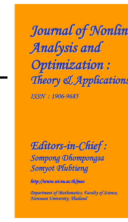

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ON THE SEMILOCAL CONVERGENCE OF A TWO STEP NEWTON METHOD UNDER THE $\gamma-$CONDITION

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ABSTRACT. We present a semilocal convergence analysis of a two-step Newton method using the $\alpha-$theory in order to approximate a locally unique solution of an equation in a Banach space setting. The new idea uses a combination of center-γ as well as a $\gamma-$ condition in the convergence analysis. This convergence criteria are weaker than the corresponding ones in the literature even in the case of the single step Newton method [3, 14, 15, 16, 17, 18, 19, 20]. Numerical examples involving a nonlinear integral equation where the older convergence criteria are not satisfied but the new convergence criteria are satisfied, are also presented in the paper.

KEYWORDS : Two-step Newton method, Newton method, Banach space, $\alpha-$ theory, semi-local convergence, γ-condition, Fréchet-derivative.
AMS Subject Classification:65G99, 65J15, 47H17, 49M15.

1. Introduction

Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces. Let $U(x,r)$ and $\overline{U}(x,r)$ stand, respectively, for the open and closed ball in $\mathcal{X}$ with center x and radius $r>0$. Denote by $\mathcal{L}(\mathcal{X},\mathcal{Y})$ the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$. In the present paper we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x)=0, \tag{1.1}$$

where F is a Fréchet continuously differentiable operator defined on $\overline{U}(x_0,R)$ for some $R>0$ with values in $\mathcal{Y}$.

A lot of problems from Computational Sciences and other disciplines can be brought in the form of equation (1.1) using Mathematical Modelling [5, 7, 8, 12, 13, 16, 17, 20]. The solution of these equations can rarely be found in closed

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form. That is why the solution methods for these equations are iterative. In particular, the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method [1]–[21]. The study about convergence matter of Newton methods is usually centered on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of Newton methods; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. We find in the literature several studies on the weakness and/or extension of the hypothesis made on the underlying operators. There is a plethora on local as well as semil-local convergence results, we refer the reader to [1]–[21]. The most famous among the semi-local convergence of iterative methods is the celebrated Kantorovich theorem for solving nonlinear equations. This theorem provides a simple and transparent convergence criterion for operators with bounded second derivatives F'' or the Lipschitz continuous first derivatives [2, 7, 10, 12, 13, 21]. Another important theorem inaugurated by Smale at the International Conference of Mathematics (cf. [16, 17]), where the concept of an approximate zero was proposed and the convergence criteria were provided to determine an approximate zero for analytic function, depending on the information at the initial point. Wang [19] generalized Smale's result by introducing the γ-condition (see $(\mathcal{H}_2)$). For more details on Smale's theory, the reader can refer to the excellent Dedieu's book [9, Chapter 3.3].

The two-step Newton's method defined by

$$\begin{array}{l} x_0 \text{ is an initial point} \\ y_n = x_n - F'(x_n)^{-1}\,F(x_n), \\ x_{n+1} = y_n - F'(x_n)^{-1}\,F(y_n) \qquad \text{for each} \quad n = 0, 1, 2, \cdots \end{array} \tag{1.2}$$

is the most popular cubically convergent iterative process for generating a sequence $\{x_n\}$ approximating $x^\star$. Here, $F'(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ denotes the Fréchet-derivative of F at $x \in \overline{U}(x_0, R)$ [2, 5, 7]. However, the convergence domain of (1.2) is usually very small. That is why it is important to extend the convergence domain without additional hypotheses.

In the present paper, motivated by the preceding observation and optimization considerations, we expand the applicability of Newton's method under the γ-condition by introducing the notion of the center γ_0-condition (to be precised in Definition $(\mathcal{H}_3)$) for some $\gamma_0 \leq \gamma$. This way we obtain tighter upper bounds on the norms of $\parallel F'(x)^{-1}\,F'(x_0) \parallel$ for each $x \in \overline{U}(x_0, R)$ (see $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$) leading to tighter majorizing sequences and more precise information on the location of the solution x^* than in earlier studies such as [3, 14, 15, 16, 17, 18, 19, 20] (see in particular, (3.3)), (3.7), Remark 3.4, Theorem 3.5 and the numerical examples in Section 4). The approach of introducing center-Lipschitz condition has already been fruitful for expanding the applicability of Newton's method under the Kantorovich-type theory [1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 21].

The rest of the paper is organized as follows: section 2 contains results on majorizing sequences. In section 3 we present the semi-local convergence analysis of (1.2). Applications and numerical examples are given in the concluding section 4.

2. Majorizing Sequences

In this section we introduce some scalar sequences that shall be shown to be majorizing for Newton's method in Section 3.

Let $\beta > 0,\ \gamma_0 > 0$ and $\gamma > 0$ be given. Define functions f_0 on $[0, \frac{1}{\gamma_0}]$ and f on $[0, \frac{1}{\gamma}]$ by

$$f_0(t) = \beta - t + \frac{\gamma_0 t^2}{1-\gamma_0 t} \tag{2.1}$$

and

$$f(t) = \beta - t + \frac{\gamma t^2}{1-\gamma t}. \tag{2.2}$$

Moreover, define scalar sequences $\{t_n\}$ and $\{s_n\}$ for each $n = 0, 1, 2, \cdots$ by

$$\begin{aligned} & t_0 = 0, \\ & s_n = t_n - f'(t_n)^{-1} f(t_n), \\ & t_{n+1} = s_n - f'(t_n)^{-1} f(s_n) \qquad \text{for each} \quad n = 0, 1, 2, \cdots . \end{aligned} \tag{2.3}$$

Notice that by direct algebraic manipulation these sequences can equivalently be written as

$$\begin{aligned} s_0 = 0, &\qquad s_0 = \beta,\ \ t_1 = s_0 - f'(t_0)^{-1} f(s_0), \\ s_{n+1} &= t_{n+1} - \frac{f(t_{n+1}) - f(s_n) - f'(t_n)(t_{n+1} - s_n)}{f'(t_{n+1})}, \\ &= t_{n+1} + \frac{\gamma(t_{n+1} - s_n)^2}{(2 - \frac{1}{(1-\gamma t_{n+1})^2})(1-\gamma t_{n+1})(1-\gamma s_n)^2} \\ t_{n+2} &= s_{n+1} - \frac{f(s_{n+1}) - f(t_{n+1}) - f'(t_{n+1})(s_{n+1} - t_{n+1})}{f'(t_{n+1})}, \\ &= s_{n+1} + \frac{\gamma(s_{n+1} - t_{n+1})^2}{(2 - \frac{1}{(1-\gamma t_{n+1})^2})(1-\gamma s_{n+1})(1-\gamma t_{n+1})^2}. \end{aligned}$$

Then, we can show the first result for majorizing sequences.

LEMMA 2.1. Suppose that

$$\alpha := \beta\gamma \le 3 - 2\sqrt{2}. \tag{2.4}$$

Then, the following items hold

(a) [19] Function f has two real zeros given by

$$t^* = \frac{1+\alpha - \sqrt{(1+\alpha)^2 - 8\alpha}}{4\gamma},\quad t^{**} = \frac{1+\alpha + \sqrt{(1+\alpha)^2 - 8\alpha}}{4\gamma}$$

which satisfy

$$\beta \le t^* \le (1 + \frac{1}{\sqrt{2}})\beta \le (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma} \le t^{**} \le \frac{1}{2\gamma}.$$

(b) Sequences $\{t_n\}$ and $\{s_n\}$ are increasingly convergent to t^* and satisfy for each $n = 0, 1, 2, \cdots$

$$0 \le t_n \le s_n \le t_{n+1} < t^*.$$

Proof. (b) Let us define functions g and g_1 by

$$g(x) = x - \frac{f(x)}{f'(x)} \ \text{ and } \ g_1(x) = g(x) - \frac{f(g(x))}{f'(x)}.$$

Then, it follows that $g(x)$ is strictly increasing on $[0, t^*)$, since

$$g'(x) = \frac{f(x)f''(x)}{f'(x)^2},\ \ f(x) > 0,\ \ f'(x) < 0,$$

$f''(x) > 0$ and $f''(x)$ is strictly increasing for $x \in [0, t^*)$. We also have that

$$\begin{aligned} g_1'(x) &= g'(x) - \frac{f'(g(x))g'(x)f'(x) - f(g(x))f''(x)}{f'(x)^2} \\ &= \frac{f(x)^2 f''(x) f''(\bar{x}) + f'(x)^2 f(g(x)) f''(x)}{f'(x)^4} > 0 \end{aligned}$$

for $\bar{x} \in (x, g(x))$, $x < g(x) < g(t^*) = t^*$, if $x \in [0, t^*)$. Therefore, g_1 is strictly increasing on $[0, t^*)$ and $g(x) < g_1(x) < g_1(t^*) = t^*$. The result now follows by induction if we set $t_0 = 0 < t^*$, $s_n = g(t_n)$ and $t_{n+1} = g_1(t_n)$ for each $n = 0, 1, 2 \cdots$.

□

Moreover, we define scalar sequences $\{q_n\}$, $\{r_n\}$ by

$$\begin{aligned} q_0 = 0, \qquad & r_0 = \beta, \ \ q_1 = r_0 - f_0'(q_0)^{-1} f(r_0), \\ r_{n+1} &= q_{n+1} - \frac{f(q_{n+1}) - f(r_n) - f'(q_n)(q_{n+1} - r_n)}{f_0'(q_{n+1})}, \\ &= q_{n+1} + \frac{\gamma(q_{n+1} - r_n)^2}{(2 - \frac{1}{(1-\gamma_0 q_{n+1})^2})(1 - \gamma q_{n+1})(1 - \gamma r_n)^2} \qquad (2.5) \\ q_{n+2} &= r_{n+1} - \frac{f(r_{n+1}) - f(q_{n+1}) - f'(q_{n+1})(r_{n+1} - q_{n+1})}{f_0'(q_{n+1})}, \\ &= r_{n+1} + \frac{\gamma(r_{n+1} - q_{n+1})^2}{(2 - \frac{1}{(1-\gamma_0 q_{n+1})^2})(1 - \gamma r_{n+1})(1 - \gamma q_{n+1})^2}. \end{aligned}$$

Next, we compare sequences $\{s_n\}$, $\{t_n\}$ with $\{q_n\}$, $\{r_n\}$ under convergence criterion (2.4).

LEMMA 2.2. Suppose that (2.4) and

$$\gamma_0 \le \gamma \tag{2.6}$$

hold. Then, the following items hold for each $n = 0, 1, 2, \cdots$

$$\begin{aligned} 0 \le q_n &\le t_n && (2.7) \\ 0 \le r_n &\le s_n && (2.8) \\ q_n \le r_n &\le q_{n+1} && (2.9) \end{aligned}$$

and

$$q^* := \lim_{n \to \infty} q_n \le t^*. \tag{2.10}$$

Moreover, strict inequality holds if $\gamma_0 < \gamma$ for each $n = 1, 2, 3, \cdots$ in (2.7) and (2.9) and for each $n = 2, 3, \cdots$ in (2.8).

Proof. Using a simple induction argument and the definition of these sequences estimates (2.7)- (2.9) follows. We then have that sequences $\{q_n\}$ and $\{r_n\}$ are increasing, bounded above by t^* and as such they converge to their unique least upper bound denoted by q^* which satisfies (2.10).

□

Notice that sequences $\{s_n\}$, $\{t_n\}$ appear in the study of two step Newton methods in connection to the $\gamma-$theory and criterion (2.4) [1]-[9], [14]-[19]. So far we showed that sequences $\{q_n\}$, $\{r_n\}$ are tighter than $\{s_n\}$, $\{t_n\}$ under criterion (2.4).

However, a direct approach to the study of the convergence of sequences $\{q_n\}$, $\{r_n\}$ leads to weaker convergence criterion than (2.4). The proof of the next result can be found in [8, Theorem 2.1 (i)].

LEMMA 2.3. Let $\lambda = \frac{\gamma_0}{\gamma}$. Denote by $\rho,\ \rho_1$ the small zeroes in $(0,1)$ of the polynomials $P_\lambda(t) = 2\sqrt{2}\lambda^3 t^4 + (3-7\sqrt{2}-2\sqrt{2}\lambda)\lambda^2 t^3 + (7\sqrt{2}-6+2(3\sqrt{2}-1)\lambda)\lambda t^2 - (2(\sqrt{2}-1)+(5\sqrt{2}-4)\lambda)t + \sqrt{2}-1$ and $P^1_\lambda(t) = 2\lambda^2 t^4 - (4+5\lambda)\lambda t^3 + (1+10\lambda+2\lambda^2)t^2 - (3+4\lambda)t + 1$, respectively. Suppose that

$$\alpha \le \begin{cases} \rho_1, & if\ \frac{\gamma_0}{\gamma} \le 1 - \frac{1}{\sqrt{2}} \\ \rho_2 := \min\{(1-\frac{1}{\sqrt{2}})\frac{\gamma}{\gamma_0}, \rho\}, & if\ \frac{\gamma_0}{\gamma} > 1 - \frac{1}{\sqrt{2}} \end{cases} \tag{2.11}$$

Inequality (2.11) must be strict if $(1-\frac{1}{\sqrt{2}})\frac{\gamma}{\gamma_0} \le \rho$. Then, scalar sequences $\{q_n\},\ \{r_n\}$ are increasingly convergent,

$$0 \le q_n \le r_n \le q_{n+1}$$

and

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty} r_n = q^*.$$

Next, we compsre the right hsnd side of inequality (2.11) for $\lambda \in [0.0001, 1]$ to the right hand side of inequality (2.4). We observe from Table 1 that our approach

λ	Right hand side of (2.11)	Right hand side of (2.4) $\delta_0 = 3 - 2\sqrt{2}$
0.0001	0.3819529609564926	0.17157287525380990
0.001	0.3818354384029206	0.17157287525380990
0.01	0.3806532896793318	0.17157287525380990
0.1	0.3681420045094538	0.17157287525380990
0.15	0.3606611353927973	0.17157287525380990
0.2	0.3528242051436774	0.17157287525380990
0.25	0.3446613843095571	0.17157287525380990
0.26	0.3429931090685413	0.17157287525380990
0.27	0.3413137650216341	0.17157287525380990
0.28	0.3396237410848502	0.17157287525380990
0.29	0.3379234409140188	0.17157287525380990
$1-\dfrac{1}{\sqrt{2}}$	0.3374296493260468	0.17157287525380909
0.3	0.33447804873307100	0.17157287525380990
0.4	0.29722914975127396	0.17157287525380990
0.5	0.26682799202395086	0.17157287525380990
0.6	0.24178390124881075	0.17157287525380990
0.7	0.22090983862630980	0.17157287525380990
0.8	0.20330124076393735	0.17157287525380990
0.9	0.18827676080151223	0.17157287525380990
0.99	0.17653626898768845	0.17157287525380990
0.999	0.17544263627916407	0.17157287525380990
1	0.17157287525380990	0.17157287525380990

TABLE 1. Comparison Table

extends the applicability of the two step Newton's method (1.2).

3. SEMILOCAL CONVERGENCE

We present semilocal convergence results for the two step Newton-like method (1.2) in this section.

First, we need an auxiliary Ostrowski-type representation for operator F [2, 7, 11, 12].

LEMMA 3.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $D \subset \mathcal{X}$ be open and convex and $F : D \to \mathcal{Y}$ be twice continuously Fréchet differentiable. Moreover, suppose that sequences $\{x_n\}$ and $\{y_n\}$ generated by two step Newton-method (1.2) are well defined. Then, the following items hold for each $n = 0, 1, 2, \cdots$.

$$F(x_{n+1}) = \int_0^1 \int_0^1 F''(x_n+\theta(y_n-x_n+t(x_{n+1}-y_n)))d\theta(y_n-x_n+t(x_{n+1}-y_n))dt(x_{n+1}-y_n) \tag{3.1}$$

and

$$F(y_n) = \int_0^1 F''(x_n + t(y_n - x_n))(1-t)dt(y_n - x_n)^2. \tag{3.2}$$

Proof. Using two-step Newton method (1.2), we get in turn that

$$\begin{aligned} F(x_{n+1}) &= F(x_{n+1}) - F(y_n) - F'(x_n)(x_{n+1} - y_n) \\ &= \int_0^1 F'(y_n + t(x_{n+1} - y_n))dt(x_{n+1} - y_n) - \int_0^1 F'(x_n)dt(x_{n+1} - y_n) \\ &= \int_0^1 [F'(y_n + t(x_{n+1} - y_n)) - F'(x_n)]dt(x_{n+1} - y_n) \\ &= \int_0^1 \int_0^1 F''(x_n + \theta(y_n - x_n + t(x_{n+1} - y_n)))d\theta(y_n - x_n + t(x_{n+1} - y_n))dt(x_{n+1} - y_n), \end{aligned}$$

which shows (3.1). Similarly, we get that

$$\begin{aligned} F(y_n) &= F(y_n) - F(x_n) - F'(x_n)(y_n - x_n) \\ &= \int_0^1 F'(x_n + t(y_n - x_n))dt(y_n - x_n) - F'(x_n)(y_n - x_n) \\ &= \int_0^1 F''(x_n + t(y_n - x_n))(1-t)dt(y_n - x_n)^2, \end{aligned}$$

which shows (3.2).

□

We shall show the main semilocal convergence result for the two-step Newton method (1.2) under conditions:

($\mathcal{H}$) Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $R > 0$, $D \subseteq \mathcal{X}$ open and convex and $F : D \to \mathcal{Y}$ be continuously twice- Fréchet-differentiable on $int(D)$. Let $x_0 \in int(D)$ with $F'(x_0)^{-1} \in L(\mathcal{Y}, \mathcal{X})$. Let $\gamma_0 > 0$, $\gamma > 0$ with $\gamma_0 \leq \gamma$ and $\beta > 0$. Set $R_0 = \min\{\frac{1}{\gamma}, (1 - \frac{1}{\sqrt{2}}\frac{1}{\gamma_0}\}$. Suppose:

($\mathcal{H}_1$): $R_0 \leq R, \quad U(x_0, R) \subseteq D$
and
$\|F'(x_0)^{-1}F(x_0)\| \leq \beta$;

($\mathcal{H}_2$): Operator F satisfies the γ-Lipschitz condition at x_0
$\|F'(x_0)^{-1}F''(x)\| \leq \dfrac{2\gamma}{(1-\gamma\|x-x_0\|)^3} = f''(\|x - x_0\|)$ for each $x \in U(x_0, R_0)$;

($\mathcal{H}_3$): Operator F satisfies the γ_0-Lipschitz condition at x_0
$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq \dfrac{\gamma_0(2-\gamma_0\|x-x_0\|)\|x-x_0\|}{(1-\gamma_0\|x-x_0\|)^2} = 1+f_0'(\|x - x_0\|)$ for each $x \in U(x_0, R_0)$;

($\mathcal{H}_4$): Condition (2.11) holds.

We need the following Banach Lemma on invertible operators.

LEMMA 3.2. Suppose that condition $(\mathcal{H}_3)$ holds. Then $F'(x)^{-1} \in L(\mathcal{Y}, \mathcal{X})$ and

$$\|F'(x)^{-1}F'(x_0)\| \leq -f_0'(\|x - x_0\|)^{-1}. \tag{3.3}$$

Proof. Using $(\mathcal{H}_3)$ and (2.1) we have in turn that

$$\begin{aligned}
\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| &\leq \frac{\gamma_0(2 - \gamma_0\|x - x_0\|)\|x - x_0\|}{(1 - \gamma_0\|x - x_0\|)^2} \\
&= \int_0^1 f_0''(t\|x - x_0\|)dt\|x - x_0\| \qquad (3.4) \\
&= f_0'(\|x - x_0\|) - f_0'(0) \\
&= f_0'(\|x - x_0\|) + 1 < 1,
\end{aligned}$$

since $f_0'(t) < 0$, if $0 \leq t < R_0$. It then follows from (3.4) and the Banach Lemma on invertible operators [2, 7, 11, 12] that $F'(x)^{-1} \in L(\mathcal{Y}, \mathcal{X})$ so that (3.3) holds.

□

Using the above auxiliary results and notation, we can show the main semilocal convergence result for two step Newton method (1.2) under the $(\mathcal{H})$ conditions.

THEOREM 3.3. Suppose that the $(\mathcal{H})$ conditions hold. Then, sequence $\{x_n\}$ generated by two-step Newton method (1.2) is well defined, remains in $\overline{U}(x_0, q^*)$ for each $n = 0, 1, 2, \cdots$ and converges to a unique solution x^* of equation $F(x) = 0$ in $\overline{U}(x_0, (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma_0})$. Moreover, the following estimates hold for each $n = 0, 1, 2, \cdots$

$$\|x_n - x^*\| \leq q^* - r_n, \tag{3.5}$$

where q^* and $\{r_n\}$ were defined in (2.5).

Proof. We shall show the following items using induction

$$\begin{aligned}
\|x_k - x_0\| &\leq q_k, \\
\|F'(x_k)^{-1}F'(x_0)\| &\leq -f_0'(q_k)^{-1}; \\
\|y_k - x_k\| &\leq r_k - q_k, \\
\|y_k - x_0\| &\leq r_k, \\
\|x_{k+1} - y_k\| &\leq q_{k+1} - r_k.
\end{aligned}$$

The preceding items hold for $k = 0$ by the initial conditions. Suppose these estimates hold for all $n \leq k$. Then, we have that

$$\|x_{k+1} - x_0\| \leq \|x_{k+1} - y_k\| + \|y_k - x_0\| \leq q_{k+1} - r_k + r_k = q_{k+1}.$$

Using Lemma 3.2, we have that

$$\|F'(x_{k+1}^{-1}F'(x_0)\| \leq -f_0'(\|x_{k+1} - x_0\|)^{-1} \leq -f_0'(q_{k+1})^{-1}.$$

Using $(\mathcal{H}_2)$ and the definitions of function f and the sequences we get in turn that

$$\begin{aligned}
\|F'(x_0)^{-1}F(x_{k+1})\| &\leq \int_0^1 \int_0^1 \|F'(x_0)^{-1}F''(x_k + \theta(y_k - x_k + t(x_{k+1} - y_k)))\|d\theta \\
&\quad \|(y_k - x_k + t(x_{k+1} - y_k))\|dt\|(x_{k+1} - y_k)\| \\
&\leq \int_0^1 \int_0^1 f''(\|x_k - x_0 + \theta(y_k - x_k + t(x_{k+1} - y_k)))\|d\theta \\
&\quad \|(y_k - x_k + t(x_{k+1} - y_k))\|dt\|(x_{k+1} - y_k)\| \\
&\leq \int_0^1 \int_0^1 \|f''(q_k + \theta(r_k - q_k + t(q_{k+1} - r_k)))d\theta
\end{aligned}$$

$$\begin{aligned}
& (r_k - q_k + t(q_{k+1} - r_k))dt(q_{k+1} - r_k) \\
= & \int_0^1 f'(r_k + t(q_{k+1} - r_k)) - f'(q_k)(q_{k+1} - r_k) \\
= & f(q_{k+1}) - f(r_k) - f'(q_k)(q_{k+1} - r_k). \qquad (3.6)
\end{aligned}$$

Then, the preceding estimates and (1.2) give that

$$\begin{aligned}
\|y_{k+1} - x_{k+1}\| &= \| - F'(x_{k+1})^{-1}F(x_{k+1})\| \\
&\leq \|F'(x_{k+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{k+1})\| \\
&\leq -f_0'(q_{k+1})^{-1}f(q_{k+1}) = r_{k+1} - q_{k+1} \\
\|y_{k+1} - x_0\| &\leq \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - x_0\| \leq r_{k+1}.
\end{aligned}$$

Moreover, we also have that

$$\begin{aligned}
\|F'(x_0)^{-1}F(y_k)\| &\leq \int_0^1 \|F'(x_0)^{-1}F''(x_k + t(y_k - x_k)\|(1-t)dt\|y_k - x_k\|^2 \\
&\leq \int_0^1 f''(q_k + t(r_k - q_k)(1-t)dt(r_k - q_k)^2 \\
&= -f'(q_k)(r_k - q_k) + \int_0^1 f'(q_k + t(r_k - q_k)dt(r_k - q_k) \\
&= -f'(q_k)(r_k - q_k) + f(r_k) - f(q_k).
\end{aligned}$$

Consequently, we deduce that

$$\begin{aligned}
\|x_{k+1} - y_k\| &= \|F'(x_k)^{-1}F(y_k)\| \\
&\leq \|F'(x_k)^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(y_k)\| \\
&\leq f_0'(q_k)^{-1}f(r_k) \\
&= q_{k+1} - r_k.
\end{aligned}$$

By Lemma 2.3 sequences $\{x_n\}$, $\{y_n\}$ are complete in the Banach space $\mathcal{X}$ and as such they converge to some $x^* \in \overline{U}(x_0, q^*)$ (since $\overline{U}(x_0, q^*)$ is a closed set). Estimate (3.5) follows from the preceding or by using standard majorization techniques [2, 7, 11, 12]. Moreover, by letting $k \to \infty$ in (3.6), we obtain that $F(x^*) = 0$. Furthermore, to show uniqueness, let $y^* \in U(x_0, (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma_0})$ be such that $F(y^*) = 0$. Then, we have by $(\mathcal{H}_3)$ that

$$\begin{aligned}
& \|F'(x_0)^{-1}\int_0^1 F'(x^* + t(y^* - x^*))dt - I\| \\
\leq & \|F'(x_0)^{-1}\int_0^1 [F'(x^* + t(y^* - x^*)) - F'(x_0)]dt\| \\
= & \int_0^1 f_0'(\|x^* - x_0 + t(y^* - x^*)\|)dt - f_0'(0) \\
= & \int_0^1 f_0'(\|(1-t)(x^* - x_0) + t(y^* - x_0)\|)dt + 1 < 1.
\end{aligned}$$

It follows by the Banach Lemma, that the inverse of $\int_0^1 F'(x^* + t(y^* - x^*))dt$ exists. Using the identity,

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))dt(y^* - x^*),$$

we deduce that $y^* = x^*$.

□

REMARK 3.4. (a) It follows from the proof of Theorem 3.3 that in the estimates for the upper bounds on $\|y_1 - x_1\|,\ \|x_2 - y_1\|$ condition $(\mathcal{H}_3)$ can be used instead of the less precise (for $\gamma_0 < \gamma$) condition $(\mathcal{H}_2)$. This observation motivates us to define more precise majorizing sequences $\{\overline{q_n}\},\ \{\overline{r_n}\}$ than $\{q_n\},\ \{r_n\}$, respectively by

$$\overline{q_0} = 0, \qquad \overline{r_0} = \beta,\ \ \overline{q_1} = \overline{r_0} - f'(\overline{q_0})^{-1} f(\overline{r_0}),$$

$$\overline{r_1} = \overline{q_1} - \frac{f_0(\overline{q_1}) - f_0(\overline{r_0}) - f_0'(\overline{q_0})(\overline{q_1} - \overline{r_0})}{f_0'(\overline{q_1})},$$

$$\overline{q_2} = \overline{r_1} - \frac{f_0(\overline{r_1}) - f_0(\overline{q_1}) - f_0'(\overline{q_1})(\overline{r_1} - \overline{q_1})}{f_0'(\overline{q_1})},$$

$$\overline{r_{n+1}} = \overline{q_{n+1}} - \frac{f(\overline{q_{n+1}}) - f(\overline{r_n}) - f'(\overline{q_n})(\overline{q_{n+1}} - \overline{r_n})}{f_0'(\overline{q_{n+1}})},$$

and

$$\overline{q_{n+2}} = \overline{r_{n+1}} - \frac{f(\overline{r_{n+1}}) - f(\overline{q_{n+1}}) - f'(\overline{q_{n+1}})(\overline{r_{n+1}} - \overline{q_{n+1}})}{f_0'(\overline{q_{n+1}})}$$

Clearly, $\{\overline{q_n}\},\ \{\overline{r_n}\}$ can replace $\{q_n\},\ \{r_n\}$ in Theorem 3.3. We also have that

$$\overline{q_n} \le q_n, \qquad \overline{q_{n+1}} - \overline{r_n} \le q_{n+1} - r_n$$
$$\overline{r_n} \le r_n, \qquad \overline{r_{n+1}} - \overline{q_{n+1}} \le r_{n+1} - q_{n+1}$$

and

$$\overline{q^*} = \lim_{n\to\infty} \overline{q_n} \le q^*.$$

(b) Notice that $(\mathcal{H}_2)$ implies $(\mathcal{H}_3)$ but not necessarily vice versa. The results in the literature [8, 14, 15, 16, 17, 18, 19] use the estimate (under $(\mathcal{H}_2)$)

$$\left\|F'(x_{n+1})^{-1} F'(x_0)\right\| \le \left(2 - \frac{1}{(1-\gamma t_{n+1})^2}\right)^{-1}$$

which is less precise than the one obtained in our Theorem if $\gamma_0 < \gamma$ given by

$$\|F(x_{n+1})^{-1} F'(x_0)\| \le (2 - \frac{1}{(1-\gamma_0 s_{n+1})^2})^{-1}$$

(see (3.3)). This observation is the motivation for the introduction of more precise majorizing sequences. Notice also that $(\mathcal{H}_3)$ is not an additional to $(\mathcal{H}_2)$ hypothesis, since in practice the computation of constant γ requires the computation of constant γ_0 as a special case.

(c) Concerning to the choice of constants γ_0 and γ, let us suppose that the following Lipschitz conditions hold

$(\mathcal{H}_2)$': Operator F satisfies the L-Lipschitz condition at x_0

$$\left\|F'(x_0)^{-1}\left[F'(x) - F'(y)\right]\right\| \le L\,\|x-y\| \quad \text{for each } x,\, y \in U(x_0, R_0).$$

$(\mathcal{H}_3)$': Operator F satisfies the L_0-Lipschitz condition at x_0

$$\left\|F'(x_0)^{-1}\left[F'(x) - F'(x_0)\right]\right\| \le L_0\,\|x-x_0\| \quad \text{for each } x \in U(x_0, R_0).$$

Then, $(\mathcal{H}_3)$' implies $(\mathcal{H}_3)$ for $\gamma_0 = \dfrac{L_0}{2}$. Moreover, if F is continuously twice-Fréchet-differentiable, then $(\mathcal{H}_2)$' implies $(\mathcal{H}_2)$ and we can set $\gamma = \dfrac{L}{2}$. Therefore, the conclusions of Theorem 3.3 hold with $(\mathcal{H}_2)$', $(\mathcal{H}_3)$' replacing $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, respectively. Examples, where $L_0 < L$

(i.e. $\gamma_0 < \gamma$) can be found in [2, 3, 4, 5, 6, 7] (see also the numerical examples in Section 4).

(d) If F is an analytic operator, then a choice for γ_0 (or γ) is given by $\gamma_0 = \sup_{n>1} \left\| \frac{F'(x_0)^{-1} F^{(n)}(x_0)}{n!} \right\|^{\frac{1}{n-1}}$. This choice is due to Smale [16] (see also [14, 15, 17, 18, 19]).

We complete this section with a useful and obvious extension of Theorem 3.3.

THEOREM 3.5. Suppose: there exists an integer $N \geq 1$ such that

$$q_0 < r_0 < q_1 < \ldots < q_N < R_0;$$

Let $\alpha_N = \gamma\beta_N$, where $\beta_N = r_N - q_N$. Conditions $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ are satisfied for α_N replacing α in Condition $(\mathcal{H}_4)$. Then, the conclusions of Theorem 3.3 hold. Consequently, the conclusions of Theorem 3.3 also hold for sequence $\{\overline{q_n}\}$, $\{\overline{r_n}\}$. Notice also if $N = 0$ Theorem 3.5 reduces to Theorem 3.3.

4. Numerical Examples

We present examples where the older convergence criterion (2.4) is not satisfied but the new convergence criterion (2.11) satisfied.

EXAMPLE 4.1. Let $\mathcal{C}[0,1]$ stand for the space of continuous functions defined on interval $[0,1]$ and be equipped with the max-norm. Let also $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$ and $\mathcal{D} = U(0,r)$ for some $r > 1$. Define F on $\mathcal{D}$ by

$$F(x)(s) = x(s) - y(s) - \mu \int_0^1 \mathcal{G}(s,t)\, x^3(t)\, dt, \quad x \in \mathcal{C}[0,1], \; s \in [0,1].$$

$y \in \mathcal{C}[0,1]$ is given, μ is a real parameter and the Kernel G is the Green's function defined by

$$\mathcal{G}(s,t) = \begin{cases} (1-s)\,t & if \quad t \leq s \\ s\,(1-t) & if \quad s \leq t. \end{cases}$$

Then, the Fréchet-derivative of F is defined by

$$(F'(x)\,(w))(s) = w(s) - 3\,\mu \int_0^1 \mathcal{G}(s,t)\, x^2(t)\, y(t)\, dt, \quad w \in \mathcal{C}[0,1], \; s \in [0,1].$$

Let us choose $x_0(s) = y(s) = 1$ and $|\mu| < 8/3$. Then, we have that

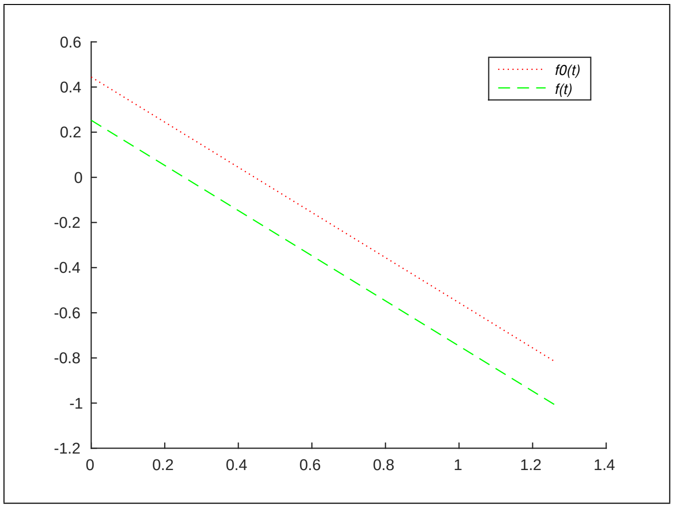
$$\| \mathcal{I} - F'(x_0) \| < \frac{3}{8}\,|\mu|, \quad F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}),$$

$$\| F'(x_0)^{-1} \| \leq \frac{8}{8 - 3\,|\mu|}, \quad \beta = \frac{|\mu|}{8 - 3\,|\mu|}, \quad L_0 = \frac{12\,|\mu|}{8 - 3\,|\mu|},$$

$$L = \frac{6\,r\,|\mu|}{8 - 3\,|\mu|}, \quad \gamma_0 = \frac{L_0}{2} and \quad \gamma = \frac{L}{2}.$$

In Table 2, we pick some values of r and we show the values of μ for which condition (2.11) is satisfied but (2.4) is not satisfied. Hence, the new sufficient semilocal convergence criteria are satisfied but the old in [3, 14, 15, 16, 17, 18, 19, 20] are not satisfied.

FIGURE 1. Plots for f_0 and f for $r = 2.5$ and $\mu = 0.755$



r	α	Interval of μ
2	$\frac{6\mu}{(8-3\mu)^2}$	$(0.849668, 0.859174)$
2.25	$\frac{6.75\mu}{(8-3\mu)^2}$	$(0.798444, 0.807768)$
2.5	$\frac{7.5\mu}{(8-3\mu)^2}$	$(0.753551, 0.762683)$
2.75	$\frac{8.25\mu}{(8-3\mu)^2}$	$(0.713806, 0.722742)$
3	$\frac{9\mu}{(8-3\mu)^2}$	$(0.678318, 0.68706)$

TABLE 2. Comparison Table

5. Conclusion

We studied the semilocal convergence of a two step method in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting using the $\alpha-$ theory. The novelty of our paper lies in the introduction of a center Lipschitz condition that leads to more precise upper bounds on the norms $\|F'(x_n)^{-1}F'(x_0)\|$ yielding to a tighter convergence analysis (see Remark 3.4) and even weaker sufficient convergence criteria (see Theorem 3.5) than in earlier studies such as [3, 14, 15, 16, 17, 18, 19, 20]. The theoretical results are illustrated using munerical examples to show that our new convergence criteria are satisfied but the old ones are not. Moreover, we show that the new error bounds are tighter than the old ones.

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