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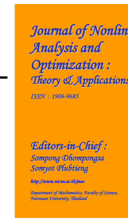

SET-VALUED PREŠIĆ-REICH TYPE CONTRACTIONS IN CONE METRIC SPACES AND FIXED POINT THEOREMS

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ABSTRACT. The purpose of this paper is to prove some fixed point theorems for set-valued mappings satisfying Prešić-Reich type contractive condition in cone metric spaces, without assuming the normality of cone. Our results generalize some known results in metric and cone metric spaces.

KEYWORDS : Prešić-Reich type; Point-set-cone metric; Cone metric space; Fixed point.
AMS Subject Classification: 47H10; 54H25

1. Introduction

Let (X, d) be a metric space and f be a self-map on X. The mapping f is called a Banach contraction if, there exists $\lambda \in [0, 1)$ such that

$$d(fx, fy) \leq \lambda d(x, y) \text{ for all } x, y \in X. \tag{1.1}$$

The Banach contraction principle states that every Banach contraction on a complete metric space has a fixed point, i.e., there exists a point $x^* \in X$ such that $fx^* = x^*$.

Let X be a nonempty set, 2^X the collection of all possible subsets of X and $f\colon X \longrightarrow 2^X$ be a mapping. Then, f is called a set-valued mapping. Let $x \in X$ be such that $fx \neq \emptyset$, then x is called a fixed point of f if $x \in fx$.

Let A be any nonempty subset of a metric space (X, d). For $x \in X$, define

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

Let $CB(X)$ denotes the set of all nonempty closed bounded subset of X. For $A, B \in CB(X)$, define

$$\delta(A, B) = \sup\{d(x, B) : x \in A\},$$
$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\}.$$

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Then H is a metric on $CB(X)$ and called Pompeiu-Hausdorff (or Hausdorff) metric. A mapping $f\colon X \longrightarrow CB(X)$ is called a Nadler contraction (or a set-valued Banach contraction), if there exists $\lambda \in [0,1)$ such that

$$H(fx, fy) \leq \lambda d(x,y) \text{ for all } x, y \in X. \tag{1.2}$$

In 1969, Nadler [10] generalized the famous Banach contraction principle for the set-valued mappings defined from a complete metric space X into the set $CB(X)$. Nadler [10] proved the following theorem:

Theorem 1.1. *Let (X,d) be a complete metric space and let f be a set-valued Banach contraction. Then there exists a point $x^* \in X$ such that $x^* \in fx^*$, i.e., f has a fixed point in X.*

On the other hand, for mappings $f : X \to X$ Kannan [6] introduced the contractive condition:

$$d(fx, fy) \leq \lambda[d(x, fx) + d(y, fy)] \tag{1.3}$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$ is a constant, and proved a fixed point theorem using (1.3) instead of contractive condition (1.1). The conditions (1.3) and (1.1) are independent, as it was shown by two examples in [7].

Reich [14], generalized the fixed point theorems of Banach and Kannan, using contractive condition: for all $x, y \in X$,

$$d(fx, fy) \leq \alpha d(x,y) + \beta d(x, fx) + \gamma d(y, fy) \tag{1.4}$$

where α, β, γ are nonnegative reals with $\alpha+\beta+\gamma < 1$. An example in [14] shows that the Reich's contractive condition is a proper generalization of contractive conditions of Banach and Kannan.

In 1965, Prešić [12, 13] generalized the Banach contraction principle in product spaces and proved the following theorem.

Theorem 1.2. *Let (X,d) be a complete metric space, k a positive integer and $f\colon X^k \to X$ be a mapping satisfying the following contractive type condition:*

$$d(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \leq \sum_{i=1}^{k} q_i d(x_i, x_{i+1}), \tag{1.5}$$

for every $x_1, x_2, \ldots, x_k, x_{k+1} \in X$, where $q_1, q_2, \ldots, q_k$ are nonnegative constants such that $q_1 + q_2 + \cdots + q_k < 1$. Then there exists a unique point $x \in X$ such that $f(x, x, \ldots, x) = x$. Moreover, if $x_1, x_2, \ldots, x_k$ are arbitrary points in X and for $n \in \mathbb{N}$, $x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and $\lim x_n = f(\lim x_n, \lim x_n, \ldots, \lim x_n)$.

A mapping $f\colon X^k \to X$ is called a Prešić type contraction if it satisfies (1.5). The mapping f is called a Prešić-Kannan type contraction if,

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \leq a \sum_{i=0}^{k} d(x_i, f(x_i, \ldots, x_i)), \tag{1.6}$$

for all $x_0, x_1, \ldots, x_k \in X$, where the real constant a is such that $0 \leq ak(k+1) < 1$.

In a similar manner to that used by S.B. Prešić [12, 13], when extending Banach contractions to product spaces, Păcurar [11] generalized the Kannan's theorem in product spaces and proved a fixed point theorem for Prešić-Kannan type contractions.

f is called a Prešić-Reich type contraction if,

$$d(f(x_0,\dots,x_{k-1}),f(x_1,\dots,x_k)) \le \sum_{i=1}^{k}\alpha_i d(x_{i-1},x_i) + \sum_{i=0}^{k}\beta_i d(x_i,f(x_i,\dots,x_i)), \tag{1.7}$$

for all $x_0,x_1,\dots,x_k \in X$, where α_i,β_i are nonnegative constants such that

$$\sum_{i=1}^{k}\alpha_i + k\sum_{i=0}^{k}\beta_i < 1.$$

Note that, for $k=1$ the above definition reduces into the definition due to Reich [14]. Also, Prešić-Banach type contraction (i.e., a mapping f satisfying (1.5)) and Prešić-Kannan type contraction are particular cases of Prešić-Reich type contractions. Malhotra et al. [9] first introduced the notion of Prešić-Reich type contractions (for single-valued case) in cone metric spaces and proved some common fixed point and fixed point results for such mappings.

In 2011, Wardowski [16] introduced the set-valued mappings in cone metric spaces and proved the cone metric version of the result of Nadler [10] (see also [1, 8, 15]). In this paper, we introduced the notion of set-valued Prešić-Reich type contractions in cone metric spaces and prove some fixed point results for such mappings, using the definitions due to Wardowski [16]. Our results generalize and extend the results of Nadler [10], Kannan [6], Reich [14], Prešić [12], Malhotra et al. [9] and Wardowski [16] in the setting of cone metric spaces for set-valued mappings. An example is provided which illustrate the main theorem of this paper.

2. Preliminaries

We use the following definitions and results, consistent with [2] and [3].

Definition 2.1. [3] Let E be a real Banach space and P be a subset of E. The set P is called a cone if

(i) P is closed, nonempty and $P \neq \{\theta\}$, here θ is the zero vector of E;
(ii) $a,b\in\mathbb{R},\ a,b\ge 0,\ x,y\in P \implies ax+by\in P$;
(iii) $x\in P$ and $-x\in P \implies x=\theta$.

Given a cone $P\subset E$, we define a partial ordering " $\preceq$ " with respect to P by $x\preceq y$ if and only if $y-x\in P$. We write $x\prec y$ to indicate that $x\preceq y$ but $x\neq y$. While $x\ll y$ if and only if $y-x\in P^\circ$, where P° denotes the interior of P.
Let P be a cone in a real Banach space E, then P is called normal, if there exist a constant $K>0$ such that for all $x,y\in E$,

$$\theta\preceq x\preceq y \text{ implies } \|x\|\le K\|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P. A cone P is called solid if $P^\circ\neq\emptyset$.

Definition 2.2. [3] Let X be a nonempty set, E be a real Banach space with cone P. Suppose that the mapping $d: X\times X\to E$ satisfies:

1. $\theta\preceq d(x,y)$, for all $x,y\in X$ and $d(x,y)=\theta$ if and only if $x=y$;
2. $d(x,y)=d(y,x)$ for all $x,y\in X$;
3. $d(x,y)\preceq d(x,z)+d(y,z)$, for all $x,y,z\in X$.

Then d is called a *cone metric* on X, and (X, d) is called a cone metric space. If the underlying cone is normal, then (X, d) is called a normal cone metric space.

The following lemma will be useful in the sequel.

Lemma 2.3. [4, 5] *Let E be a real Banach space, P a solid cone in E. Then:*

(i) *If $\{a_n\}$ is a sequence in P, $a_n \to \theta$ then, for every $c \in P^\circ$ there exists $n \in \mathbb{N}$ such that, $a_n \ll c$ for all $n > n_0$.*
(ii) *If $u, v, w \in P$ and $u \preceq v$, $v \ll w$ then $u \ll w$.*
(iii) *If $u, v, w \in P$ and $u \ll v$, $v \preceq w$ then $u \ll w$.*
(iv) *If $u \in P$ and $u \ll c$ for each $c \in P^\circ$, then $u = \theta$.*

Let (X, d) be a cone metric space with cone P. A subset $A \subset X$ is called closed if for any sequence $\{x_n\} \subset A$ convergent to x, we have $x \in A$.

Denote by $N(X)$ a collection of all nonempty subsets of X and by $C(X)$ a collection of all nonempty closed subsets of X.

The following definitions can be found in [16].

Definition 2.4. [16] Let (X, d) be a cone metric space and let $\mathcal{A}$ be a collection of nonempty subsets of X. A map $H\colon \mathcal{A} \times \mathcal{A} \longrightarrow E$ is called a $H-$cone metric with respect to d if for any $A_1, A_2 \in \mathcal{A}$ the following conditions hold:

(H1) $H(A_1, A_2) = \theta \implies A_1 = A_2$;
(H2) $H(A_1, A_2) = H(A_2, A_1)$;
(H3) $\forall_{c\in E, \theta \ll c}\ \forall_{x\in A_1}\ \exists_{y\in A_2}\quad d(x, y) \preceq H(A_1, A_2) + c$;
(H4) One of the following is satisfied:
  (i) $\forall_{c\in E, \theta \ll c}\ \exists_{x\in A_1}\ \forall_{y\in A_2}\ H(A_1, A_2) \preceq d(x, y) + c$;
  (ii) $\forall_{c\in E, \theta \ll c}\ \exists_{x\in A_2}\ \forall_{y\in A_1}\ H(A_1, A_2) \preceq d(x, y) + c$.

The following are some examples of $H-$cone metrics.

Example 2.5. [16] Let (X, d) be a cone metric space and let $\mathcal{A} = \{\{x\} : x \in X\}$. Define the mapping $H\colon \mathcal{A} \times \mathcal{A} \longrightarrow E$ by the formula

$$H(\{x\}, \{y\}) = d(x, y) \text{ for all } x, y \in X,$$

is a $H-$cone metric with respect to d.

Example 2.6. [16] Let (X, d) be a metric space and let $\mathcal{A}$ be the family of all nonempty, closed bounded subsets of X. Then the mapping $H\colon \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{R}^+$ given by the formula

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A), A, B \in \mathcal{A}\right\}$$

which is called a Hausdorff metric, is a $H-$cone metric with respect to d.

The following lemma shows that a $H-$cone metric with respect to the cone metric d, is itself a cone metric when $\mathcal{A} \subset N(X)$.

Lemma 2.7. [16] *Let (X, d) be a cone metric space and let $\mathcal{A} \subset N(X)$, $\mathcal{A} \neq \emptyset$. If $H\colon \mathcal{A} \times \mathcal{A} \longrightarrow E$ is a $H-$cone metric with respect to d then a pair $(\mathcal{A}, H)$ is a cone metric space.*

Wardowski [16] proved the following cone metric version of result of Nadler [10].

Theorem 2.1. [16] *Let (X, d) be a complete cone metric space with a normal cone P with a normal constant K, $\mathcal{A}$ be a nonempty collection of nonempty closed subsets* *of X and let $H: \mathcal{A} \times \mathcal{A} \longrightarrow E$ be a $H-$cone metric with respect to d. If for a map $f: X \longrightarrow \mathcal{A}$ there exists $\lambda \in (0,1)$ such that*

$$H(fx, fy) \preceq \lambda d(x,y) \text{ for all } x, y \in X, \tag{2.1}$$

then the set of all fixed points of f is nonempty.

3. Main results

In this section, we introduce various types of set-valued Prešić type contractions, the point-set-cone metric and prove some fixed point results for set-valued Prešić type contractions in cone metric spaces. In further discussion, we assume that the cones under consideration are solid cones, i.e., $P^\circ \neq \emptyset$.

First, we define the point-set-cone metric between a point and a subset of cone metric spaces which is an extension and generalization of the distance of point from a set in ordinary metric spaces.

Definition 3.1. Let (X,d) be a cone metric space and let $\mathcal{A}$ be a nonempty collection of nonempty subsets of X. A map $d_s: X \times \mathcal{A} \longrightarrow E$ is called the point-set-cone metric with respect to d if for all $x \in X$, $A \in \mathcal{A}$ the following conditions hold:

(PS1) $\theta \preceq d_s(x,A)$ and $d_s(x,A) = \theta \implies x \in A$;
(PS2) $\forall_{a \in A}\ d_s(x,A) \preceq d(x,a)$.

Let us observe that for each cone metric d the family of point-set-cone metrics with respect to d is nonempty and each point-set-cone metric depends on the shape of the family $\mathcal{A}$. See the following examples:

Example 3.2. Let (X,d) be a cone metric space and let

$$\mathcal{A} = \{\{x\}: x \in X\}.$$

Then, the mapping $d_s: X \times \mathcal{A} \longrightarrow E$ defined by the formula

$$d_s(x, \{y\}) = d(x,y) \text{ for all } x, y \in X,$$

is a point-set-cone metric with respect to d.

Example 3.3. Let (X,d) be a metric space and let $\mathcal{A}$ be the family of all nonempty, closed and bounded subsets of X. Then the mapping $d_s: X \times \mathcal{A} \longrightarrow \mathbb{R}^+$ given by the formula

$$d_s(x,A) = \inf\{d(x,a): a \in A\}$$

which is called the distance of point x from the set A, is a point-set-cone metric with respect to d.

Example 3.4. Let $E = \mathbb{R}^2$, the Euclidean plane, $P = \{(x,y) \in \mathbb{R}^2: x, y \geq 0\}$ be the cone in E and $X = \{(x,0) \in \mathbb{R}^2: 0 \leq x \leq 1\} \cup \{(0,x) \in \mathbb{R}^2: 0 \leq x \leq 1\}$ Let $X = \{(x,0),(0,x): 0 \leq x \leq 1\}$ and the mapping $d: X \times X \longrightarrow E$ by defined by

$$d((x,0),(y,0)) = \mid x-y \mid (0,1)\,, d((0,x),(0,y)) = \mid x-y \mid (1,0)\,,$$

$$d((x,0),(0,y)) = d((0,y),(x,0)) = (y,x)\,.$$

Then (X,d) is a cone metric space. Let $\mathcal{A} = \{\{(0,0),(x,0),(0,x)\}: 0 \leq x \leq 1\}$, then the mapping $d_s: X \times \mathcal{A} \longrightarrow \mathbb{R}^+$ given by the formula

$$d_s\big((z,0), \{(0,0),(x,0),(0,x)\}\big) = \big(0, [z \cdot \mid z-x \mid]^p\big),$$

$$d_s((0,z), \{(0,0),(x,0),(0,x)\}) = \big([z \cdot \mid z-x \mid]^p, 0\big)$$

where $p \in \mathbb{N}$, is a point-set-cone metric with respect to d.

Remark 3.5. It is obvious from (PS2) that if $x \in A$, then $d_s(x,A) = \theta$. Therefore, from (PS1) we conclude the double implication: $d_s(x,A) = \theta \iff x \in A$.

Lemma 3.6. *Let (X,d) be a cone metric space and let $\mathcal{A}$ be a collection of nonempty subsets of X. Let H be a $H-$cone metric and d_s be a point-set-cone metric with respect to d. Then*

$$\forall_{A,B\in\mathcal{A}}\ \forall_{a\in A}\ d_s(a,B) \preceq H(A,B).$$

Proof. Let $A,B \in \mathcal{A}$ and $a \in A$. Suppose, $\{c_n\}$ be a sequence in P° such that $c_n \longrightarrow \theta$ as $n \longrightarrow \infty$ and $\theta \ll c_n$ for all $n \in \mathbb{N}$. Then, by (H3) we have there exists $b \in B$ such that

$$d(a,b) \preceq H(A,B) + c_n \text{ for all } n \in \mathbb{N}.$$

By (PS2) we have $d_s(a,B) \preceq d(a,b)$, so by the above inequality we obtain $d_s(a,B) \preceq H(A,B) + c_n$ for all $n \in \mathbb{N}$, i.e., $H(A,B) + c_n - d_s(a,B) \in P$ for all $n \in \mathbb{N}$. Since P is closed, by choice of the sequence $\{c_n\}$ we have $H(A,B) - d_s(a,B) \in P$, i.e., $d_s(a,B) \preceq H(A,B)$. □

Let (X,d) be a cone metric space and $\mathcal{A}$ be a nonempty collection of nonempty subsets of X. In further discussion, $H\colon \mathcal{A}\times\mathcal{A} \longrightarrow E$ will represent the $H-$cone metric and $d_s\colon X\times\mathcal{A} \longrightarrow E$ will represent the point-set-cone metric with respect to d.

Now we can define various set-valued Prešić type contractions in cone metric spaces.

Let (X,d) be a cone metric space, k a positive integer, $\mathcal{A}$ a nonempty collection of nonempty closed subsets of X and let $f\colon X^k \to \mathcal{A}$ be a mapping. Then, f is said to be Lipschitzian on X if there exist nonnegative constants α_i such that

$$H(f(x_0,x_1,\dots,x_{k-1}), f(x_1,x_2,\dots,x_k)) \preceq \sum_{i=1}^{k} \alpha_i d(x_{i-1},x_i), \tag{3.1}$$

for all $x_0,x_1,\dots,x_k \in X$. If $\sum\limits_{i=1}^{k} \alpha_i < 1$, then the mapping f is said to be a set-valued Prešić type contractions on X.

The mapping f is called a set-valued Prešić-Kannan type contraction on X if,

$$H(f(x_0,\dots,x_{k-1}), f(x_1,\dots,x_k)) \preceq a\sum_{i=0}^{k} d_s(x_i, f(x_i,\dots,x_i)) \tag{3.2}$$

for all $x_0,x_1,\dots,x_k \in X$, where the real constant a is such that $0 \le ak(k+1) < 1$.

The mapping f is called a set-valued Prešić-Reich type contraction on X if,

$$\begin{aligned} H(f(x_0,\dots,x_{k-1}), f(x_1,\dots,x_k)) \preceq & \sum_{i=1}^{k} \alpha_i d(x_{i-1},x_i) \\ & + \sum_{i=0}^{k} \beta_i d_s(x_i, f(x_i,\dots,x_i)), \end{aligned} \tag{3.3}$$

for all $x_0,x_1,\dots,x_k \in X$, where α_i, β_i are nonnegative constants such that

$$\sum_{i=1}^{k} \alpha_i + k\sum_{i=0}^{k} \beta_i < 1. \tag{3.4}$$

We denote the set of all fixed points of f by Fix$f$ and

$$\text{Fix}f = \{x \in X : x \in f(x, \dots, x)\}.$$

The following theorem is the main result of this paper.

Theorem 3.1. *Let (X, d) be a complete cone metric space, k a positive integer and $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X. If $f : X^k \longrightarrow \mathcal{A}$ be a set-valued Prešić-Reich type contraction, then Fix$f \neq \emptyset$.*

Proof. Let $\{c_n\}$ be an arbitrary sequence in E which satisfies $\theta \ll c_n$ for all $n \in \mathbb{N}$. Let x_0 be an arbitrary point of X. Because $f(x_0, \dots, x_0) \in \mathcal{A}$, let $x_1 \in f(x_0, \dots, x_0)$. From (H3) there exists $x_2 \in f(x_1, \dots, x_1)$ such that

$$d(x_1, x_2) \preceq H(f(x_0, \dots, x_0), f(x_1, \dots, x_1)) + c_1.$$

Similarly, there exists $x_3 \in f(x_2, \dots, x_2)$ such that

$$d(x_2, x_3) \preceq H(f(x_1, \dots, x_1), f(x_2, \dots, x_2)) + c_2.$$

Continuing this procedure we obtain $x_{n+1} \in f(x_n, \dots, x_n)$ and

$$d(x_n, x_{n+1}) \preceq H(f(x_{n-1}, \dots, x_{n-1}), f(x_n, \dots, x_n)) + c_n \tag{3.5}$$

for all $n \in \mathbb{N}$.
As H is a metric on $N(X)$, for any $n \in \mathbb{N}$ it follows from (3.5) that

$$\begin{aligned} d(x_n, x_{n+1}) &\preceq H(f(x_{n-1}, \dots, x_{n-1}), f(x_n, \dots, x_n)) + c_n \\ &\preceq H(f(x_{n-1}, \dots, x_{n-1}), f(x_{n-1}, \dots, x_{n-1}, x_n)) \\ &\quad + H(f(x_{n-1}, \dots, x_{n-1}, x_n), f(x_{n-1}, \dots, x_{n-1}, x_n, x_n)) \\ &\quad + \cdots + H(f(x_{n-1}, x_n, \dots, x_n), f(x_n, \dots, x_n)) + c_n. \end{aligned}$$

Since f is a set-valued Prešić-Reich type contraction, it follows from the above inequality that

$$\begin{aligned} d(x_n, x_{n+1}) \preceq\ & \alpha_k d(x_{n-1}, x_n) + \beta_0 d_s(x_{n-1}, f(x_{n-1}, \dots, x_{n-1})) + \cdots \\ & + \beta_{k-1} d_s(x_{n-1}, f(x_{n-1}, \dots, x_{n-1})) + \beta_k d_s(x_n, f(x_n, \dots, x_n)) \\ & + \alpha_{k-1} d(x_{n-1}, x_n) + \beta_0 d_s(x_{n-1}, f(x_{n-1}, \dots, x_{n-1})) + \cdots \\ & + \beta_{k-1} d_s(x_n, f(x_n, \dots, x_n)) + \beta_k d_s(x_n, f(x_n, \dots, x_n)) \\ & + \cdots + \alpha_1 d(x_{n-1}, x_n) + \beta_0 d_s(x_{n-1}, f(x_{n-1}, \dots, x_{n-1})) \\ & + \beta_1 d_s(x_n, f(x_n, \dots, x_n)) + \cdots + \beta_k d_s(x_n, f(x_n, \dots, x_n)) \\ & + c_n. \end{aligned}$$

Since $x_n \in f(x_{n-1}, \dots, x_{n-1})$ for all $n \in \mathbb{N}$, it follows from the definition of point-set-cone metric and the above inequality that

$$\begin{aligned} d(x_n, x_{n+1}) \preceq & \left[\sum_{i=1}^{k} \alpha_i\right] d(x_{n-1}, x_n) + \beta_0 d(x_{n-1}, x_n) + \cdots \\ & + \beta_{k-1} d(x_{n-1}, x_n) + \beta_k d(x_n, x_{n+1}) + \beta_0 d(x_{n-1}, x_n) + \cdots \\ & + \beta_{k-1} d(x_n, x_{n+1}) + \beta_k d(x_n, x_{n+1}) + \cdots + \beta_0 d(x_{n-1}, x_n) \\ & + \beta_1 d(x_n, x_{n+1}) + \cdots + \beta_k d(x_n, x_{n+1}) + c_n. \end{aligned}$$

Rearranging the terms in the above expression, we obtain

$$d(x_n, x_{n+1}) \quad \preceq \quad \left[\sum_{i=1}^{k} \alpha_i + \sum_{i=0}^{k-1} (k-i)\beta_i\right] d(x_{n-1}, x_n) + \left[\sum_{i=1}^{k} i\beta_i\right] d(x_n, x_{n+1}) + c_n.$$

Thus, we have

$$d(x_n, x_{n+1}) \preceq \frac{\sum_{i=1}^{k} \alpha_i + \sum_{i=0}^{k} (k-i)\beta_i}{1 - \sum_{i=0}^{k} i\beta_i} d(x_{n-1}, x_n) + \frac{1}{1 - \sum_{i=0}^{k} i\beta_i} c_n. \tag{3.6}$$

For simplicity, set $A = \sum_{i=1}^{k} \alpha_i, B = k\sum_{i=0}^{k} \beta_i, C = \sum_{i=0}^{k} i\beta_i$ and $\lambda = \frac{A+B-C}{1-C}$, then in view of (3.4) we have,

$$A + B = \sum_{i=1}^{k} \alpha_i + k\sum_{i=0}^{k} \beta_i < 1, \quad C < 1, \text{ also } C \leq B,$$

and so, $0 \leq \lambda < 1$. Thus, from (3.6) it follows that

$$d(x_n, x_{n+1}) \preceq \lambda d(x_{n-1}, x_n) + \frac{c_n}{1-C} \text{ for all } n \in \mathbb{N}. \tag{3.7}$$

From the successive applications of the inequality (3.7) we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &\preceq \lambda d(x_{n-1}, x_n) + \frac{c_n}{1-C} \\ &\preceq \lambda \left[\lambda d(x_{n-2}, x_{n-1}) + \frac{c_{n-1}}{1-C} \right] + \frac{c_n}{1-C} \\ &= \lambda^2 d(x_{n-2}, x_{n-1}) + \lambda \frac{c_{n-1}}{1-C} + \frac{c_n}{1-C} \\ &\preceq \lambda^2 \left[\lambda d(x_{n-3}, x_{n-2}) + \frac{c_{n-2}}{1-C} \right] + \lambda \frac{c_{n-1}}{1-C} + \frac{c_n}{1-C} \\ &= \lambda^3 d(x_{n-3}, x_{n-2}) + \lambda^2 \frac{c_{n-2}}{1-C} + \lambda \frac{c_{n-1}}{1-C} + \frac{c_n}{1-C}, \end{aligned}$$

which yields

$$d(x_n, x_{n+1}) \preceq \lambda^n d(x_0, x_1) + \frac{1}{1-C} \sum_{i=0}^{n-1} \lambda^i c_{n-i}.$$

Let $\omega \in P^\circ$, i.e., $\omega \in E, \theta \ll \omega$ be given. Since the sequence $\{c_n\}$ was arbitrary, choose c_n such that $\theta \ll c_n \ll \lambda^n \omega$ for all $n \in \mathbb{N}$. Therefore, it follows from the above inequality that

$$d(x_n, x_{n+1}) \ll \lambda^n d(x_0, x_1) + \frac{n\lambda^n}{1-C} \omega. \tag{3.8}$$

Let $n, m \in \mathbb{N}$ be such that $m > n$, then using inequality (3.8) we obtain

$$\begin{aligned} d(x_n, x_m) &\preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \\ &\ll \sum_{j=n}^{m-1} \left[\lambda^j d(x_0, x_1) + \frac{j\lambda^j}{1-C} \omega \right] \\ &= d(x_0, x_1) \sum_{j=n}^{m-1} \lambda^j + \frac{\omega}{1-C} \sum_{j=n}^{m-1} j\lambda^j. \end{aligned}$$

Since $0 \leq \lambda < 1$, therefore both the series $\sum_{n=1}^{\infty} \lambda^n$ and $\sum_{n=1}^{\infty} n\lambda^n$ are convergent series of nonnegative terms, and so, we have $\sum_{j=n}^{m-1} \lambda^j \longrightarrow 0$ and $\sum_{j=n}^{m-1} j\lambda^j \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore, the quantity on the right of the above inequality must tends to θ as $n \longrightarrow \infty$. Now, using Lemma 2.3 and the last inequality we obtain, for each $c \in P^\circ$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n > n_0$. Therefore, $\{x_n\}$ is a Cauchy sequence.

By completeness of X, there exists $x^* \in X$ such that $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$. We shall show that x^* is a fixed point of f. Using similar calculations to the previous one we obtain

$$\begin{aligned} & H(f(x_n, \ldots, x_n), f(x^*, \ldots, x^*)) \\ & \preceq\ Ad(x_n, x^*) + (B - C)d_s(x_n, f(x_n, \ldots, x_n)) + Cd_s(x^*, f(x^*, \ldots, x^*)) \end{aligned}$$

which with the fact $x_{n+1} \in f(x_n, \ldots, x_n)$ and the definition of point-set-cone metric gives

$$\begin{aligned} H(f(x_n, \ldots, x_n), f(x^*, \ldots, x^*)) \quad \preceq \quad & Ad(x_n, x^*) + (B - C)d_s(x_n, x_{n+1}) \\ & + Cd_s(x^*, f(x^*, \ldots, x^*)). \end{aligned} \tag{3.9}$$

Suppose $c \in P^\circ$ be given, then since $x_{n+1} \in f(x_n, \ldots, x_n)$, by (H3) for all $n \in \mathbb{N}$, there exists $y_n \in f(x^*, \ldots, x^*)$ such that

$$d(x_{n+1}, y_n) \preceq H(f(x_n, \ldots, x_n), f(x^*, \ldots, x^*)) + c'_n, \tag{3.10}$$

where $\{c'_n\}$ is a sequence in P° such that $c'_n \ll \dfrac{(1 - C)c}{4}$ for all $n \in \mathbb{N}$. Again, since $y_n \in f(x^*, \ldots, x^*)$ we have

$$d_s(x^*, f(x^*, \ldots, x^*)) \preceq d(x^*, y_n) \preceq d(x^*, x_{n+1}) + d(x_{n+1}, y_n)$$

which with (3.9) and (3.10) gives

$$\begin{aligned} d_s(x^*, f(x^*, \ldots, x^*)) \quad \preceq \quad & d(x^*, x_{n+1}) + Ad(x_n, x^*) + (B - C)d(x_n, x_{n+1}) \\ & + Cd_s(x^*, f(x^*, \ldots, x^*)) + c'_n. \end{aligned}$$

Since $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$ and $c'_n \ll \dfrac{(1 - C)c}{4}$ for all $n \in \mathbb{N}$, we can choose $n_1 \in \mathbb{N}$ such that $d(x^*, x_{n+1}) \ll \dfrac{(1 - C)c}{4}$, $d(x_n, x^*) \ll \dfrac{(1 - C)c}{4A}$ and $d(x_n, x_{n+1}) \ll \dfrac{(1 - C)c}{4(B - C)}$ for all $n > n_1$. Therefore, the above inequality yields

$$d_s(x^*, f(x^*, \ldots, x^*)) \ll c \quad \text{for all} \quad n > n_1.$$

Therefore, it follows from Lemma 2.3 and the above inequality that

$$d_s(x^*, f(x^*, \ldots, x^*)) = \theta.$$

Thus, $x^* \in f(x^*, \ldots, x^*)$, i.e., x^* if a fixed point of f. □

Taking $k = 1$ in the above theorem, we obtain the following fixed point theorem which generalize the result of Wardowski [16] without assuming the normality of the underlying cone.

Corollary 3.7. *Let (X, d) be a complete cone metric space and $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X. If for a map $f\colon X \longrightarrow \mathcal{A}$ there exist nonnegative constants α, β_0, β_1 such that $\alpha + \beta_0 + \beta_1 < 1$ and*

$$H(fx, fy) \preceq \alpha d(x, y) + \beta_0 d_s(x, fx) + \beta_1 d_s(y, fy)$$

for all $x, y \in X$, then Fix$f \neq \emptyset$.

Corollary 3.8. *Let (X,d) be a complete cone metric space and $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X. If for a map $f\colon X \longrightarrow \mathcal{A}$ there exist nonnegative constants β_0, β_1 such that $\beta_0 + \beta_1 < 1$ and*

$$H(fx, fy) \preceq \beta_0 d_s(x, fx) + \beta_1 d_s(y, fy)$$

for all $x, y \in X$, then Fix$f \neq \emptyset$.

Corollary 3.9. *Let (X,d) be a complete cone metric space and $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X. If for a map $f\colon X \longrightarrow \mathcal{A}$ there exists $\alpha \in [0,1)$ such that*

$$H(fx, fy) \preceq \alpha d(x, y)$$

for all $x, y \in X$, then Fix$f \neq \emptyset$.

Corollary 3.10. *Let (X,d) be a complete cone metric space, k a positive integer and $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X. If $f\colon X^k \longrightarrow \mathcal{A}$ be a set-valued Prešić-Kannan type contraction, then Fix$f \neq \emptyset$.*

Proof. Taking $\alpha_i = 0$ for $i = 1, 2, \ldots, k$ and $\beta_i = a$ (say) for $i = 0, 1, \ldots, k$ in Theorem 3.1, we obtain the desired result. □

Corollary 3.11. *Let (X,d) be a complete cone metric space, k a positive integer and $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X. If $f\colon X^k \longrightarrow \mathcal{A}$ be a set-valued Prešić type contraction, then Fix$f \neq \emptyset$.*

Proof. Taking $\beta_i = 0$ for $i = 0, 1, \ldots, k$ in Theorem 3.1, we obtain the desired result. □

Example 3.12. Let $X = [0,1]$, $E = C^1_{\mathbb{R}}[0,1]$ with the norm $\|\psi\| = \|\psi\|_\infty + \|\psi'\|_\infty$ and $P = \{\psi \in E\colon \psi(t) \geq 0, t \in [0,1]\}$. Define $d\colon X \times X \longrightarrow E$ by

$$d(x,y) = \mid x - y \mid \phi(t) \text{ for all } x, y \in X,$$

where $\phi(t) = e^t$, $t \in [0,1]$. Then, (X,d) is a complete cone metric space. Let $\mathcal{A}$ be the family of subsets of X of the form $\mathcal{A} = \{[0,x]\colon x \in X\} \cup \{\{x\}\colon x \in X\}$, and define the functions $H\colon \mathcal{A} \times \mathcal{A} \longrightarrow E$ and $d_s\colon X \times \mathcal{A} \longrightarrow E$ by

$$H(A,B) = \begin{cases} \mid x - y \mid \cdot e^t, & \text{for } A = [0,x], B = [0,y]; \\ \mid x - y \mid \cdot e^t, & \text{for } A = \{x\}, B = \{y\}; \\ \max\{y, \mid x - y \mid\} \cdot e^t, & \text{for } A = [0,x], B = \{y\}; \\ \max\{x, \mid x - y \mid\} \cdot e^t, & \text{for } A = \{x\}, B = [0,y] \end{cases}$$

and

$$d_s(x, A) = \min\{\mid x - a \mid \colon a \in A\} \cdot e^t, \ t \in [0,1] \text{ for all } x \in X.$$

Then, H is a $H-$cone metric and d_s is a point-set-cone metric with respect to d. For $k = 2$, define a mapping $f\colon X^2 \longrightarrow \mathcal{A}$ as follows:

$$f(x,y) = \begin{cases} \left[0, \frac{1}{5}(x + y - 1)^2\right], & \text{if } x, y \in \left(\frac{1}{2}, 1\right]; \\ \{0\}, & \text{otherwise.} \end{cases}$$

Now, by some routine calculations one can see that the mapping f is a set-valued Prešić-Reich type contraction on X^2 with $\alpha_1 = \alpha_2 = \frac{2}{5}$ and $\beta_1 = \beta_2 = \beta_3 = \frac{1}{36}$. All the conditions of Theorem 3.1 are satisfied and $0 \in$ Fix$f$.

In the next theorem, we replace the completeness of cone metric space by an additional condition on the set-valued Prešić-Reich type contractions.

Theorem 3.2. *Let (X, d) be a cone metric space, k a positive integer, $\mathcal{A}$ a nonempty collection of nonempty closed subsets of X and $f\colon X^k \longrightarrow \mathcal{A}$ be a set-valued Prešić-Reich type contraction. Suppose there exists $x^* \in X$ such that*

$$d_s(x^*, f(x^*, \dots, x^*)) \preceq d_s(x, f(x, \dots, x)) \text{ for all } x \in X.$$

Then Fix$f \neq \emptyset$.

Proof. Let $D(x) = d_s(x, f(x, \dots, x))$ for all $x \in X$. Then by assumption we have

$$D(x^*) \preceq D(x) \text{ for all } x \in X. \tag{3.11}$$

If $x^* \in f(x^*, \dots, x^*)$, then $u \in$ Fix$f$. Suppose $x^* \notin f(x^*, \dots, x^*)$, then $D(x^*) = d_s(x^*, f(x^*, \dots, x^*)) \neq \theta$. Let $x_0 = x^*$, then following similar arguments to those in Theorem 3.1, the sequence $\{x_n\}$, where $x_n \in f(x_{n-1}, \dots, x_{n-1})$ for all $n \in \mathbb{N}$ is a Cauchy sequence in X. Now, by Lemma 3.6, we have

$$\begin{aligned} D(x_n) &= d_s(x_n, f(x_n, \dots, x_n)) \\ &\preceq H(f(x_{n-1}, \dots, x_{n-1}), f(x_n, \dots, x_n)) \\ &\preceq H(f(x_{n-1}, \dots, x_{n-1}), f(x_{n-1}, \dots, x_{n-1}, x_n)) \\ &\quad + H(f(x_{n-1}, \dots, x_{n-1}, x_n), f(x_{n-1}, \dots, u, x_n, x_n)) \\ &\quad + \cdots + H(f(x_{n-1}, x_n, \dots, x_n), f(x_n, \dots, x_n)) \\ &\preceq A d(x_{n-1}, x_n) + (B - C) d_s(x_{n-1}, f(x_{n-1}, \dots, x_{n-1})) \\ &\quad + C d_s(x_n, f(x_n, \dots, x_n)), \end{aligned}$$

where $A = \sum_{i=1}^{k} \alpha_i$, $B = k \sum_{i=0}^{k} \beta_i$ and $C = \sum_{i=0}^{k} i\beta_i$.
Since $x_n \in f(x_{n-1}, \dots, x_{n-1})$ for all $n \in \mathbb{N}$, by (PS2) we have

$$d_s(x_{n-1}, f(x_{n-1}, \dots, x_{n-1})) \preceq d(x_{n-1}, x_n) \text{ for all } n \in \mathbb{N}.$$

Therefore, it follows from the above inequality that

$$D(x_n) \preceq \frac{A + B - C}{1 - C}\, d(x_{n-1}, x_n) \text{ for all } n \in \mathbb{N}.$$

As, $A + B < 1, C \leq B$, and $\{x_n\}$ is a Cauchy sequence, for each $c \in P$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_{n-1}, x_n) \ll \dfrac{(1 - C)c}{A + B - C}$ for all $n > n_0$. So, it follows from the above inequality that $D(x_n) \ll c$ for all $n > n_0$. Using the inequality (3.11) and the Remark 2.3 we have

$$D(x^*) \ll c \text{ for all } n \in \mathbb{N}.$$

Therefore, we must have $D(x^*) = \theta$, i.e., $d_s(x^*, f(x^*, \dots, x^*)) = \theta$, or, $x^* \in f(x^*, \dots, x^*)$. Thus $x^* \in$ Fix$f$. □

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