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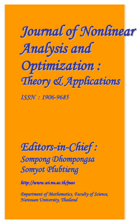

OPERATOR THEORETIC TECHNIQUES IN THE THEORY OF NONLINEAR ORDINARY HYBRID DIFFERENTIAL EQUATIONS WITH MAXIMA

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ABSTRACT. In this paper author proves the algorithms for the existence as well as approximation of the solutions for a couple of initial value problems of nonlinear p^{th} order ordinary differential equations with maxima using the operator theoretic techniques in a partially ordered normed linear space. The main results rely on some recent hybrid fixed point theorems of Dhage (2013) in a partially ordered normed linear space and the approximation of the solutions of the considered nonlinear differential equations with maxima are obtained under weaker mixed partial continuity and partial Lipschitz conditions. Our hypotheses and results are also illustrated by some numerical examples.

KEYWORDS : Hybrid differential equation; Hybrid fixed point theorem; Dhage iteration method; Existence and approximation theorems.
AMS Subject Classification: 34A12, 34H34, 47H07, 47H10

1. INTRODUCTION

The operator theory is a vast developed branch of the subject of mathematics which has important applications to the problems of several areas of mathematics. The area of differential equations with maxima is not an exception. More specifically the area of nonlinear differential equations with maxima totally depends upon nonlinear operator theory and applications. It is known that even for a simple nonlinear differential equation there is no method to solve and obtain the exact solution. However, if we use nonlinear operator theory, then we can have more information about the solutions of the nonlinear problems such as existence, uniqueness, stability, attractivity, positivity, monotonicity and multiplicity results to mention a few. Therefore, there is considerable development of this area under the title nonlinear analysis and has been discussed all over the world. Again, fixed point theory is an important branch of nonlinear analysis which concerns with the solutions of the operator equation $\mathcal{T}x = x$, where $\mathcal{T}$ is a nonlinear operator in an abstract space under

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consideration, and the celebrated mathematicians Schauder(1914), Banach(1922) and Tarski(1940) proved three basic fixed point principles and provided the foundation stones for the subject of nonlinear analysis (cf. Granas and Dugundji [19] and the references therein). Each of these three basic fixed point theorems has some advantages and disadvantages over the others. There are several extensions and generalizations of the above three basic fixed point principles, called the fixed point theoretic techniques or operator theoretic techniques and have been widely used in the literature on nonlinear equations for proving the different aspects of the solutions. There are nonlinear equations related to some dynamic systems for which the above fixed point theorems are not applicable but the fixed point theorems with mixed arguments from these theorems may be applicable for proving the existence as well as some other information about the solutions. Therefore, a new area of operator theoretic techniques under the title hybrid fixed point theory is developed (see Dhage [3, 4, 6, 7, 8, 9, 10] and the references therein). Actually the origin of hybrid fixed point theory lies in the works of Krasnoselskii [21] and Dhage [3, 4], however the theory gained momentum after the publication of the papers of Ran and Reurings [24] and Dhage [6]. Like Picard iteration method, the operator theoretic technique involved in the hybrid fixed pint theorem of Dhage [5] is commonly known as Dhage iteration method and a few details of the hybrid fixed point theory may be found in Dhage [6, 7, 8] and Dhage and Dhage [12, 13]. In the present paper we employ some hybrid fixed point theorems in the study of initial value problems of nonlinear higher order ordinary differential equations with maxima .

The rest of the paper will be organized as follows. In Section 2 we give some preliminaries and key fixed point theorem that will be used in subsequent part of the paper. In Section 3 we discuss the existence result for the initial value problems and in Section 4 we discuss the existence result for initial value problems of hybrid differential differential equations with maxima with linear perturbation of first type.

2. Auxiliary Results

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$ in which the addition and the scalar multiplication by positive real numbers is preserved by $\preceq$. The details of such spaces appear in Dhage [4] and the references therein. Two elements x and y in E are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Deimling [2] and Heikkilä and Lakshmikantham [20] and the references therein.

We need the following definitions (see Dhage [4, 5, 6] and the references therein) in what follows.

Definition 2.1. A mapping $\mathcal{T} : E \longrightarrow E$ is called **isotone** or **monotone nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called **monotone nonincreasing** if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called **monotonic** or simply **monotone** if it is either monotone nondecreasing or monotone nonincreasing on E.

The following terminologies may be found in any book on nonlinear analysis and applications. See Deimling [2], Granas and Dugundji [19], Zeidler [25] and the references therein.

Definition 2.2. An operator $\mathcal{T}$ on a normed linear space E into itself is called **compact** if $\mathcal{T}(E)$ is a relatively compact subset of E. $\mathcal{T}$ is called **totally bounded** if for any bounded subset S of E, $\mathcal{T}(S)$ is a relatively compact subset of E. If $\mathcal{T}$ is continuous and totally bounded, then it is called **completely continuous** on E.

Definition 2.3 (Dhage [5])**.** A mapping $\mathcal{T} : E \longrightarrow E$ is called **partially continuous** at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ is called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E and vice-versa.

Definition 2.4. An operator $\mathcal{T}$ on a partially normed linear space E into itself is called **partially bounded** if $\mathcal{T}(C)$ is bounded for every chain C in E. $\mathcal{T}$ is called **uniformly partially bounded** if all chains $\mathcal{T}(C)$ in E are bounded by a unique constant. $\mathcal{T}$ is called **partially compact** if $\mathcal{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{T}$ is called **uniformly partially compact** if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called **partially totally bounded** if for any totally ordered and bounded subset C of E, $\mathcal{T}(C)$ is a relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

Remark 2.5. Note that every compact mapping on a partially normed linear space is partially compact and every partially compact mapping is partially totally bounded, however the reverse implications do not hold. Again, every completely continuous mapping is partially completely continuous and every partially completely continuous mapping is partially continuous and partially totally bounded, but the converse may not be true.

Definition 2.6. The order relation $\preceq$ and the metric d on a non-empty set E are said to be **$\mathcal{D}$-compatible** if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nondecreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the whole sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are **$\mathcal{D}$-compatible**. A subset S of E is called **Janhavi** if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are compatible in it. In particular, if $S = E$, then E is called a **Janhavi metric** or **Janhavi Banach space**.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property. In general every finite dimensional Banach space with a standard norm and an order relation is a **Janhavi Banach space**.

The essential idea of **Dhage iteration principle** may be described as " **the monotonic convergence of the sequence of successive approximations to**

the solutions of a nonlinear equation beginning with a lower or an upper solution of the equation as its initial or first approximation" and it is a very powerful tool in the existence theory of nonlinear analysis. The procedure involved in the application of Dhage iteration principle to nonlinear equation is called the " **Dhage iteration method**." It is clear that Dhage iteration method embodied in hybrid fixed point theorems is different for different nonlinear problems and also different from the usual Picard's successive iteration method. A few other hybrid fixed point theorems involving the Dhage iteration method may be found in Dhage [8, 9, 10].

Theorem 2.1 (Dhage [7]). *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that every compact chain C in E is Janhavi. Let $\mathcal{T} : E \longrightarrow E$ be a partially continuous, nondecreasing and partially compact operator. If there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $\mathcal{T}x_0 \preceq x_0$, then the operator equation $\mathcal{T}x = x$ has a solution x^* in E and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Remark 2.7. The regularity of E in above Theorem 2.1 may be replaced with a stronger continuity condition of the operator $\mathcal{T}$ on E which is a result proved in Dhage [6].

The Dhage iteration method involved in the following hybrid fixed point theorems are employed for proving the existence and uniqueness of the solutions for the IVP considered in the subsequent section of the paper. Before stating these results, we need the following definitions (see Dhage [4, 5, 6] and the references therein) in what follows.

Definition 2.8. An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. An operator $\mathcal{T} : E \longrightarrow E$ is called partially nonlinear $\mathcal{D}$-contraction if there exists a $\mathcal{D}$-function ψ such that

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi\big(\|x - y\|\big) \tag{2.1}$$

for all comparable elements $x, y \in E$, where $\psi(r) < r$ for $r > 0$. In particular if $\psi(r) = k\,r$, $\mathcal{T}$ is a partially linear contraction on E with a contraction constant k.

Theorem 2.2 (Dhage [7]). *Let $(E, \preceq, \|\cdot\|)$ be a partially ordered Banach space and let $\mathcal{T} : E \longrightarrow E$ be a nondecreasing and partial nonlinear $\mathcal{D}$-contraction. Suppose that there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$. If $\mathcal{T}$ is continuous or E is regular, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*. Moreover, the fixed point x^* is unique if every pair of elements in E has a lower and an upper bound.*

Theorem 2.3 (Dhage [8]). *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that every compact chain C in E is Janhavi. Let $\mathcal{A}, \mathcal{B} : E \longrightarrow E$ be two nondecreasing operators such that*

(a) *$\mathcal{A}$ is partially bounded and partially nonlinear $\mathcal{D}$-contraction,*
(b) *$\mathcal{B}$ is partially continuous and partially compact, and*
(c) *there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0 + \mathcal{B}x_0$.*

Then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, n=0,1,..., converges monotonically to x^*.

Remark 2.9. The compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E holds if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$. This simple fact has been utilized to prove the main results of this paper.

Note that the Dhage iteration method presented in the above hybrid fixed point theorems have been employed in Dhage and Dhage [12, 13, 14] and Dhage *et.al.* [17] for approximating the solutions of initial value problems of nonlinear first order ordinary differential equation under some natural hybrid conditions. In the following section we approximate the solutions of certain IVPs of nonlinear higher order ordinary differential equations with maxima via successive approximations.

3. Initial Value Problems

Given a closed and bounded interval $J = [t_0, t_0 + a]$ of the real line $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $t_0 \geq 0$ and $a > 0$ and given a positive integer p, consider the initial value problem (in short IVP) of p^{th} order ordinary nonlinear hybrid differential equation with maxima,

$$\left.\begin{aligned} x^{(p)}(t) &= f(t, x(t), X(t)),\ t \in J, \\ x(t_0) = \alpha_0, x'(t_0) &= \alpha_1, \ldots, x^{(p-1)}(t_0) = \alpha_{p-1}, \end{aligned}\right\} \tag{3.1}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and $X(t) = \max\limits_{t_0 \leq \xi \leq t} x(\xi)$ for $t \in J$.

By a solution of the IVP (3.1) we mean a function $x \in C^p(J, \mathbb{R})$ that satisfies equation (3.1), where $C^p(J, \mathbb{R})$ is the space of p - times continuously differentiable real-valued functions defined on J.

The differential equations with maxima occur in the regulated systems of automatic control and a few details of such equations appears in Bainov and Hristova [1]. The study of first order ordinary nonlinear differential equations is exploited in Dhage [11], Dhage and Dhage [15, 16] and Dhage and Otrocol [18] via Dhage iteration method for existence and approximation theorems. Therefore it is desirable to extend the Dhage iteration method to other nonlinear higher order differential equations. The IVP (3.1) is new to the literature and not discussed for existence as well as any other aspects of the solutions. In the present paper it is proved that the existence of the solutions may be proved under weaker partial continuity and partial compactness type conditions.

The equivalent integral form of the IVP (3.1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.2}$$

and

$$x \leq y \iff x(t) \leq y(t) \tag{3.3}$$

for all $t \in J$. Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ is regular and lattice so that every pair of elements of E has a lower and an upper bound in it.

Lemma 3.1. *Let $\big(C(J,\mathbb{R}), \leq, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3.1) and (3.2) respectively. Then every partially compact subset S of $C(J,\mathbb{R})$ is Janhavi, i.e., $\|\cdot\|$ and $\leq$ are compatible in every compact chain C in S.*

Proof. The lemma mentioned in Dhage [6, 8], but the proof appears in Dhage [10, 11] and Dhage and Dhage [12, 13, 14]. Since the proof is not well-known, we give the details of the proof for completeness. Let S be a partially compact subset of $C(J,\mathbb{R})$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \leq x_2(t) \leq \cdots \leq x_n(t) \leq \cdots , \tag{$*$}$$

for each $t \in J$.

Suppose that a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}_{k\in\mathbb{N}}$ of the monotone real sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent. By monotone characterization, the whole sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent and converges to a point $x(t)$ in $\mathbb{R}$ for each $t \in J$. This shows that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x point-wise on J. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence by Arzelá-Ascoli theorem. Hence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges uniformly to x. As a result $\|\cdot\|$ and $\leq$ are compatible in S. This completes the proof. □

We need the following definition in what follows.

Definition 3.2. A function $u \in C^{(p)}(J,\mathbb{R})$ is said to be a lower solution of the IVP (3.1) if it satisfies

$$\left.\begin{aligned} u^{(p)}(t) &\leq f(t,u(t),U(t)),\ t \in J, \\ u(t_0) \leq \alpha_0,\ u'(t_0) &\leq \alpha_1, \ldots, u^{(p-1)}(t_0) \leq \alpha_{p-1}, \end{aligned}\right\} \tag{$*$}$$

for all $t \in J$, where $U(t) = \max\limits_{t_0 \leq \xi \leq t} u(\xi)$ for $t \in J$. Similarly, an upper solution $\in C^{(p)}(J,\mathbb{R})$ to the IVP (3.1) is defined on J by the above inequalities with reverse sign.

We consider the following set of assumptions in what follows:

- (H_1) There exists a constant $M_f > 0$ such that $|f(t,x,y)| \leq M_f$ for all $t \in J$ and $x, y \in \mathbb{R}$.
- (H_2) The function $f(t,x,y)$ is monotone nondecreasing in x and y for each $t \in J$.
- (H_3) The IVP (3.1) has a lower solution $u \in C^p(J,\mathbb{R})$.

Lemma 3.3. *For a given integrable function $h : J \longrightarrow \mathbb{R}$, a function $u \in C^p(J,\mathbb{R})$ is a solution of the IVP*

$$\left.\begin{aligned} x^{(p)}(t) &= h(t),\ t \in J, \\ x(t_0) = \alpha_0, x'(t_0) &= \alpha_1, \ldots, x^{(p-1)}(t_0) = \alpha_{p-1}, \end{aligned}\right\} \tag{3.4}$$

if and only if it is a solution of the nonlinear integral equation,

$$x(t) = \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!}\, h(s)\, ds,\ t \in J. \tag{3.5}$$

Theorem 3.1. *Assume that the hypotheses (H_1) through (H_3) hold. Then the IVP (3.1) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_0 = u,$$
$$x_{n+1}(t) = \sum_{i=0}^{p-1} \frac{\alpha_i(t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x_n(s), X_n(s))\, ds, \tag{3.6}$$

for all $t \in \mathbb{R}$, converges monotonically to x^*.

Proof. By Lemma 3.3, the IVP (3.1) is equivalent to the nonlinear integral equation

$$x(t) = \sum_{i=0}^{p-1} \frac{\alpha_i(t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x(s), X(s))\, ds,\ t \in J. \tag{3.7}$$

Set $E = C(J, \mathbb{R})$ and define the operator $\mathcal{T}$ by

$$\mathcal{T}x(t) = \sum_{i=0}^{p-1} \frac{\alpha_i(t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x(s), X(s))\, ds,\ t \in J. \tag{3.8}$$

From the continuity of the integral, it follows that $\mathcal{T}$ defines the map $\mathcal{T} : E \longrightarrow E$. Then, the IVP (3.1) is equivalent to the operator equation

$$\mathcal{T}x(t) = x(t),\ \ t \in J. \tag{3.9}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.1. This is achieved in the series of following steps.

Step I: *$\mathcal{T}$ is nondecreasing on E.*

Let $x, y \in E$ be such that $x \geq y$. Then $x(t) \geq y(t)$ for all $t \in J$. Since y is continuous on $[t_0, t]$, there exists a $\xi^* \in [t_0, t]$ such that $y(\xi^*) = \max_{t_0 \leq \xi \leq t} y(\xi)$. By definition of $\leq$, one has $x(\xi^*) \geq y(\xi^*)$. Consequently, we obtain

$$\max_{t_0 \leq \xi \leq t} x(\xi) \geq x(\xi^*) \geq y(\xi^*) = \max_{t_0 \leq \xi \leq t} y(\xi).$$

Then by hypothesis (H_2), we obtain

$$\begin{aligned} \mathcal{T}x(t) &= \sum_{i=0}^{p-1} \frac{\alpha_i(t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x(s), X(s))\, ds \\ &\geq \sum_{i=0}^{p-1} \frac{\alpha_i(t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, y(s), Y(s))\, ds \\ &= \mathcal{T}y(t), \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{T}$ is nondecreasing operator on E into E.

Step II: *$\mathcal{T}$ is partially continuous on E.*

Let $\{x_n\}$ be a sequence of points of an arbitrary chain C in E such that $x_n \longrightarrow x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned} \lim_{n \longrightarrow \infty} \mathcal{T}x_n(t) &= \lim_{n \longrightarrow \infty} \left[\sum_{i=0}^{p-1} \frac{\alpha_i(t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x(s), X(s))\, ds \right] \\ &= \sum_{i=0}^{p-1} \frac{\alpha_i(t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} \left[\lim_{n \longrightarrow \infty} f(s, x_n(s), X_n(s)) \right] ds \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x(s), X(s))\, ds \\
&= \mathcal{T}x(t),
\end{aligned}
$$

for all $t \in J$. This shows that $\{\mathcal{T}x_n\}$ converges to $\mathcal{T}x$ pointwise on J.

Next, we will show that $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$
\begin{aligned}
|\mathcal{T}x_n(t_2) - \mathcal{T}x_n(t_1)| &\leq \left| \sum_{i=0}^{p-1} \frac{\alpha_i (t_2-t_0)^i}{i!} - \sum_{i=0}^{p-1} \frac{\alpha_i (t_1-t_0)^i}{i!} \right| \\
&\quad + \bigg| \int_{t_0}^{t_2} \frac{(t_2-s)^{p-1}}{(p-1)!} f(s, x_n(s), X_n(s))\, ds \\
&\qquad\quad - \int_{t_0}^{t_1} \frac{(t_1-s)^{p-1}}{(p-1)!} f(s, x_n(s), X_n(s))\, ds \bigg| \\
&\leq \sum_{i=0}^{p-1} \frac{|\alpha_i|}{i!} \big| (t_2-t_0)^i - (t_1-t_0)^i \big| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_0}^{t_2} (t_2-s)^{p-1}\, ds - \int_{t_0}^{t_1} (t_1-s)^{p-1}\, ds \right| \\
&\leq \sum_{i=0}^{p-1} \frac{|\alpha_i|}{i!} \big| (t_2-t_0)^i - (t_1-t_0)^i \big| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_0}^{t_2} (t_2-s)^{p-1}\, ds - \int_{t_0}^{t_2} (t_1-s)^{p-1}\, ds \right| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_0}^{t_2} (t_1-s)^{p-1}\, ds - \int_{t_0}^{t_1} (t_1-s)^{p-1}\, ds \right| \\
&\leq \sum_{i=0}^{p-1} \frac{|\alpha_i|}{i!} \big| (t_2-t_0)^i - (t_1-t_0)^i \big| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_0}^{t_0+a} \big| (t_2-s)^{p-1} - (t_1-s)^{p-1} \big|\, ds \right| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_1}^{t_2} \big| (t_1-s)^{p-1} \big|\, ds \right| \\
&\leq \sum_{i=0}^{p-1} \frac{|\alpha_i|}{i!} \big| (t_2-t_0)^i - (t_1-t_0)^i \big| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_0}^{t_0+a} \big| (t_2-s)^{p-1} - (t_1-s)^{p-1} \big|\, ds \right| \\
&\quad + \frac{M_f\, a^{p-1}}{(p-1)!} |t_1 - t_2| \\
&\longrightarrow 0 \quad as \quad t_1 \longrightarrow t_2,
\end{aligned}
$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{T}x_n \longrightarrow \mathcal{T}x$ is uniform and hence $\mathcal{T}$ is partially continuous on E.

Step III: *$\mathcal{T}$ is partially compact on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{T}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{T}(C)$ is uniformly bounded. Let $x \in C$ be arbitrary. Then,

$$\begin{aligned}
|\mathcal{T}x(t)| &\leq \sum_{i=0}^{p-1} \left| \frac{\alpha_i (t-t_0)^i}{i!} \right| + \left| \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x(s), X(s))\, ds \right| \\
&\leq \sum_{i=0}^{p-1} \frac{|\alpha_i|\, a^i}{i!} + \left| \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} \big| f(s, x(s), X(s)) \big|\, ds \right| \\
&\leq \sum_{i=0}^{p-1} \frac{|\alpha_i|\, a^i}{i!} + \frac{a^p}{p!} M_f \\
&= r,
\end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{T}x\| \leq r$ for all $x \in C$. Hence $\mathcal{T}(C)$ is a uniformly bounded subset of E. Next, we will show that $\mathcal{T}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned}
|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)| &\leq \left| \sum_{i=0}^{p-1} \frac{\alpha_i (t_2-t_0)^i}{i!} - \sum_{i=0}^{p-1} \frac{\alpha_i (t_1-t_0)^i}{i!} \right| \\
&\quad + \left| \int_{t_0}^{t_2} \frac{(t_2-s)^{p-1}}{(p-1)!} f(s, x(s), X(s))\, ds \right. \\
&\qquad\qquad \left. - \int_{t_0}^{t_1} \frac{(t_1-s)^{p-1}}{(p-1)!} f(s, x(s), X(s))\, ds \right| \\
&\leq \sum_{i=0}^{p-1} \frac{|\alpha_i|}{i!} \big| (t_2-t_0)^i - (t_1-t_0)^i \big| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_0}^{t_2} (t_2-s)^{p-1}\, ds - \int_{t_0}^{t_1} (t_1-s)^{p-1}\, ds \right| \\
&\leq \sum_{i=0}^{p-1} \frac{|\alpha_i|}{i!} \big| (t_2-t_0)^i - (t_1-t_0)^i \big| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_0}^{t_2} (t_2-s)^{p-1}\, ds - \int_{t_0}^{t_2} (t_1-s)^{p-1}\, ds \right| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_0}^{t_2} (t_1-s)^{p-1}\, ds - \int_{t_0}^{t_1} (t_1-s)^{p-1}\, ds \right| \\
&\leq \sum_{i=0}^{p-1} \frac{|\alpha_i|}{i!} \big| (t_2-t_0)^i - (t_1-t_0)^i \big| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_0}^{t_0+a} \big| (t_2-s)^{p-1} - (t_1-s)^{p-1} \big|\, ds \right| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_1}^{t_2} (t_1-s)^{p-1}\, ds \right| \\
&\leq \sum_{i=0}^{p-1} \frac{|\alpha_i|}{i!} \big| (t_2-t_0)^i - (t_1-t_0)^i \big| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_0}^{t_0+a} \big| (t_2-s)^{p-1} - (t_1-s)^{p-1} \big|\, ds \right|
\end{aligned}$$

$$+\frac{M_f\, a^{p-1}}{(p-1)!}|t_1-t_2|$$
$$\longrightarrow 0 \quad as \quad t_1 \longrightarrow t_2,$$

uniformly for all $x \in C$. Hence $\mathcal{T}(C)$ is compact subset of E and consequently $\mathcal{T}$ is a partially compact operator on E into itself.

Step IV: *u satisfies the operator inequality* $u \leq \mathcal{T}u$.

Since the hypothesis (H_3) holds, u is a lower solution of (3.1) defined on J. Then

$$u^{(p)}(t) \leq f(t, u(t), U(t)),\ t \in J, \tag{3.10}$$

satisfying,

$$u(t_0) \leq \alpha_0,\, u'(t_0) \leq \alpha_1, \ldots, u^{(p-1)}(t_0) \leq \alpha_{p-1}, \tag{3.11}$$

for all $t \in J$.

Integrating (3.10) from t_0 to t, we obtain

$$u^{(p-1)}(t) \leq \alpha_{p-1} + \int_{t_0}^{t} f(s, u(s), U(s)),\, ds. \tag{3.12}$$

for all $t \in J$. Again, integrating (3.12) from t_0 to t,

$$u^{(p-2)}(t) \leq \alpha_{p-2} + \alpha_{p-1}(t-t_0) + \int_{t_0}^{t} \frac{(t-s)^2}{2} f(s, u(s), U(s))\, ds$$

for all $t \in J$.

Proceeding in this way, by induction, we obtain

$$u(t) \leq \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, u(s), U(s))\, ds = \mathcal{T}u(t)$$

for all $t \in J$. This show that u is a lower solution of the operator equation $x = \mathcal{T}x$.

Thus $\mathcal{T}$ satisfies all the conditions of Theorem 2.1 in view of Remark 2.9 and we apply to conclude that the operator equation $\mathcal{T}x = x$ has a solution. Consequently the integral equation and the IVP (3.1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (3.6) converges monotonically to x^*. This completes the proof. □

Remark 3.4. The conclusion of Theorem 3.1 also remains true if we replace the hypothesis (H_3) with the following one:

(H_3') The IVP (3.1) has an upper solution $v \in C^p(J, \mathbb{R})$.

Example 3.5. Given a closed and bounded interval $J = [0, 1]$, consider the IVP,

$$\left.\begin{aligned} x^{(p)}(t) &= \tanh x(t) + \tanh X(t),\ t \in J, \\ x(0) = 0,\, x'(0) = 1, \ldots, x^{(p-1)}(0) &= \frac{1}{p-1}, \end{aligned}\right\} \tag{3.13}$$

where $X(t) = \max\limits_{0 \leq \xi \leq t} x(\xi)$.

Here, $f(t, x, y) = \tanh x + \tanh y$. Clearly, the functions f is continuous on $J \times \mathbb{R} \times \mathbb{R}$. The function f satisfies the hypothesis (H_1) with $M_f = 2$. Moreover, the function $f(t, x, y) = \tanh x + \tanh y$ is nondecreasing in x and y for each $t \in J$ and so the hypothesis (H_2) is satisfied.

Finally, the IVP (3.1) has a lower solution

$$u(t) = \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} - 2\int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!}\, ds$$

defined on J. Thus all the hypotheses of Theorem 3.1 are satisfied. Hence we conclude that the IVP (3.13) has a solution x^* defined on J and the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_0 &= \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} - 2\int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!}\, ds, \\ x_{n+1}(t) &= \sum_{i=1}^{p-1} \frac{1}{i}\cdot\frac{t^i}{i!} + \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\big[\tanh x_n(s) + \tanh x_n(s)\big]\, ds, \end{aligned} \tag{3.14}$$

for all $t \in J$, converges monotonically to x^*. □

Next, we prove the uniqueness theorem for the IVP (3.1) under weak Lipschitz condition. We need the following hypothesis in what follows.

(H_4) There exists a $\mathcal{D}$-function φ such that

$$0 \le f(t, x_1, x_2) - f(t, y_1, y_2) \le \varphi(\max\{x_1 - y_1\,,\, x_2 - y_2\})$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$, $x_1 \ge y_1$, $x_2 \ge y_2$. Moreover, $\dfrac{a^p}{p!}\varphi(r) < r$ for $r > 0$.

Theorem 3.2. *Assume that hypotheses (H_3) and (H_4) hold. Then the IVP (3.1) has a unique solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by* (3.6) *converges monotonically to x^*.*

Proof. Set $E = C(J, \mathbb{R})$. Clearly, E is a lattice w.r.t. the order relation $\le$ and so the lower and the upper bound for every pair of elements in E exist. Define the operator $\mathcal{T}$ by (3.7). Then, the IVP (3.1) is equivalent to the operator equation (3.9). We shall show that $\mathcal{T}$ satisfies all the conditions of Theorem 2.2 in E.

Clearly, $\mathcal{T}$ is a nondecreasing operator on E into itself. We shall simply show that the operator $\mathcal{T}$ is a partial nonlinear $\mathcal{D}$-contraction on E. Then, we have

$$|x(t) - y(t)| \le |X(t) - Y(t)|$$

and that

$$\begin{aligned} |X(t) - Y(t)| &= X(t) - Y(t) \\ &= \max_{t_0 \le \xi \le t} x(\xi) - \max_{t_0 \le \xi \le t} y(\xi) \\ &\le \max_{t_0 \le \xi \le t} [x(\xi) - y(\xi)] \\ &= \max_{t_0 \le \xi \le t} |x(\xi) - y(\xi)| \\ &\le \|x - y\| \end{aligned}$$

for each $t \in J$. As a result, we obtain by hypothesis (H_4),

$$\begin{aligned} |\mathcal{T}x(t) - \mathcal{T}y(t)| &\le \int_{t_0}^{t} \left| \frac{(t-s)^{p-1}}{(p-1)!} f(s, x(s), X(s)) - \frac{(t-s)^{p-1}}{(p-1)!} f(s, y(s), Y(s)) \right| ds \\ &\le \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} \left| f(s, x(s), X(s)) - f(s, y(s), Y(s)) \right| ds \end{aligned}$$

$$\leq \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!}\, \varphi\Big(\max\{|x(s)-y(s)|\,, |X(t)-Y(t)|\}\Big)\, ds$$

$$\leq \frac{(t-t_0)^p}{p!}\, \varphi(\|x-y\|)$$

$$\leq \psi(\|x-y\|) \tag{3.15}$$

for all $t \in J$, where $\psi(r) = \dfrac{a^p}{p!}\varphi(r) < r$, $r > 0$.

Taking the supremum over t, we obtain

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x-y\|)$$

for all $x, y \in E$, $x \geq y$. As a result $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-contraction on E. Now a direct application of Theorem 2.2 yields that the IVP (3.1) has a unique solution x^* and the sequence $\{x_n\}$ of successive approximations defined by (3.6) converges monotonically to x^*. □

Remark 3.6. The conclusion of Theorem 3.2 also remains true if we replace the hypothesis (H_3) with the following one:

(H_3') The IVP (3.1) has an upper solution $v \in C^p(J, \mathbb{R})$.

Example 3.7. Given a closed and bounded interval $J = [0, 1]$, consider the IVP,

$$\left.\begin{aligned} x^{(p)}(t) &= \frac{1}{2}\Big[\tan^{-1} x(t) + \tan^{-1} X(t)\Big],\ t \in J,\\ x(0) = 0,\ x'(0) &= 1, \ldots, x^{(p-1)}(0) = \frac{1}{p-1}, \end{aligned}\right\} \tag{3.16}$$

where $X(t) = \max\limits_{0 \leq \xi \leq t} x(\xi)$, $t \in J$.

Here, $f(t, x, y) = \tan^{-1} x + \tan^{-1} y$. Clearly, the functions f is continuous on $J \times \mathbb{R} \times \mathbb{R}$. The function f satisfies the hypothesis (H_1) with $M_f = \dfrac{\pi}{2}$. We show that f satisfies the hypothesis (H_4) on $J \times \mathbb{R} \times \mathbb{R}$. Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ be such that $x_1 \geq y_1$ and $x_2 \geq y_2$. Then, we have

$$\begin{aligned} 0 &\leq f(t, x_1, x_2) - f(t, y_1, y_2)\\ &\leq \frac{1}{2}\big[\tan^{-1} x_1 - \tan^{-1} y_1\big] + \frac{1}{2}\big[\tan^{-1} x_2 - \tan^{-1} y_2\big]\\ &\leq \varphi\Big(\max\{x_1 - y_1\,, x_2 - y_2\}\Big) \end{aligned}$$

for all $t \in J$, where φ is a $\mathcal{D}$-function defined by $\varphi(r) = \dfrac{r}{1+\xi^2} < r$, $0 < \xi < r$.

Finally, the IVP (3.1) has a lower solution

$$u(t) = \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} - 2\int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!}\, ds$$

defined on J. Thus, if $\dfrac{a^p}{p!} \cdot \dfrac{1}{1+\xi^2} < 1$ for each $\xi > 0$, then all the hypotheses of Theorem 3.2 are satisfied. Hence we conclude that the IVP (3.16) has a unique solution x^* defined on J and the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_0 &= \sum_{i=0}^{p-1} \frac{\alpha_i(t-t_0)^i}{i!} - 2\int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!}\, ds, \\ x_{n+1}(t) &= \sum_{i=1}^{p-1} \frac{1}{i}\cdot\frac{t^i}{i!} + \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\, \tan^{-1} x_n(s)\, ds, \end{aligned} \tag{3.17}$$

for all $t \in J$, converges monotonically to x^*.

4. HDE of Linear Perturbations of First Type

Next, with usual notation, we consider the following nonlinear hybrid differential equation with maxima,

$$\left.\begin{aligned} x^{(p)}(t) &= f(t, x(t), X(t)) + g(t, x(t), X(t)),\ t \in J, \\ x(t_0) &= \alpha_0, x'(t_0) = \alpha_1, \ldots, x^{(p-1)}(t_0) = \alpha_{p-1}, \end{aligned}\right\} \tag{4.1}$$

where $f, g : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions and $X(t) = \max\limits_{t_0 \le \xi \le t} x(\xi)$.

By a *solution* of the IVP (4.1) we mean a function $x \in C^p(J, \mathbb{R})$ that satisfies equation (4.1), where $C^p(J, \mathbb{R})$ is the space of p-times continuously differentiable real-valued functions defined on J.

The IVP (4.1) is a hybrid differential equation with a linear perturbation of first type. See Dhage [5], Krasnoselskii [21] and the references therein. The IVP (4.1) is new to the literature and not discussed for existence as well asany other aspects of the solutions. In the present discussion, it is proved that the existence and approximation of the solutions may be proved under mixed partial Lipschitz and partial compactness type conditions.

We need the following definition in what follows.

Definition 4.1. A function $u \in C^{(p)}(J, \mathbb{R})$ is said to be a lower solution of the IVP (4.1) if it satisfies

$$\left.\begin{aligned} u^{(p)}(t) &\le f(t, u(t), U(t)) + g(t, u(t), U(t)),\ t \in J, \\ u(t_0) &\le \alpha_0,\ u'(t_0) \le \alpha_1, \ldots, u^{(p-1)}(t_0) \le \alpha_{p-1}, \end{aligned}\right\} \tag{**}$$

where $U(t) = \max\limits_{t_0 \le \xi \le t} u(\xi)$ for $t \in J$. Similarly, an upper solution v to the IVP (4.1) is defined on J, by the above inequalities with reverse sign.

We consider the following set of assumptions in what follows:

- (H_5) There exists a constant $M_g > 0$ such that $|g(t, x, y)| \le M_g$ for all $t \in J$ and $x, y \in \mathbb{R}$.
- (H_6) The mapping $g(t, x, y)$ is monotone nondecreasing in x and y for each $t \in J$.
- (H_7) The IVP (4.1) has a lower solution $u \in C^p(J, \mathbb{R})$.

Our main existence and approximation theorem for the IVP (4.1) is as follows.

Theorem 4.1. *Assume that the hypotheses (H_1) and (H_4) through (H_7) hold. Then the IVP (4.1) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive* *approximations defined by*

$$
\begin{aligned}
x_0 &= u,\\
x_{n+1}(t) &= \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x_n(s), X_n(s))\, ds \\
&\quad + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} g(s, x_n(s), X_n(s))\, ds,
\end{aligned} \tag{4.2}
$$

for all $t \in J$, *converges monotonically to* x^*.

Proof. By Lemma 3.3, the IVP (4.1) is equivalent to the nonlinear integral equation

$$
\begin{aligned}
x(t) &= \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x(s), X(s))\, ds \\
&\quad + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} g(s, x_n(s), X(s))\, ds, \ t \in J.
\end{aligned} \tag{4.3}
$$

Set $E = C(J, \mathbb{R})$ and define the operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$
\mathcal{A}x(t) = \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x(s), X(s))\, ds, \ t \in J, \tag{4.4}
$$

and

$$
\mathcal{B}x(t) = \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} g(s, x(s), X(s))\, ds, \ t \in J. \tag{4.5}
$$

From the continuity of the integrals, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the operators $\mathcal{A}, \mathcal{B} : E \longrightarrow E$. Now, the IVP (4.1) is equivalent to the operator equation

$$
\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \ t \in J. \tag{4.6}
$$

Then following the arguments similar to those given in Theorems 3.1 and 3.2, it can be shown that the operator $\mathcal{A}$ is partially bounded and partial nonlinear $\mathcal{D}$-contraction and $\mathcal{B}$ is partially continuous and partially compact operator on E into E. Now by a direct application of Theorem 2.3 we conclude that the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^*. Consequently the IVP (4.1) has a solution x^* and the sequence $\{x_n\}_{n=1}^{\infty}$ defined by (4.2) converges monotonically to x^*. This completes the proof. □

The conclusion of Theorems 4.1 also remains true if we replace the hypothesis (H_7) with the following one:

(H'_7) The IVP (4.1) has an upper solution $v \in C^p(J, \mathbb{R})$.

Example 4.2. Given a closed and bounded interval $J = [0, 1]$, consider the IVP,

$$
\left.
\begin{aligned}
x^{(p)}(t) &= \tan^{-1} x(t) + g(t, x(t), X(t)), \ t \in J, \\
x(0) = 0,\ x'(0) &= 1, \ldots, x^{(p-1)}(0) = \frac{1}{p-1},
\end{aligned}
\right\} \tag{4.7}
$$

where $X(t) = \max_{0 \le \xi \le t} x(\xi)$ and $g : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function defined by

$$
g(t, x, y) = \begin{cases} 1, & \text{if } y \le 0, \\ 1 + \dfrac{y}{1+y}, & \text{if } x > 0. \end{cases}
$$

for $t \in J$.

Here, $f(t,x,y) = \tan^{-1} x$. The function f satisfies the hypothesis (H_1) with $M_f = \dfrac{\pi}{2}$. Moreover, f satisfies (H_4) with $\varphi(r) = \dfrac{r}{1+\xi^2} < r$, $0 < \xi < r$. Next, g satisfies (H_5) with $M_g = 2$. Again, the function $y \mapsto g(t,x,y)$ is nondecreasing in y for each $t \in J$ and so the hypothesis (H_6) is satisfied. Finally, the function

$$u(t) = \sum_{i=1}^{p-1} \frac{1}{i} \cdot \frac{t^i}{i!} - 2 \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\, ds,$$

for all $t \in J$ is a lower solution of the IVP (4.7) on J. Hence, if $\dfrac{a^p}{p!} \cdot \dfrac{1}{1+\xi^2} < 1$ for each $\xi > 0$, we apply Theorem 4.1 and conclude that the IVP (4.7) has a solution x^* on J and the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$\begin{aligned} x_1 &= \sum_{i=1}^{p-1} \frac{1}{i} \cdot \frac{t^i}{i!} - 2 \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\, ds, \\ x_{n+1}(t) &= \sum_{i=1}^{p-1} \frac{1}{i} \cdot \frac{t^i}{i!} + \int_0^t \frac{(t-s)^{p-1}}{(p-1)!} \tan^{-1} x_n(s))\, ds \\ &\quad + \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\, g(s, x_n(s), X_n(s))\, ds, \end{aligned}$$

for each $t \in J$ converges monotonically x^*.

Example 4.3. Given a closed and bounded interval $J = [0,1]$ and given a positive integer p, consider the IVP,

$$\left.\begin{aligned} &x^{(p)}(t) = \tan^{-1} x(t) + \tanh X(t), \ t \in J, \\ &x(0) = 0,\ x'(0) = 1, \dots, x^{(p-1)}(0) = \frac{1}{p-1}. \end{aligned}\right\} \tag{4.8}$$

where $X(t) = \max\limits_{0 \le \xi \le t} x(\xi)$.

Here $f(t,x,y) = \tan^{-1} x$ and $g(t,x,y) = \tanh y$ for all $t \in J$ and $x, y \in \mathbb{R}$. Then proceeding with the arguments that given in Examples 3.5 and 3.7, it is proved that the function f satisfies the hypotheses (H_1) and (H_4) and g satisfies the hypotheses (H_5)-(H_6) on $J \times \mathbb{R} \times \mathbb{R}$. Similarly, the hypothesis (H_7) is held with a lower solution u given by

$$u(t) = \sum_{i=1}^{p-1} \frac{1}{i} \cdot \frac{t^i}{i!} - 2 \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\, ds - \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\, ds,$$

for all $t \in J$. Hence, if $\dfrac{a^p}{p!} \cdot \dfrac{1}{1+\xi^2} < 1$ for each $\xi > 0$, we apply Theorem 4.1 and prove that the IVP (4.8) has a solution x^* on J and the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$\begin{aligned} x_1 &= \sum_{i=1}^{p-1} \frac{1}{i} \cdot \frac{t^i}{i!} - 2 \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\, ds - \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\, ds, \\ x_{n+1}(t) &= \sum_{i=1}^{p-1} \frac{1}{i} \cdot \frac{t^i}{i!} + \int_0^t \frac{(t-s)^{p-1}}{(p-1)!} \tan^{-1} x_n(s))\, ds \\ &\quad + \int_0^t \frac{(t-s)^{p-1}}{(p-1)!} \tanh x_n(s))\, ds, \end{aligned} \tag{4.9}$$

for each $t \in J$ converges monotonically x^*.

Remark 4.4. The study of the present paper may be extended with essentially the same approach but under appropriate modifications to the general higher order IVPs of the type

$$\left.\begin{aligned} \mathcal{L}x(t) = \mathcal{N}x(t)),\ t \in J, \\ x(t_0) = \alpha_0, x'(t_0) = \alpha_1, \dots, x^{(n-1)}(t_0) = \alpha_{n-1}, \end{aligned}\right\} \tag{4.10}$$

where $\mathcal{L}$ is a linear n^{th} order differential operator of the type

$$\mathcal{L} = a_0 D^n + a_1 D^{n-1} + \cdots + a_n$$

and $\mathcal{N}$ is a Nemytsky operator defined by

$$\mathcal{N}x(t) = f\big(t, x(t), x'(t), \dots, x^{n-1}(t), X(t)\big),\ t \in J,$$

for some constants $a_0, \dots, a_n$, where $X(t) = \max_{t_0 \le \xi \le t} x(\xi)$.

5. Conclusion

From the foregoing discussion it is clear that unlike Schauder fixed point principle, the proofs of Theorems 3.1 and 4.1 do not invoke the construction of a nonempty, closed, convex and bounded subset of the Banach space of navigation which is mapped into itself by the operators related to the given differential equations with maxima. The convexity hypothesis is altogether omitted from the discussion and still we have proved the existence and approximation of the solutions for the differential equations with maxima considered in this paper with stronger conclusion. Similarly, unlike the use of Banach fixed point theorem, Theorems 3.1 and 4.1 do not make any use of usual Lipschitz condition on the nonlinearity involved in the differential equations with maxima (3.1) and (4.1), but even then the algorithms for the solutions of the differential equations with maxima (3.1) and (4.1) are proved in terms of the new Dhage iteration scheme which is different from Picard iterations. Again, unlike Tarski fixed point theorem, we do not need the partially ordered space under consideration to be a complete lattice, however one needs the additional condition of regularity together with partially continuity of the mappings on it. But the advantage of our results over Tarski lies in the fact that we obtain the algorithms for the solutions of the considered nonlinear problems. Unlike Picard iteration method for the nonlinear differential equations with maxima, the new so called Dhage iteration method does not start with the initial data but starts at the given lower or upper solution of the related problems and uses compactness type arguments instead of Lipschitz condition which is usually needed for the Picard iteration scheme. The advantage of hybrid fixed point theoretic techniques over classical ones is that we obtain the algorithms along with the existence of solutions with strong conclusion of monotone convergence of the algorithms to the solutions. The nature of the convergence of the algorithms is not geometrical and so we are not able to obtain the rate of convergence of the algorithms to the solutions of the related problems. However, in a way we have been able to prove the existence as well as approximation results for the IVPs (3.1) and (4.1) under much weaker conditions with a stronger conclusion of the monotone convergence of successive approximations to the solution than those proved in the existing literature on nonlinear differential equations with maxima. The study of this paper may be extended to other nonlinear higher order differential equations such as higher order boundary value problems with appropriate modifications and some of the results in this directions will be reported elsewhere.

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