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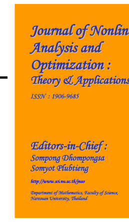

HYPERSTABILITY OF A CAUCHY FUNCTIONAL EQUATION

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ABSTRACT. The aim of this paper is to offer hyperstability results for the Cauchy functional equation

$$f\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} f(x_i)$$

in Banach spaces. Namely, we show that a function satisfying the equation approximately must be actually a solution to it.

KEYWORDS : Hyperstability, Cauchy equation, Fixed point theorem.
AMS Subject Classification: Primary 39B82, 39B62; Secondary 47H14, 47H10.

1. INTRODUCTION

Let X and Y be Banach spaces. A mapping $X :\longrightarrow Y$ is called, additive function, if it satisfies the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \qquad \text{for all} \qquad x, y \in X.$$

In 1940, S. M. Ulam [15] raised the question concerning the stability of group homomorphisms: "when is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation?". The first answer to Ulams question, concerning the Cauchy equation, was given by D. H. Hyers [10]. Thus we speak about the Hyers-Ulam stability. This terminology is also applied to the case of other functional equations. Th. M. Rassias [14] generalized the theorem of Hyers for approximately linear mappings [14]. The stability phenomena that was proved by Th. M. Rassias [14] is called the Hyers-Ulam-Rassias stability. The modified Ulams stability problem with the generalized control function was proved by P. Găvruta [8].

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In 1994, J. M. Rassias [13] studied the Ulams problem of the following equation

$$f\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} f(x_i) \tag{1.1}$$

for all $x_1, x_2, ..., x_n \in X$.

We say a functional equation $\mathfrak{D}$ is *hyperstable* if any function f satisfying the equation $\mathfrak{D}$ approximately is a true solution of $\mathfrak{D}$. It seems that the first hyperstability result was published in [2] and concerned the ring homomorphisms. However, The term *hyperstability* has been used for the first time in [11]. Quite often the hyperstability is confused with superstability, which admits also bounded functions.

The hyperstability problem of various types of functional equations have been investigated by a number of authors, we refer, for example, to [1], [6], [4], [5], [9] and [12]. Throughout this paper, we present the hyperstability results for the additive functional equation (1.1).

The method of the proofs used in the main results is based on a fixed point result that can be derived from [3, Theorem 1]. To present it we need the following three hypotheses:

(H1) X is a nonempty set, Y is a Banach space, $f_1, ..., f_k : X \longrightarrow Y$ and $L_1, ..., L_k : X \longrightarrow \mathbb{R}_+$ are given.

(H2) $\mathcal{T} : Y^X \longrightarrow Y^X$ is an operator satisfying the inequality

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \leq \sum_{i=1}^{k} L_i(x) \|\xi(f_i(x)) - \mu(f_i(x))\|, \qquad \xi, \mu \in Y^X, \quad x \in X.$$

(H3) $\Lambda : \mathbb{R}_+^X \longrightarrow \mathbb{R}_+^X$ is a linear operator defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)), \qquad \delta \in \mathbb{R}_+^X, \quad x \in X.$$

The following theorem is the basic tool in this paper. We use it to assert the existence of a unique fixed point of operator $\mathcal{T} : Y^X \longrightarrow Y^X$.

Theorem 1.1. *Let hypotheses (**H1**)-(**H3**) be valid and functions $\varepsilon : X \longrightarrow \mathbb{R}_+$ and $\varphi : X \longrightarrow Y$ fulfil the following two conditions*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \varepsilon(x), \qquad x \in X,$$

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \qquad x \in X.$$

Then there exits a unique fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \qquad x \in X.$$

Moreover,

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x), \qquad x \in X.$$

Numerous papers on this subject have been published and we refer to [1], [6], [4], [5], [9], [12].

2. Hyperstability results

The following theorems and corollaries are the main results in this paper and concern the hyperstability of equation (1.1).

Theorem 2.1. *Let X be a normed space, Y be a Banach space, $c \geq 0$, $p < 0$ and let $f : X \longrightarrow Y$ satisfy*

$$\left\| f\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_i) \right\| \leq c\left(\sum_{i=1}^{n} \|x_i\|^p\right) \tag{2.1}$$

for all $x_1, x_2, ..., x_n \in X \setminus \{0\}$ where n is an integer with $n \geq 2$. Then f is additive on $X \setminus \{0\}$.

Proof. We study two cases as follows:
Case 1: n is even
In this case, let $n = 2r + 2$ where $r \in \mathbb{N}$. Then, the inequality (2.1) can be written as follows

$$\left\| f\left(\sum_{i=1}^{2r+2} x_i\right) - \sum_{i=1}^{2r+2} f(x_i) \right\| \leq c\left(\sum_{i=1}^{2r+2} \|x_i\|^p\right) \tag{2.2}$$

where $r \in \mathbb{N}$.
Replacing $x_{(2r+2)}$ by $((2r+1)m+1)\,x$, x_i by $\left(-m - \frac{i}{2}\right)x$ where $i = 2, 4, ..., 2r$ and x_j by $\left(-m + \frac{j-1}{2}\right)$ where $j = 1, 3, ..., (2r+1)$ and $m \in \mathbb{N}$ in (2.2), we obtain that

$$\left\| f(x) - f\Big(\big((2r+1)m+1\big)x\Big) - f(-mx) - \sum_{\ell=1}^{r} f\big((-m+\ell)x\big) - \sum_{\ell=1}^{r} f\big((-m-\ell)x\big) \right\|$$
$$\leq c\left(\Big((2r+1)m+1\Big)^p + m^p + \sum_{\ell=1}^{r} \big|\ell - m\big|^p + \sum_{\ell=1}^{r} \big|\ell + m\big|^p\right)\|x\|^p \tag{2.3}$$

for all $x \in X \setminus \{0\}$.
Further put

$$\mathcal{T}_m\xi(x) := \xi\Big(\big((2r+1)m+1\big)x\Big) + \xi(-mx) + \sum_{\ell=1}^{r} \xi\big((-m+\ell)x\big) + \sum_{\ell=1}^{r} \xi\big((-m-\ell)x\big)$$

and

$$\varepsilon_m(x) := c\left(\Big((2r+1)m+1\Big)^p + m^p + \sum_{\ell=1}^{r} \big|\ell - m\big|^p + \sum_{\ell=1}^{r} \big|\ell + m\big|^p\right)\|x\|^p$$

for all $x \in X \setminus \{0\}$ and all $\xi \in Y^{X\setminus\{0\}}$. The inequality (2.3) now takes the following form

$$\|\mathcal{T}_m f(x) - f(x)\| \leq \varepsilon_m(x), \qquad x \in X \setminus \{0\}.$$

The following operator

$$\Lambda_m\delta(x) := \delta\Big(\big((2r+1)m+1\big)x\Big) + \delta(-mx) + \sum_{\ell=1}^{r} \delta\big((-m+\ell)x\big) + \sum_{\ell=1}^{r} \delta\big((-m-\ell)x\big)$$

for all $x \in X\setminus\{0\}$ and all $\delta \in \mathbb{R}_+^{X\setminus\{0\}}$, has the form described in (**H3**) with $k = 2r+2$, and

$$\begin{aligned} f_i(x) &= (-m \pm i)x, \quad i = 1, 2, ..., r, \\ f_{(2r+1)}(x) &= \big((2r+1)m+1\big)x, \\ f_{(2r+2)}(x) &= -mx, \\ L_i(x) &= 1, \quad i = 1, 2, ..., (2r+2). \end{aligned}$$

Moreover, for every $\xi, \mu \in Y^{X\setminus\{0\}}$

$$\begin{aligned} \Big\| \mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x) \Big\| &= \Big\| \xi\Big(\big((2r+1)m+1\big)x\Big) + \xi(-mx) \\ &+ \sum_{\ell=1}^{r} \xi\big((-m+\ell)x\big) + \sum_{\ell=1}^{r} \xi\big((-m-\ell)x\big) \\ &- \mu\Big(\big((2r+1)m+1\big)x\Big) - \mu(-mx) \\ &- \sum_{\ell=1}^{r} \mu\big((-m+\ell)x\big) - \sum_{\ell=1}^{r} \mu\big((-m-\ell)x\big) \Big\| \\ &\le \Big\| (\xi-\mu)\Big(\big((2r+1)m+1\big)x\Big) \Big\| + \|(\xi-\mu)(-mx)\| \\ &+ \sum_{\ell=1}^{r} \big\|(\xi-\mu)\big((-m+\ell)x\big)\big\| + \sum_{\ell=1}^{r} \big\|(\xi-\mu)\big((-m-\ell)x)\big)\big\| \\ &= \sum_{i=1}^{(2r+2)} L_i(x) \left\| \xi\big(f_i(x)\big) - \mu\big(f_i(x)\big) \right\|, \end{aligned}$$

and so **(H2)** is valid. Next, we can find $m_0 \in \mathbb{N}$ such that

$$\alpha_m = \Big((2r+1)m+1\Big)^p + m^p + \sum_{\ell=1}^{r} \big|\ell - m\big|^p + \sum_{\ell=1}^{r} \Big(m+\ell\Big)^p < 1$$

for all $m \ge m_0$. Therefore, we have

$$\begin{aligned} \varepsilon_m^*(x) &:= \sum_{s=0}^{\infty} \Lambda_m^s \varepsilon_m(x) \\ &= c\,\alpha_m \sum_{s=0}^{\infty} \alpha_m^s \|x\|^p \\ &= \frac{c\,\alpha_m}{1-\alpha_m}\|x\|^p, \qquad x \in X\setminus\{0\}, m \ge m \ge m_0. \end{aligned}$$

Thus, according to Theorem 1.1, for each $m \ge m_0$ there exists a unique solution $F_m : X\setminus\{0\} \longrightarrow Y$ of the equation

$$F_m(x) = F_m\Big(\big((2r+1)m+1\big)x\Big) + F_m(-mx) + \sum_{\ell=1}^{r} F_m\big((-m+\ell)x\big) + \sum_{\ell=1}^{r} F_m\big((-m-\ell)x\big)$$

such that

$$\|f(x) - F_m(x)\| \le \frac{c\,\alpha_m}{1-\alpha_m}\|x\|^p, \qquad x \in X\setminus\{0\}, m \ge m \ge m_0.$$

Moreover,

$$F_m(x) := \lim_{s\to\infty} \mathcal{T}_m^s f(x), \qquad x \in X\setminus\{0\}.$$

To prove that $F_m(x)$ satisfies the Cauchy equation (1.1) on $X\setminus\{0\}$ observe that

$$\left\| \mathcal{T}_m^s f\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n \mathcal{T}_m^s f(x_i) \right\| \leq c\, \alpha_m^s \left(\sum_{i=1}^n \|x_i\|^p\right) \tag{2.4}$$

for every $x_1, x_2, ..., x_n \in X \setminus \{0\}$ and $s \in \mathbb{N}_0$.
Indeed, if $s = 0$, then (2.4) is simply (2.1). So, take $t \in \mathbb{N}_0$ and suppose that (2.4) holds for $s = t$ and $x_1, x_2, ..., x_n \in X \setminus \{0\}$. Then

$$\begin{aligned}
&\left\| \mathcal{T}_m^{t+1} f\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n \mathcal{T}_m^{t+1} f(x_i) \right\| = \Bigg\| \mathcal{T}_m^t f\left(\sum_{i=1}^n \Big((2r+1)m+1\Big)x_i\right) \\
&+\mathcal{T}_m^t f\left(\sum_{i=1}^n (-mx_i)\right) + \sum_{\ell=1}^r \mathcal{T}_m^t f\left(\sum_{i=1}^n \Big(-m+\ell\Big)x_i\right) + \sum_{\ell=1}^r \mathcal{T}_m^t f\left(\sum_{i=1}^n \Big(-m-\ell\Big)x_i\right) \\
&- \sum_{i=1}^n \mathcal{T}_m^t f\Big(\big((2r+1)m+1\big)x_i\Big) - \sum_{i=1}^n \mathcal{T}_m^t f(-mx_i) \\
&- \sum_{\ell=1}^r \left(\sum_{i=1}^n \mathcal{T}_m^t f\Big((-m+\ell)x_i\Big)\right) - \sum_{\ell=1}^r \left(\sum_{i=1}^n \mathcal{T}_m^t f\Big(\big(-m-\ell\big)x_i\Big)\right) \Bigg\| \\
&\leq \left\| \mathcal{T}_m^t f\left(\sum_{i=1}^n \Big((2r+1)m+1\Big)x_i\right) - \sum_{i=1}^n \mathcal{T}_m^t f\Big(\big((2r+1)m+1\big)x_i\Big) \right\| \\
&+ \left\| \mathcal{T}_m^t f\left(\sum_{i=1}^n (-mx_i)\right) - \sum_{i=1}^n \mathcal{T}_m^t f(-mx_i) \right\| \\
&+ \left\| \sum_{\ell=1}^r \mathcal{T}_m^t f\left(\sum_{i=1}^n (-m+\ell)x_i\right) - \sum_{\ell=1}^r \left(\sum_{i=1}^n \mathcal{T}_m^t f\Big((-m+\ell)x_i\Big)\right) \right\| \\
&+ \left\| \sum_{\ell=1}^r \mathcal{T}_m^t f\left(\sum_{i=1}^n (-m-\ell)x_i\right) - \sum_{\ell=1}^r \left(\sum_{i=1}^n \mathcal{T}_m^t f\Big((-m-\ell)x_i\Big)\right) \right\| \\
&\leq c\, \alpha_m^t \left(\big((2r+1)m+1\big)^p + m^p + \sum_{\ell=1}^r \big|\ell - m\big|^p + \sum_{\ell=1}^r \big|-m-\ell\big|^p\right) \sum_{i=1}^n \|x_i\|^p \\
&= c\, \alpha_m^{t+1} \sum_{i=1}^n \|x_i\|^p.
\end{aligned}$$

By induction, we have shown that (2.4) holds for all $x_1, x_2, ..., x_n \in X \setminus \{0\}$ and $s \in \mathbb{N}_0$. Letting $s \longrightarrow \infty$ in (2.4), we obtain that

$$F_m\left(\sum_{i=1}^n x_i\right) = \sum_{i=1}^n F_m(x_i), \qquad x_1, x_2, ..., x_n \in X \setminus \{0\}.$$

Thus, we have proved that for every $m \geq m_0$ there exists a unique function $F_m : X \setminus \{0\} \longrightarrow Y$ such that F_m is a solution of the Cauchy equation (1.1) on $X \setminus \{0\}$ and

$$\|f(x) - F_m(x)\| \leq \frac{c\, \alpha_m}{1 - \alpha_m} \|x\|^p, \qquad x \in X \setminus \{0\}.$$

Since $p < 0$, the sequence

$$\left\{ \frac{c\, \alpha_m}{1 - \alpha_m} \|x\|^p \right\}_{m \geq m_0}$$

tends to zero when $m \longrightarrow \infty$. Consequently, f satisfies the Cauchy equation (1.1) on $X \setminus \{0\}$ as the pointwise of $(F_m)_{m \geq m_0}$.

Case 2: n is odd

Letting $n = 2r+1$ where $r \in \mathbb{N}$, we can rewrite the inequality (2.1) as follows

$$\left\| f\left(\sum_{i=1}^{2r+1} x_i\right) - \sum_{i=1}^{2r+1} f(x_i) \right\| \leq c\left(\sum_{i=1}^{2r+1} \|x_i\|^p\right). \tag{2.5}$$

Replacing x_{2r+1} by $(2rm+1)x$, x_i by $(-m - \frac{i}{2})x$ where $i = 2, 4, ..., 2r$ and x_j by $(-m + \frac{j+1}{2})$ where $j = 1, 3, ..., (2r-1)$ and $m \in \mathbb{N}$ in (2.5), we get that

$$\begin{aligned} &\left\| f(x) - f\Big((2rm+1)x\Big) - \sum_{\ell=1}^{r} f\Big((-m+\ell)x\Big) - \sum_{\ell=1}^{r} f\Big((-m-\ell)x\Big) \right\| \\ &\qquad \leq c\left(\big|2rm+1\big|^p + \sum_{\ell=1}^{r} \big|-m+\ell\big|^p + \sum_{\ell=1}^{r} \big|m+\ell\big|^p \right) \|x\|^p \end{aligned} \tag{2.6}$$

for all $x \in X \setminus \{0\}$.

Further put

$$\mathcal{T}_m\xi(x) := \xi\Big((2rm+1)x\Big) + \sum_{\ell=1}^{r} \xi\Big((-m+\ell)x\Big) + \sum_{\ell=1}^{r} \xi\Big((-m-\ell)x\Big)$$

and

$$\varepsilon_m(x) := c\left(\big|2rm+1\big|^p + \sum_{\ell=1}^{r} \big|-m+\ell\big|^p + \sum_{\ell=1}^{r} \big|m+\ell\big|^p \right) \|x\|^p$$

for all $x \in X \setminus \{0\}$ and all $\xi \in Y^{X\setminus\{0\}}$. Then the inequality (2.6) takes the form

$$\|\mathcal{T}_m f(x) - f(x)\| \leq \varepsilon_m(x), \qquad x \in X \setminus \{0\}.$$

The following operator

$$\Lambda_m\delta(x) := \delta\Big((2rm+1)x\Big) + \sum_{\ell=1}^{r} \delta\Big((-m+\ell)x\Big) + \sum_{\ell=1}^{r} \delta\Big((-m-\ell)x\Big)$$

for all $x \in X \setminus \{0\}$ and all $\delta \in \mathbb{R}_+^{X\setminus\{0\}}$, has the form described in (**H3**) with $k = 2r+1$, and

$$\begin{aligned} f_i(x) &= (-m \pm i)x, \qquad i = 1, 2, ..., r, \\ f_{2r+1}(x) &= (2rm+1)x, \\ L_i(x) &= 1, \qquad i = 1, 2, ..., (2r+1). \end{aligned}$$

Moreover, for every $\xi, \mu \in Y^{X\setminus\{0\}}$

$$\begin{aligned} \big\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\big\| &= \Bigg\| \xi\big((2rm+1)x\big) + \sum_{\ell=1}^{r} \xi\big((-m+\ell)x\big) + \sum_{\ell=1}^{r} \xi\big((-m-\ell)x\big) \\ &\qquad -\mu\big((2rm+1)x\big) - \sum_{\ell=1}^{r} \mu\big((-m+\ell)x\big) - \sum_{\ell=1}^{r} \mu\big((-m-\ell)x\big) \Bigg\| \\ &\leq \Big\|(\xi-\mu)\big((2rm+1)x\big)\Big\| + \sum_{\ell=1}^{r} \Big\|(\xi-\mu)\big((-m+\ell)x\big)\Big\| + \sum_{\ell=1}^{r} \Big\|(\xi-\mu)\Big((-m-\ell)x\Big)\Big\| \end{aligned}$$

$$= \sum_{i=1}^{(2r+1)} L_i(x)\Big\|\xi\big(f_i(x)\big) - \mu\big(f_i(x)\big)\Big\|,$$

so (**H2**) is valid. Now,we can find $m_0 \in \mathbb{N}$ such that

$$\alpha_m = \big|2rm+1\big|^p + \sum_{\ell=1}^{r} \big| -m+\ell\big|^p + \sum_{\ell=1}^{r} \big|m+\ell\big|^p < 1$$

for all $m \geq m_0$. Therefore, we have

$$\begin{aligned}
\varepsilon_m^*(x) \quad &:= \quad \sum_{s=0}^{\infty} \Lambda_m^s \varepsilon_m(x) \\
&= \quad c\ \alpha_m \sum_{s=0}^{\infty} \alpha_m^s \|x\|^p \\
&= \quad \frac{c\ \alpha_m}{1-\alpha_m}\|x\|^p, \qquad x \in X\setminus\{0\}, m \geq m_0.
\end{aligned}$$

The rest of the proof is similar to the proof of case 1. □

Corollary 2.2. *Let X be a normed space, Y be a Banach space, $c \geq 0$, $p < 0$ and let $f : X \longrightarrow Y$ satisfy*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\big(\|x\|^p + \|y\|^p\big) \tag{2.7}$$

for all $x, y \in X \setminus \{0\}$ where $n \in \mathbb{N}_0$. Then f is additive on $X \setminus \{0\}$.

Theorem 2.3. *Let X be a normed space, Y be a Banach space, $c \geq 0$, $p_i \in \mathbb{R}$ with $\sum_{i=1}^{n} p_i < 0$ and let $f : X \longrightarrow Y$ satisfy*

$$\left\| f\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_i)\right\| \leq c\left(\prod_{i=1}^{n} \|x_i\|^{p_i}\right) \tag{2.8}$$

for all $x_1, x_2, ..., x_n \in X \setminus \{0\}$ where $n \in \mathbb{N}_0$. Then f is additive on $X \setminus \{0\}$.

Proof. Since $\sum_{i=1}^{n} p_i < 0$, some of p_i must be negative. Assume that these are $p_j < 0$ where $1 \leq j \leq n$. By using the same technic of the proof of Theorem 2.1, we study two cases as follows:

Case 1: n is even

In this case, suppose that $n = 2r+2$ where $r \in \mathbb{N}$. Then the inequality (2.8) can be written as follows

$$\left\| f\left(\sum_{i=1}^{2r+2} x_i\right) - \sum_{i=1}^{2r+2} f(x_i)\right\| \leq c\left(\prod_{i=1}^{2r+2} \|x_i\|^{p_i}\right). \tag{2.9}$$

Replacing $x_{(2r+2)}$ by $\left((2r+1)m+1\right)x$, x_i by $\left(-m-\frac{i}{2}\right)x$ where $i = 2, 4, ..., 2r$ and x_j by $\left(-m+\frac{j-1}{2}\right)x$ where $j = 1, 3, ..., (2r+1)$ and $m \in \mathbb{N}$ in (2.9), we obtain that

$$\left\| f(x) - f\Big(\big((2r+1)m+1\big)x\Big) - f(-mx) - \sum_{\ell=1}^{r} f\Big((-m+\ell)x\Big) - \sum_{\ell=1}^{r} f\Big((-m-\ell)x\Big)\right\|$$

$$\leq c|m|^{p_1}\cdot\big|(2r+1)m+1\big|^{p_{2r+2}}\cdot\prod_{\ell=1}^{r}\Big(\big|m+\ell\big|^{p_{2\ell}}\cdot\big|-m+\ell\big|^{p_{2\ell+1}}\Big)\|x\|^{\beta} \tag{2.10}$$

for all $x \in X \setminus \{0\}$ where $\beta = \sum_{i=1}^{n} p_i$.

Further put

$$\mathcal{T}_m\xi(x) := \xi\Big(\big((2r+1)m+1\big)x\Big) + \xi(-mx) + \sum_{\ell=1}^{r}\xi\Big((-m+\ell)x\Big) + \sum_{\ell=1}^{r}\xi\Big((-m-\ell)x\Big)$$

and

$$\varepsilon_m(x) := \theta|m|^{p_1}\cdot\big|(2r+1)m+1\big|^{p_{2r+2}}\cdot\prod_{\ell=1}^{r}\Big(\big|m+\ell\big|^{p_{2\ell}}\cdot\big|-m+\ell\big|^{p_{2\ell+1}}\Big)\|x\|^{\beta}$$

for all $x \in X\setminus\{0\}$ and all $\xi \in Y^X$. Thus, the inequality (2.10) takes the following form

$$\|\mathcal{T}_m f(x) - f(x)\| \leq \varepsilon_m(x), \qquad x \in X\setminus\{0\}.$$

The following operator

$$\Lambda_m\delta(x) := \delta\Big(\big((2r+1)m+1\big)x\Big) + \delta(-mx) + \sum_{\ell=1}^{r}\delta\big((-m+\ell)x\big) + \sum_{\ell=1}^{r}\delta\big((-m-\ell)x\big)$$

for all $x \in X\setminus\{0\}$ and all $\delta \in \mathbb{R}_+^{X\setminus\{0\}}$, has the form described in (**H3**) with $k = 2r+2$ and

$$\begin{aligned} f_i(x) &= (-m\pm i)x, \qquad i = 1,2,...,r,\\ f_{(2r+1)}(x) &= \big((2r+1)m+1\big)x,\\ f_{(2r+2)}(x) &= -mx,\\ L_i(x) &= 1, \qquad i = 1,2,...,(2r+2). \end{aligned}$$

Moreover, for every $\xi,\mu \in Y^{X\setminus\{0\}}$

$$\begin{aligned} \Big\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\Big\| &= \Bigg\|\xi\Big(\big((2r+1)m+1\big)x\Big) + \xi(-mx) + \sum_{\ell=1}^{r}\xi\big((-m+\ell)x\big)\\ &+ \sum_{\ell=1}^{r}\xi\big((-m-\ell)x\big) - \mu\Big(\big((2r+1)m+1\big)x\Big) - \mu(-mx)\\ &- \sum_{\ell=1}^{r}\mu\big((-m+\ell)x\big) - \sum_{\ell=1}^{r}\mu\big((-m-\ell)x\big)\Bigg\|\\ &\leq \Big\|(\xi-\mu)\Big(\big((2r+1)m+1\big)x\Big)\Big\| + \big\|(\xi-\mu)(-mx)\big\|\\ &+ \sum_{\ell=1}^{r}\big\|(\xi-\mu)\big((-m+\ell)x\big)\big\| + \sum_{\ell=1}^{r}\big\|(\xi-\mu)\big((-m-\ell)x\big)\big\|\\ &= \sum_{i=1}^{(2r+2)} L_i(x)\Big\|\xi\Big(f_i(x)\Big) - \mu\Big(f_i(x)\Big)\Big\|, \end{aligned}$$

and so (**H2**) is valid. Next, we can find $m_0 \in \mathbb{N}$ such that

$$\lambda_m = |m|^{p_1}\cdot\big|(2r+1)m+1\big|^{p_{2r+2}}\cdot\prod_{\ell=1}^{r}\Big(\big|m+\ell\big|^{p_{2\ell}}\cdot\big|-m+\ell\big|^{p_{2\ell+1}}\Big) < 1$$

for all $m \geq m_0$. Therefore, we have

$$\varepsilon_m^*(x) \;:=\; \sum_{s=0}^{\infty}\Lambda_m^s\varepsilon_m(x)$$

$$= \frac{\theta\,\lambda_m}{1-\lambda_m}\|x\|^{\beta},$$

for all $x \in X \setminus \{0\}$ where $m \geq m_0$.

The rest of the proof is similar to the proof of Theorem 2.1.

Case 2: n is odd

Let $n = 2r+1$ where $r \in \mathbb{N}$. Then, we can rewrite the inequality (2.8) as follows

$$\left\| f\left(\sum_{i=1}^{2r+1} x_i\right) - \sum_{i=1}^{2r+1} f(x_i)\right\| \leq c\left(\prod_{i=1}^{2r+1}\|x_i\|^{p}\right). \tag{2.11}$$

Replacing x_{2r+1} by $(2rm+1)x$, x_i by $(-m-\frac{i}{2})x$ where $i = 2,4,...,2r$ and x_j by $(-m+\frac{j+1}{2})x$ where $j = 1,3,...,(2r-1)$ and $m \in \mathbb{N}$ in (2.13), we get that

$$\left\| f(x) - f\big((2rm+1)x\big) - \sum_{\ell=1}^{r} f\big((-m+\ell)x\big) - \sum_{\ell=1}^{r} f\big((-m-\ell)x\big)\right\|$$

$$\leq c\,\big|2rm+1\big|^{p_{2r+1}} \cdot \prod_{\ell=1}^{r}\Big(\big|m+\ell\big|^{p_{2\ell}} \cdot \big|-m+\ell\big|^{p_{2\ell+1}}\Big)\|x\|^{\beta} \tag{2.12}$$

where $\sum_{i=1}^{2r+1} p_i = \beta$ for all $x \in X \setminus \{0\}$.

Further put

$$\mathcal{T}_m\xi(x) := \xi\big((2rm+1)x\big) + \sum_{\ell=1}^{r}\xi\big((-m+\ell)x\big) + \sum_{\ell=1}^{r}\xi\big((-m-\ell)x\big)$$

and

$$\varepsilon_m(x) := \theta\,\big|2rm+1\big|^{p_{2r+1}} \cdot \prod_{\ell=1}^{r}\Big(\big|m+\ell\big|^{p_{2\ell}} \cdot \big|-m+\ell\big|^{p_{2\ell+1}}\Big)\|x\|^{\beta}$$

for all $x \in X \setminus \{0\}$ and all $\xi \in Y^{X\setminus\{0\}}$. Then the inequality (2.12) takes the form

$$\|\mathcal{T}_m f(x) - f(x)\| \leq \varepsilon_m(x), \qquad x \in X \setminus \{0\}.$$

The following operator

$$\Lambda_m\delta(x) := \delta\big((2rm+1)x\big) + \sum_{\ell=1}^{r}\delta\big((-m+\ell)x\big) + \sum_{\ell=1}^{r}\delta\big((-m-\ell)x\big)$$

for all $x \in X\setminus\{0\}$ and all $\delta \in \mathbb{R}_+^{X\setminus\{0\}}$, has the form described in (**H3**) with $k = 2r+1$, and

$$\begin{aligned} f_i(x) &= (-m \pm i)x, \qquad i = 1,2,...,r,\\ f_{2r+1}(x) &= (2rm+1)x,\\ L_i(x) &= 1, \qquad i = 1,2,...,(2r+1). \end{aligned}$$

Moreover, for every $\xi, \mu \in Y^{X\setminus\{0\}}$

$$\begin{aligned} \big\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\big\| &= \Bigg\|\xi\Big((2rm+1)x\Big) + \sum_{\ell=1}^{r}\xi\Big((-m+\ell)x\Big) + \sum_{\ell=1}^{r}\xi\Big((-m-\ell)x\Big)\\ &\quad -\mu\Big((2rm+1)x\Big) - \sum_{\ell=1}^{r}\mu\Big((-m+\ell)x\Big) \end{aligned}$$

$$
\begin{aligned}
& -\sum_{\ell=1}^{r} \mu\Big((-m-\ell)x\Big)\Big\| \\
& \leq \Big\|(\xi-\mu)\Big((2rm+1)x\Big)\Big\| + \sum_{\ell=1}^{r} \Big\|(\xi-\mu)\Big((-m+\ell)x\Big)\Big\| \\
& + \sum_{\ell=1}^{r} \Big\|(\xi-\mu)\Big((-m-\ell)x\Big)\Big\| \\
& = \sum_{i=1}^{(2r+1)} L_i(x) \Big\|\xi\Big(f_i(x)\Big) - \mu\Big(f_i(x)\Big)\Big\|,
\end{aligned}
$$

and so (**H2**) is valid. Now,we can find $m_0 \in \mathbb{N}$ such that

$$\lambda_m = \big|2rm+1\big|^{p_{2r+1}} \cdot \prod_{\ell=1}^{r} \Big(\big|m+\ell\big|^{p_{2\ell}} \cdot \big|-m+\ell\big|^{p_{2\ell+1}}\Big) < 1$$

for all $m \geq m_0$. Therefore, we have

$$
\begin{aligned}
\varepsilon_m^*(x) &:= \sum_{s=0}^{\infty} \Lambda_m^s \varepsilon_m(x) \\
&= c\left(\big|2rm+1\big|^{p_{2r+1}} \cdot \prod_{\ell=1}^{r} \Big(\big|m+\ell\big|^{p_{2\ell}} \cdot \big|-m+\ell\big|^{p_{2\ell+1}}\Big)\right) \sum_{s=0}^{\infty} \Lambda_m^s \|x\|^{\beta} \\
&= \frac{c\,\lambda_m}{1-\lambda_m}\|x\|^{\beta}, \qquad x \in X \setminus \{0\}, m \geq m_0.
\end{aligned}
$$

The rest of the proof is similar to the proof of case 1. □

Corollary 2.4. *Let X be a normed space, Y be a Banach space, $c \geq 0$, $p, q \in \mathbb{R}$, $p+q < 0$ and let $f : X \longrightarrow Y$ satisfy*

$$\|f(x+y) - f(x) - f(y)\| \leq c\big(\|x\|^p \cdot \|y\|^q\big) \tag{2.13}$$

for all $x, y \in X \setminus \{0\}$ where $n \in \mathbb{N}_0$. Then f is additive on $X \setminus \{0\}$.

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