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SOME FIXED POINT RESULTS FOR $\acute{C}$IRI$\acute{C}$ OPERATORS IN b-METRIC SPACES

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ABSTRACT. Fixed points for $\acute{C}$irić operators satisfying some generalized contractive type conditions are obtained in the setting of b-metric spaces. Moreover, we investigate that these operators satisfy property P. Finally, strength of hypothesis made in main theorem has been weighed through an illustrative example.

KEYWORDS : b-metric; $\acute{C}$irić operator; property P; fixed point.
AMS Subject Classification: 54H25, 47H10

1. INTRODUCTION

Fixed point theory plays an important role in applications of many branches of mathematics and applied sciences. The study of metric fixed point theory has been at the centre of vigorous research activity. There has been a number of generalizations of the usual notion of a metric space(see [2, 5, 12, 16, 17, 18, 21, 22, 23]). One such generalization is a b-metric space introduced and studied by Bakhtin[4] and Czerwik [10]. After that a series of articles have been dedicated to the improvement of fixed point theory for single valued and multivalued operators in b-metric spaces and cone b-metric spaces(see [3, 6, 7, 9, 11, 13, 14, 19, 20, 24, 25, 26, 27]). In this paper, we introduce the concept of $\acute{C}$irić operators in b-metric spaces. In fact, a $\acute{C}$irić operator in b-metric spaces may not possess a fixed point and attempts have been in progress to frame appropriate conditions to be satisfied by a $\acute{C}$irić operator to attract a fixed point. The aim of this work is to invite some kind of contractive conditions to be satisfied by the operator just appropriate to possess its fixed point.

Let X be a nonempty set and $T : X \longrightarrow X$ be an operator with nonempty fixed point set $F(T)$. Then T is said to satisfy property P if $F(T) = F(T^n)$ for each

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$n \in \mathbb{N}$. We prove that Ćirić operators in b-metric spaces satisfy property P.

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2. PRELIMINARIES

In this section we need to recall some basic notations, definitions, and necessary results from existing literature.

Definition 2.1. [10] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:

(i) $d(x,y) = 0$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(iii) $d(x,y) \leq s\,(d(x,z) + d(z,y))$ for all $x, y, z \in X$.

The pair (X,d) is called a b-metric space.

Observe that if $s = 1$, then the ordinary triangle inequality in a metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of b-metric spaces is effectively larger than that of the ordinary metric spaces. That is, every metric space is a b-metric space, but the converse need not be true. The following examples illustrate the above remarks.

Example 2.2. Let $X = \{-1, 0, 1\}$. Define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) = d(y,x)$ for all $x, y \in X$, $d(x,x) = 0$, $x \in X$ and $d(-1,0) = 3$, $d(-1,1) = d(0,1) = 1$. Then (X,d) is a b-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that

$$d(-1,1) + d(1,0) = 1 + 1 = 2 < 3 = d(-1,0).$$

It is easy to verify that $s = \frac{3}{2}$.

Example 2.3. [15] Let $X = \mathbb{R}$ and $d : X \times X \to \mathbb{R}^+$ be such that

$$d(x,y) = \mid x - y \mid^2 \;\; for\ any\ x, y \in X.$$

Then (X,d) is a b-metric space with $s = 2$, but not a metric space.

Definition 2.4. [8] Let (X,d) be a b-metric space, $x \in X$ and (x_n) be a sequence in X. Then

(i) (x_n) converges to x if and only if $\lim\limits_{n\to\infty} d(x_n, x) = 0$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$.
(ii) (x_n) is Cauchy if and only if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.
(iii) (X,d) is complete if and only if every Cauchy sequence in X is convergent.

Definition 2.5. Let (X,d) be a b-metric space and let $T : X \to X$ be a given mapping. We say that T is continuous at $x_0 \in X$ if for every sequence (x_n) in X, we have $x_n \to x_0$ as $n \to \infty \Longrightarrow T(x_n) \to T(x_0)$ as $n \to \infty$. If T is continuous at each point $x_0 \in X$, then we say that T is continuous on X.

Theorem 2.6. [1]*Let (X,d) be a b-metric space and suppose that (x_n) and (y_n) converge to $x, y \in X$, respectively. Then, we have*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$.

Moreover, for each $z \in X$, we have

$$\frac{1}{s}d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq sd(x,z).$$

3. MAIN RESULTS

In this section, we will present some fixed point theorems for Ćirić operators satisfying some contractive type conditions in the setting of b-metric spaces. Furthermore, we will give an example to examine the strength of hypothesis made in main theorem.

We begin with the following definition.

Definition 3.1. Let (X,d) be a b-metric space with coefficient $s \geq 1$. An operator $T : X \to X$ is said to be a Ćirić operator if there are non-negative real valued functions q and δ over $X \times X$ satisfying

$$d(T^n(x),T^n(y)) \leq q^n(x,y)\delta(x,y),\ n = 1,2,\cdots,$$

for all $x,y \in X$, where $q(x,y) < 1$ with $\sup\limits_{x,y\in X} q(x,y) = 1$.

Let $CI(X)$ denote the set of all Ćirić operators on X.

Theorem 3.2. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$, and $T \in CI(X)$ be such that*

$$d(T(x),T(y)) \leq \alpha\, d(x,T(x)) + \beta\, max \left\{ \begin{array}{l} d(x,y) + d(y,T(y)), \\ \\ d(x,T(y)),\ d(y,T(x)) \end{array} \right\} \tag{3.1}$$

for all $x,y \in X$, where $\alpha \geq 0$ and $0 \leq \beta < \frac{1}{s^2}$. Then T has a unique fixed point (say u) in X and T is continuous at u. Moreover, T has property P.

Proof. Let $x_0 \in X$ be arbitrary and define sequence (x_n) by $x_n = T^n(x_0)$. Then for all $n,m \in \mathbb{N}$, we have by using (3.1) that

$$\begin{aligned} d(x_n,x_m) &\leq \alpha\, d(x_{n-1},x_n) + \beta\, max \left\{ \begin{array}{l} d(x_{n-1},x_{m-1}) + d(x_{m-1},x_m), \\ \\ d(x_{n-1},x_m),\ d(x_{m-1},x_n) \end{array} \right\} \\ &\leq \alpha\, d(x_{n-1},x_n) + \beta\, max \left\{ \begin{array}{l} sd(x_{n-1},x_n) + s^2 d(x_n,x_m) \\ \\ +s^2 d(x_m,x_{m-1}) + d(x_{m-1},x_m), \\ \\ sd(x_{n-1},x_n) + sd(x_n,x_m), \\ \\ sd(x_{m-1},x_m) + sd(x_m,x_n) \end{array} \right\} \\ &= \alpha\, d(x_{n-1},x_n) \\ &\quad +\beta \left\{ sd(x_{n-1},x_n) + s^2 d(x_n,x_m) + (1+s^2)d(x_{m-1},x_m) \right\}. \end{aligned}$$

Since $0 \leq \beta < \frac{1}{s^2}$, we have

$$\begin{aligned} d(x_n,x_m) &\leq \frac{\alpha+\beta s}{1-\beta s^2}\, d(x_{n-1},x_n) + \frac{\beta(1+s^2)}{1-\beta s^2}\, d(x_{m-1},x_m) \\ &\leq \frac{\alpha+\beta s}{1-\beta s^2}\, q^{n-1}(x_0,T(x_0))\, \delta(x_0,T(x_0)) \end{aligned}$$

$$+\frac{\beta(1+s^2)}{1-\beta s^2}\,q^{m-1}(x_0,T(x_0))\,\delta(x_0,T(x_0))$$
$$\longrightarrow 0 \;as\; m,n\longrightarrow\infty.$$

Therefore, (x_n) becomes a Cauchy sequence in (X,d). By completeness of (X,d), it follows that the sequence (x_n) is convergent. So, there exists $u\in X$ such that $\lim_{n\to\infty} d(x_n,u)=0$. Then

$$\begin{aligned}
d(u,T(u)) &\leq s[d(u,x_n)+d(x_n,T(u))]\\
&\leq sd(u,x_n)+s\alpha\, d(x_{n-1},x_n)\\
&\quad +s\beta\, max\left\{\begin{array}{l} d(x_{n-1},u)+d(u,T(u)),\\ \\ d(x_{n-1},T(u)),\ d(u,x_n)\end{array}\right\}\\
&\leq sd(u,x_n)+s\alpha\, q^{n-1}(x_0,T(x_0))\,\delta(x_0,T(x_0))\\
&\quad +s\beta\, max\left\{\begin{array}{l} d(x_{n-1},u)+d(u,T(u)),\\ \\ sd(x_{n-1},u)+sd(u,T(u)),\ d(u,x_n)\end{array}\right\}.
\end{aligned}$$

Taking the limit as $n\to\infty$, it follows from above that

$$d(u,T(u))\leq \beta s^2 d(u,T(u)).$$

Since $0\leq\beta<\frac{1}{s^2}$, we have $d(u,T(u))=0$ and so, $u=T(u)$.

For uniqueness of u, suppose that $T(v)=v$ for some $v\in X$. Then

$$\begin{aligned}
d(u,v)=d(T^n(u),T^n(v)) &\leq q^n(u,v)\,\delta(u,v)\\
&\longrightarrow 0,\ \ as\ n\to\infty.
\end{aligned}$$

This gives that $u=v$. Therefore, T has a unique fixed point u in X.

To show that T is continuous at u, let (y_n) be any sequence in X such that (y_n) is convergent to u. For $n\in\mathbb{N}$, we have

$$\begin{aligned}
d(u,T(y_n)) &= d(T(u),T(y_n))\\
&\leq \alpha\, d(u,u)+\beta\, max\left\{\begin{array}{l} d(u,y_n)+d(y_n,T(y_n)),\\ \\ d(u,T(y_n)),\ d(y_n,T(u))\end{array}\right\}\\
&\leq \beta\,[d(u,y_n)+sd(y_n,u)+sd(u,T(y_n))],
\end{aligned}$$

which implies that

$$d(u,T(y_n))\leq\frac{(1+s)\beta}{1-\beta s}d(u,y_n).$$

Taking the limit as $n\to\infty$, we see that $d(u,y_n)\to 0$ and so, $(T(y_n))$ is convergent to $u=T(u)$. Therefore T is continuous at u.

To show that T has property P, let $u\in F(T)$. Then, $u\in F(T^n)$ for each $n\in\mathbb{N}$. Therefore, $F(T)\subseteq F(T^n)$ for each $n\in\mathbb{N}$.

Since T has a fixed point, $F(T^n)\neq\emptyset$ for each $n\in\mathbb{N}$. Let $n>1$ be fixed and assume that $p\in F(T^n)$. Using (3.1), we have

$$\begin{aligned}
d(p,T(p)) &= d(T^n(p),T^{n+1}(p))\\
&\leq \alpha\, d(T^{n-1}(p),T^n(p))
\end{aligned}$$

$$+\beta\, max \left\{ \begin{array}{l} d(T^{n-1}(p), T^n(p)) + d(T^n(p), T^{n+1}(p)), \\ \\ d(T^{n-1}(p), T^{n+1}(p)),\; d(T^n(p), T^n(p)) \end{array} \right\}$$
$$\leq \quad \alpha\, d(T^{n-1}(p), T^n(p)) + \beta s \left\{ d(T^{n-1}(p), T^n(p)) + d(T^n(p), T^{n+1}(p)) \right\}.$$

Thus, it must be the case that

$$\begin{aligned} d(p, T(p)) &\leq \frac{\alpha + \beta s}{1 - \beta s}\, d(T^{n-1}(p), T^n(p)) \\ &\leq \frac{\alpha + \beta s}{1 - \beta s}\, q^{n-1}(p, T(p))\, \delta(p, T(p)) \\ &\longrightarrow 0 \; as \; n \to \infty \end{aligned}$$

which gives that, $d(p, T(p)) = 0$ and so, $T(p) = p$. Therefore, $p \in F(T)$ and T has property P. □

Corollary 3.3. *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$. Suppose that for some $m \in \mathbb{N}$, $T^m \in CI(X)$ satisfying*

$$d(T^m(x), T^m(y)) \leq \alpha\, d(x, T^m(x)) + \beta\, max \left\{ \begin{array}{l} d(x, y) + d(y, T^m(y)), \\ \\ d(x, T^m(y)),\; d(y, T^m(x)) \end{array} \right\}$$

for all $x, y \in X$, where $\alpha \geq 0$ and $0 \leq \beta < \frac{1}{s^2}$. Then T has a unique fixed point (say u) in X and T^m is continuous at u. Moreover, T^m has property P.

Proof. It follows from Theorem 3.2 that T^m has a unique fixed point (say u) in X and T^m is continuous at u. Moreover, T^m has property P. We have

$$T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u)).$$

This shows that $T(u)$ is also a fixed point of T^m. By uniqueness of u, we get $T(u) = u$. □

Remark 3.4. It is worth mentioning that Theorem 3.2 is a generalization of Theorem 1[28]. In [28], the authors considered the following class of mappings in a metric space (X, d):

$$d(T(x), T(y)) \leq \alpha[d(x, T(x)) + d(y, T(y))] + \beta d(x, y) + \gamma\, max\{d(x, T(y)), d(y, T(x))\} \tag{3.2}$$

for all $x, y \in X$, where $\alpha,\ \beta,\ \gamma \geq 0$ with $max\{\alpha,\ \beta\} + \gamma < 1$.

The class of mappings satisfying condition (3.1) is strictly larger than the class satisfying condition (3.2).

Theorem 3.5. *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$, and $T \in CI(X)$ be such that*

$$d(T(x), T(y)) \leq \alpha\, d(x, T(x)) + \beta\, max \left\{ \begin{array}{l} d(x, y) + d(y, T(y)), \\ \\ d(x, T(y)) + d(y, T(x)) \end{array} \right\} \tag{3.3}$$

for all $x, y \in X$, where $\alpha \geq 0$ and $0 \leq \beta < \frac{1}{s(s+1)}$. Then T has a unique fixed point (say u) in X and T is continuous at u. Moreover, T has property P.

Proof. The proof is similar to the Theorem 3.2. □

Theorem 3.6. *Let (X,d) be a b-metric space with coefficient $s \geq 1$, and $T \in CI(X)$ be such that*

$$d(T(x),T(y)) \leq \alpha\, d(x,T(x)) + \beta\, max \left\{ \begin{array}{l} d(x,y)+d(y,T(y)), \\ \\ d(x,T(y))+d(y,T(x)) \end{array} \right\} \tag{3.4}$$

for all $x,y \in X$, where $\alpha \geq 0$ and $0 \leq \beta < \frac{1}{s^2}$. If $(T^n(x_0))$ for some $x_0 \in X$, has a subsequence $(T^{n_k}(x_0))$ with $\lim\limits_k T^{n_k}(x_0) = u \in X$, then u is the unique fixed point of T in X and $\lim\limits_n T^n(x_0) = u$. Moreover, T is continuous at u and T has property P.

Proof. Let $\lim\limits_k T^{n_k}(x_0) = u \in X$. Then

$$\begin{aligned} d(u,T(u)) &\leq s[d(u,T^{n_k+1}(x_0)) + d(T^{n_k+1}(x_0),T(u))] \\ &\leq sd(u,T^{n_k+1}(x_0)) + s\alpha\, d(T^{n_k}(x_0),T^{n_k+1}(x_0)) \\ &\quad + s\beta\, max \left\{ \begin{array}{l} d(T^{n_k}(x_0),u)+d(u,T(u)), \\ \\ d(T^{n_k}(x_0),T(u))+d(u,T^{n_k+1}(x_0)) \end{array} \right\} \\ &\leq s^2 d(u,T^{n_k}(x_0)) + s^2 d(T^{n_k}(x_0),T^{n_k+1}(x_0)) \\ &\quad + s\alpha\, d(T^{n_k}(x_0),T^{n_k+1}(x_0)) \\ &\quad + s\beta\, max \left\{ \begin{array}{l} d(T^{n_k}(x_0),u)+d(u,T(u)), \\ \\ sd(T^{n_k}(x_0),u)+sd(u,T(u)) \\ \\ +sd(u,T^{n_k}(x_0))+sd(T^{n_k}(x_0),T^{n_k+1}(x_0)) \end{array} \right\} \\ &\leq s^2 d(u,T^{n_k}(x_0)) + (s^2+s\alpha)\, q^{n_k}(x_0,T(x_0))\,\delta(x_0,T(x_0)) \\ &\quad + s\beta\{2sd(T^{n_k}(x_0),u) + sd(u,T(u)) + sq^{n_k}(x_0,T(x_0))\,\delta(x_0,T(x_0))\}. \end{aligned}$$

Taking the limit as $k \to \infty$, we have

$$d(u,T(u)) \leq \beta s^2\, d(u,T(u)).$$

Since $0 \leq \beta < \frac{1}{s^2}$, this implies that $d(u,T(u)) = 0$ and so, $u = T(u)$.

For uniqueness of u, suppose that $T(v) = v$ for some $v \in X$. Then

$$\begin{aligned} d(u,v) = d(T^n(u),T^n(v)) &\leq q^n(u,v)\,\delta(u,v) \\ &\longrightarrow 0,\ as\ n \to \infty. \end{aligned}$$

This gives that $u = v$. Therefore, T has a unique fixed point u in X.

Finally,

$$d(u,T^n(x_0)) = d(T^n(u),T^n(x_0)) \leq q^n(u,x_0)\,\delta(u,x_0).$$

Thus,

$$d(u,T^n(x_0)) \longrightarrow 0\ as\ n \to \infty.$$

Consequently, it follows that $\lim\limits_n T^n(x_0) = u$.

By an argument similar to that used in Theorem 3.2, we can show that T is continuous at u and T has property P.

□

Remark 3.7. Theorem 3.6 is a generalization of Theorem 2[28].

As an application of Theorem 3.6, we have the following Corollary.

Corollary 3.8. *Let (X, d) be a b-metric space with coefficient $s \geq 1$, and $T \in CI(X)$ be such that*

$$d(T(x), T(y)) \leq \alpha\, d(x, T(x)) + \beta\, max \left\{ \begin{array}{l} d(x, y) + d(y, T(y)), \\ \\ d(x, T(y)),\ d(y, T(x)) \end{array} \right\}$$

for all $x, y \in X$, where $\alpha \geq 0$ and $0 \leq \beta < \frac{1}{s^2}$. If $(T^n(x_0))$ for some $x_0 \in X$, has a subsequence $(T^{n_k}(x_0))$ with $\lim_k T^{n_k}(x_0) = u \in X$, then u is the unique fixed point of T in X and $\lim_n T^n(x_0) = u$. Moreover, T is continuous at u and T has property P.

As an application of Theorem 3.2, we have the following result.

Theorem 3.9. *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$, and $T_j \in CI(X)$ satisfy*

$$d(T_j(x), T_j(y)) \leq \alpha\, d(x, T_j(x)) + \beta\, max \left\{ \begin{array}{l} d(x, y) + d(y, T_j(y)), \\ \\ d(x, T_j(y)),\ d(y, T_j(x)) \end{array} \right\} \tag{3.5}$$

for all $x, y \in X$, where $\alpha \geq 0$ and $0 \leq \beta < \frac{1}{s^5}$ with fixed points u_j $(j = 1, 2, \cdots)$. Suppose that $T^n(x) = \lim_{j\to\infty} T_j^n(x)$ for all $x \in X (n = 1, 2, \cdots)$. Then T has the unique fixed point u in X if and only if $u = \lim_j u_j$. Moreover, T is continuous at u and T has property P.

Proof. Let $T^n(x) = \lim_{j\to\infty} T_j^n(x)$ for $x \in X (n = 1, 2, \cdots)$.

For a positive integer n,

$$\begin{aligned} d(T^n(x), T^n(y)) &\leq sd(T^n(x), T_j^n(x)) + s^2 d(T_j^n(x), T_j^n(y)) + s^2 d(T_j^n(y), T^n(y)) \\ &\leq sd(T^n(x), T_j^n(x)) + s^2 q^n(x, y)\, \delta(x, y) + s^2 d(T_j^n(y), T^n(y)). \end{aligned}$$

Taking the limit as $j \to \infty$, we have

$$d(T^n(x), T^n(y)) \leq q^n(x, y)\, \delta_1(x, y),\ \ n = 1, 2, \cdots,$$

for all $x, y \in X$, where $\delta_1(x, y) = s^2 \delta(x, y)$. Therefore, $T \in CI(X)$.
Now, using (3.5), we obtain

$$\begin{aligned} d(T(x), T(y)) &\leq sd(T(x), T_j(x)) + s^2 d(T_j(x), T_j(y)) + s^2 d(T_j(y), T(y)) \\ &\leq sd(T(x), T_j(x)) + s^2 d(T_j(y), T(y)) + s^2 \alpha\, d(x, T_j(x)) \\ &\quad + s^2 \beta\, max \left\{ \begin{array}{l} d(x, y) + d(y, T_j(y)), \\ \\ d(x, T_j(y)),\ d(y, T_j(x)) \end{array} \right\} \\ &\leq sd(T(x), T_j(x)) + s^2 d(T_j(y), T(y)) \\ &\quad + s^3 \alpha\, d(x, T(x)) + s^3 \alpha d(T(x), T_j(x)) \\ &\quad + s^2 \beta\, max \left\{ \begin{array}{l} d(x, y) + sd(y, T(y)) + sd(T(y), T_j(y)), \\ \\ sd(x, T(y)) + sd(T(y), T_j(y)), \\ \\ sd(y, T(x)) + sd(T(x), T_j(x)) \end{array} \right\}. \end{aligned}$$

Taking the limit as $j \to \infty$, we have

$$\begin{aligned} d(T(x),T(y)) &\leq s^3\alpha\, d(x,T(x)) + s^2\beta\, max\left\{\begin{array}{l} d(x,y)+sd(y,T(y)), \\ \\ sd(x,T(y)),\ sd(y,T(x)) \end{array}\right\} \\ &\leq s^3\alpha\, d(x,T(x)) + s^2\beta\, max\left\{\begin{array}{l} sd(x,y)+sd(y,T(y)), \\ \\ sd(x,T(y)),\ sd(y,T(x)) \end{array}\right\} \\ &\leq s^3\alpha\, d(x,T(x)) + s^3\beta\, max\left\{\begin{array}{l} d(x,y)+d(y,T(y)), \\ \\ d(x,T(y)),\ d(y,T(x)) \end{array}\right\} \end{aligned}$$

for all $x,y \in X$, where $s^3\alpha \geq 0$ and $0 \leq s^3\beta < \frac{1}{s^2}$. So by Theorem 3.2, T has a unique fixed point (say u) in X and T is continuous at u. Moreover, T has property P.

Now u is the unique fixed point of T and $u_j = T_j(u_j)$, $j = 1,2,\cdots$. Then, we have

$$\begin{aligned} d(u,u_j) &= d(T(u),T_j(u_j)) \\ &\leq sd(T(u),T_j(u)) + sd(T_j(u),T_j(u_j)) \\ &\leq sd(T(u),T_j(u)) + s\alpha\, d(u,T_j(u)) \\ &\quad + s\beta\, max\left\{\begin{array}{l} d(u,u_j)+d(u_j,T_j(u_j)), \\ \\ d(u,T_j(u_j)),\ d(u_j,T_j(u)) \end{array}\right\} \\ &\leq sd(T(u),T_j(u)) + s\alpha\, d(u,T_j(u)) \\ &\quad + s\beta\, max\,\{d(u,u_j),\ sd(u_j,u)+sd(u,T_j(u))\} \\ &\leq sd(T(u),T_j(u)) + s\alpha\, d(u,T_j(u)) \\ &\quad + s\beta\{sd(u_j,u)+sd(u,T_j(u))\}. \end{aligned}$$

This gives that

$$\begin{aligned} d(u,u_j) &\leq \frac{s(1+\alpha+s\beta)}{1-s^2\beta} d(T(u),T_j(u)) \\ &\longrightarrow 0\ as\ j \to \infty. \end{aligned}$$

Thus, $u = \lim_j u_j$.

Conversely, suppose that $u = \lim_j u_j$. Then

$$\begin{aligned} d(u,T(u)) &\leq sd(u,u_j) + s^2d(u_j,T_j(u)) + s^2d(T_j(u),T(u)) \\ &= sd(u,u_j) + s^2d(T_j(u),T(u)) + s^2d(T_j(u_j),T_j(u)) \\ &\leq sd(u,u_j) + s^2d(T_j(u),T(u)) + s^2\alpha d(u_j,T_j(u_j)) \\ &\quad + s^2\beta\, max\{d(u_j,u)+d(u,T_j(u)), d(u_j,T_j(u)), d(u,T_j(u_j))\} \\ &\leq sd(u,u_j) + s^2d(T_j(u),T(u)) \\ &\quad + s^2\beta\, max\{d(u_j,u)+d(u,T_j(u)), sd(T_j(u),u)+sd(u,u_j), d(u_j,u)\} \\ &= sd(u,u_j) + s^2d(T_j(u),T(u)) + s^3\beta[d(u,T_j(u))+d(u_j,u)] \\ &\leq sd(u,u_j) + s^2d(T_j(u),T(u)) \\ &\quad + s^3\beta[d(u_j,u)+sd(T_j(u),T(u))+sd(T(u),u)] \end{aligned}$$

$$\begin{aligned} &\leq \quad sd(u,u_j)+s^2d(T_j(u),T(u)) \\ &\quad\ +s^3\beta[sd(u_j,u)+sd(T_j(u),T(u))+sd(T(u),u)] \\ &= \quad sd(u,u_j)+s^2d(T_j(u),T(u)) \\ &\quad\ +s^4\beta[d(u_j,u)+d(T_j(u),T(u))+d(T(u),u)]. \end{aligned}$$

Taking the limit as $j\to\infty$, we have

$$d(u,T(u))\leq s^4\beta\, d(u,T(u)).$$

Since $0\leq s^4\beta<\frac{1}{s}$, we have $d(u,T(u))=0$ and so, $u=T(u)$.

□

Remark 3.10. Theorem 3.9 is a generalization of Theorem 3[28] in metric spaces to b-metric spaces.

As an application of Theorem 3.5, we have the following result which can be obtained by the argument similar to that used in Theorem 3.9.

Theorem 3.11. *Let (X,d) be a complete b-metric space with coefficient $s\geq 1$, and $T_j\in CI(X)$ satisfy*

$$d(T_j(x),T_j(y))\leq \alpha\, d(x,T_j(x))+\beta\, max\left\{\begin{array}{l} d(x,y)+d(y,T_j(y)), \\ \\ d(x,T_j(y))+d(y,T_j(x)) \end{array}\right\}$$

for all $x,y\in X$, where $\alpha\geq 0$ and $0\leq\beta<\frac{1}{s^4(s+1)}$ with fixed points u_j $(j=1,2,\cdots)$. Suppose that $T^n(x)=\lim_{j\to\infty}T_j^n(x)$ for all $x\in X(n=1,2,\cdots)$. Then T has the unique fixed point u in X if and only if $u=\lim_j u_j$. Moreover, T is continuous at u and T has property P.

We now examine the strength of hypothesis made in Theorem 3.2. In fact, we furnish Example 3.12 below to show that Theorem 3.2 shall fall through by dropping the condition that $T\in CI(X)$.

Example 3.12. Let $X=\left\{0,\ \frac{1}{6},\ \frac{1}{4},\ \frac{1}{2}\right\}\cup[1,\infty)$ with b-metric d defined by

$$d(x,y)=\mid x-y\mid^2$$

for all $x,y\in X$.

Then (X,d) is a complete b-metric space with coefficient $s=2$. Define $T:X\to X$ by

$$\begin{aligned} T(x) &= 0,\ \ for\, x\in X\setminus\{\frac{1}{4},0\} \\ &= 1,\ \ for\, x\in\{\frac{1}{4},0\}. \end{aligned}$$

It is easy to verify that T satisfies condition (3.1) for $\alpha=36,\ \beta=0$.
But T is not a Ćirić operator, because, for $x\in X\setminus\{\frac{1}{4},0\}$ and $y=\frac{1}{4}$, we have

$$\begin{aligned} d(T^n(x),T^n(y)) &= d(T^{n-1}(0),T^{n-1}(1)) \\ &= d(T^{n-2}(1),T^{n-2}(0)) \\ &\ \ \vdots \end{aligned}$$

$$\begin{aligned} &= d(1,0) \\ &= 1 > q^n(x,y)\,\delta(x,y)\ for\ large\ n, \end{aligned}$$

where q and δ are non-negative real valued functions over $X \times X$ with $q(x,y) < 1\,(x \neq y)$ and $\sup\limits_{x,y \in X} q(x,y) = 1$. Clearly, T possesses no fixed point in X. We note that Theorem 3.2 does not hold without the condition that $T \in CI(X)$.

The following example supports our main result.

Example 3.13. Let $X = [0,\infty)$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) = \mid x - y \mid^2$ for all $x, y \in X$. Then (X,d) is a complete b-metric space with coefficient $s = 2$. Let $T : X \to X$ be defined by

$$T(x) = 0,\ for\ all\ x \in X\ except\ x = \frac{1}{5^i},\ i = 1,2,3,\cdots$$

and

$$T(\frac{1}{5^i}) = \frac{1}{5^{i+1}},\ i \geq 1.$$

Let us take $q(x,y) = 1 - \frac{1}{xy+2}$ and $\delta(x,y) = 5 + xy$ for all $x, y \in X$. Then $q(x,y) < 1$ with $\sup\limits_{x,y \in X} q(x,y) = 1$. It is to be noted that $q(x,y) \geq \frac{1}{2}$. We now verify that $T \in CI(X)$.

Case-I If $x \neq \frac{1}{5^i},\ y = \frac{1}{5^i},\ i = 1,2,\cdots$, then

$$\begin{aligned} d(T^n(x),T^n(y)) &= d(0,\frac{1}{5^{n+i}}) \\ &= \frac{1}{5^{2n+2i}} \\ &< \frac{1}{2^n} \\ &< q^n(x,y)\,\delta(x,y). \end{aligned}$$

Case-II If $x = \frac{1}{5^i},\ y = \frac{1}{5^j},\ j \geq i$, then

$$\begin{aligned} d(T^n(x),T^n(y)) &= d(\frac{1}{5^{n+i}},\frac{1}{5^{n+j}}) \\ &= \frac{1}{5^{2n+2i}}(1 - \frac{1}{5^{j-i}})^2 \\ &< \frac{1}{2^n} \\ &< q^n(x,y)\,\delta(x,y). \end{aligned}$$

Case-III If $x, y \in X$ with $x, y \neq \frac{1}{5^i},\ i = 1,2,\cdots$, then

$$\begin{aligned} d(T^n(x),T^n(y)) &= d(0,0) \\ &= 0 \\ &< \frac{1}{2^n} \\ &< q^n(x,y)\,\delta(x,y). \end{aligned}$$

Thus,

$$d(T^n(x),T^n(y)) < q^n(x,y)\delta(x,y),\ n = 1,2,\cdots,$$

for all $x, y \in X$, where $q(x, y) < 1$ with $\sup_{x,y\in X} q(x, y) = 1$ and so $T \in CI(X)$.

We now verify that T satisfies condition (3.1) for any $\alpha \geq 0$, $\beta = \frac{1}{16} < \frac{1}{s^2}$. If $x \neq \frac{1}{5^i}$, $y = \frac{1}{5^i}$, $i = 1, 2, \cdots$, then

$$\begin{aligned} d(T(x), T(y)) &= d(0, \frac{1}{5^{i+1}}) \\ &= \frac{1}{5^{2+2i}} \\ &= \frac{1}{25} \cdot \frac{1}{5^{2i}} \\ &= \frac{1}{16} \cdot \frac{16}{25} \cdot \frac{1}{5^{2i}} \\ &= \beta d(y, T(y)) \\ &< \beta\{d(y, T(y)) + d(x, y)\} \\ &\leq \alpha\, d(x, T(x)) + \beta\, max \left\{ \begin{array}{l} d(x, y) + d(y, T(y)), \\ \\ d(x, T(y)),\ d(y, T(x)) \end{array} \right\}. \end{aligned}$$

The other cases may be treated similarly. Thus, T satisfies condition (3.1) for any $\alpha \geq 0$, $\beta = \frac{1}{16} < \frac{1}{s^2}$. We see that all the conditions of Theorem 3.2 are satisfied and 0 is the unique fixed point of T in X, T is continuous at 0 and T has property P.

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