$d(a, b) - d(z, w) \leqslant d(a, z) + d(w, b),$$

$$d(z, w) - d(a, b) \leqslant d(a, z) + d(w, b).$$

Therefore

$$\|d(a, b)\| - \|d(z, w)\| < \epsilon,$$

$$\|d(z, w)\| - \|d(a, b)\| < \epsilon$$

and so $|f(a, b) - f(z, w)| < \epsilon$. Then f is continues and since A, B are compact, $f(A, B)$ is compact and so there exists $(a_0, b_0) \in A \times B$ such that

$$\|d(a_0, b_0)\| = \inf_{(a,b)\in A\times B} \|d(a, b)\|.$$

□

Definition 2.7. Let A and B be nonempty subsets of a cone metric space (X, d) and P a cone of normed space E. A map $T : A \cup B \to A \cup B$ is a cone contractive map if
i) $\|d(Tx, Ty)\| \leq \|d(x, y)\|$
ii) $\|d(Tx, Ty)\| < \|d(x, y)\|, \ if \ d(x, y) = d(A, B)$
iii) $T(A) \subseteq B \ and \ T(B) \subseteq A.$

In the following we give different version of the main result in [5].

Theorem 2.8. *Let A and B be nonempty compact subsets of a cone metric space X, P a normal cone with normal constant $K = 1$ and $T : A\cup B \to A\cup B$ cone contractive map. Then there exists $(a_0, b_0) \in A \times B$ such that $\|d(a_0, b_0)\| = \|d(a_0, Ta_0)\| = \|d(b_0, Tb_0)\| = d(A, B)$.*

Proof. Define $f : A \times B \to [0, \infty)$ by $f(a, b) = \|d(Ta, b)\| + \|d(Tb, a)\|$. Since f is continues (similar to proof Proposition 2.5) and A and B are compact, it attains minimum at some element, say (a_0, b_0), in $A \times B$. If $Ta_0 \neq b_0$, since

$$\begin{aligned} f(Ta_0, Tb_0) &= \|d(TTa_0, Tb_0)\| + \|d(Ta_0, TTb_0)\| \\ &< \|d(Ta_0, b_0)\| + \|d(a_0, Tb_0)\| \\ &= f(a_0, b_0) \end{aligned}$$

which is a contrary to the fact that f attains minimum at (a_0, b_0). Then $Ta_0 = b_0$. A similar argument can be given to show that $a_0 = Tb_0$. If $d(A, B) < \|d(a_0, b_0)\|$, then $\|d(a_0, b_0)\| = \|d(Ta_0, Tb_0)\| < \|d(a_0, b_0)\|$ which is contradiction. □

Corollary 2.9. *Let A and B be nonempty compact subsets of a cone metric space X, P a normal cone with normal constant $K = 1$ and $T : A \cup B \to A \cup B$ cone contractive map. Then T^2 has a fixed point.*

Proof. By Theorem 2.7 $Ta_0 = b_0$ and $a_0 = Tb_0$ and so $T^2 a_0 = a_0$. □

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