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ON THE EXISTENCE OF SOLUTIONS FOR A HADAMARD-TYPE FRACTIONAL INTEGRO-DIFFERENTIAL INCLUSION

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ABSTRACT. We study a class of fractional integro-differential inclusions with nonlocal fractional integral boundary conditions and we establish a Filippov type existence result in the case of nonconvex set-valued maps.

KEYWORDS : Differential inclusion; Fractional derivative; Boundary value problem.
AMS Subject Classification: 34A60, 34A08

1. INTRODUCTION

This note is concerned with the following problem

$$D^q x(t) \in F(t, x(t), I^\gamma x(t)) \quad a.e.\ ([1,e]), \tag{1.1}$$

$$x(1) = 0, \quad \sum_{i=1}^{m} \lambda_i I^{\alpha_i} x(\eta_i) = \sum_{j=1}^{n} \mu_i (I^{\beta_j} x(e) - I^{\beta_j} x(\xi_j)), \tag{1.2}$$

where D^q is the Hadamard fractional derivative of order q, $q \in (1,2]$, I^γ is the Hadamard integral of order γ, $\gamma > 0$, $\alpha_i, \beta_j > 0$, $\eta_i, \xi_j \in (1,e)$, $\lambda_i \in \mathbf{R}$, $\mu_j \in \mathbf{R}$, $i = \overline{1,m}$, $j = \overline{1,n}$, $\eta_1 < \eta_2 < ... < \eta_m$, $\xi_1 < \xi_2 < ... < \xi_n$ and $F : [1,e] \times \mathbf{R} \times \mathbf{R} \longrightarrow \mathcal{P}(\mathbf{R})$ is a set-valued map.

If F is single-valued and does not depend on the last variable, fractional inclusion (1.1) reduces to the fractional equation

$$D^q x(t) = f(t, x(t)), \tag{1.3}$$

where $f : [1,e] \times \mathbf{R} \longrightarrow \mathbf{R}$.

In the last years we may see a strong development of the study of boundary value problems associated to fractional differential equations and inclusions. Most of the results in this framework are obtained for problems defined by Riemann-Liouville or Caputo fractional derivatives. Another type of fractional derivative is the one introduced by Hadamard ([6]) which differs from the others in the sense

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that the kernel of the integral contains a logarithmic function of arbitrary exponent. Recently, several papers were devoted to fractional differential equations and inclusions defined by Hadamard fractional derivative [1,2,4,9] etc.

The present note is motivated by a recent paper of Thiramanus, Ntouyas and Taribon ([9]) where existence results for problem (1.3)-(1.2) are obtained using fixed point techniques.

Our aim is to extend the study in [9] to the set-valued framework; moreover, our right-hand side contains an integral term. We show that Filippov's ideas ([5]) can be suitably adapted in order to obtain the existence of solutions for problem (1.1)-(1.2). Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([5]) consists in proving the existence of a solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the "quasi" solution and the solution obtained. In this way we extend an existence result in [4].

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our result.

2. PRELIMINARIES

Let (X,d) be a metric space. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A,B\subset X$ is defined by

$$d_H(A,B)=\max\{d^*(A,B),d^*(B,A)\},\ d^*(A,B)=\sup\{d(a,B);a\in A\},$$

where $d(x,B)=\inf_{y\in B}d(x,y)$.

Let $I=[1,e]$, we denote by $C(I,\mathbf{R})$ the Banach space of all continuous functions from I to $\mathbf{R}$ with the norm $||x(.)||_C=\sup_{t\in I}|x(t)|$ and $L^1(I,\mathbf{R})$ is the Banach space of integrable functions $u(.):I\longrightarrow\mathbf{R}$ endowed with the norm $||u(.)||_1=\int_1^e|u(t)|dt$.

The Hadamard fractional integral of order $q>0$ of a Lebesgue integrable function $f:[1,\infty)\longrightarrow\mathbf{R}$ is defined by

$$I^qf(t)=\frac{1}{\Gamma(q)}\int_1^t\left(\ln\frac{t}{s}\right)^{q-1}\frac{f(s)}{s}ds$$

provided the integral exists and Γ is the (Euler's) Gamma function defined by $\Gamma(q)=\int_0^\infty t^{q-1}e^{-t}dt$.

The Hadamard fractional derivative of order $q>0$ of a function $f:[1,\infty)\longrightarrow\mathbf{R}$ is defined by

$$D^qf(t)=\frac{1}{\Gamma(n-q)}\left(t\frac{d}{dt}\right)^n\int_1^t\left(\ln\frac{t}{s}\right)^{n-q-1}\frac{f(s)}{s}ds,$$

where $n=[q]+1$, $[q]$ is the integer part of q.

Details and properties of Hadamard fractional derivative may be found in [8,9].

The next technical result is proved in [9]. Set

$$\Lambda:=\sum_{i=1}^m\lambda_i\frac{\Gamma(q)}{\Gamma(q+\alpha_i)}(\ln\eta_i)^{q+\alpha_i-1}-\sum_{j=1}^n\mu_i\frac{\Gamma(q)}{\Gamma(q+\beta_j)}(1-(\ln\xi_j)^{q+\beta_j-1}).$$

Lemma 2.1. *Assume that* $\Lambda\neq 0$. *For a given* $f(.)\in C(I,\mathbf{R})$, *the unique solution* $x(.)$ *of problem* $D^qx(t)=f(t)\quad a.e.\ ([1,e])$ *with boundary conditions (1.2) is given by*

$$x(t)=I^qf(t)+\frac{(\ln t)^{q-1}}{\Lambda}[\sum_{j=1}^n\mu_j(I^{q+\beta_j}f(e)-I^{q+\beta_j}f(\xi_j))-\sum_{i=1}^m\lambda_iI^{q+\alpha_i}f(\eta_i)].$$

Remark 2.2. If we denote $A(t,s)=\frac{1}{\Gamma(q)}(\ln\frac{t}{s})^{q-1}\frac{1}{s}\chi_{[1,t]}(s)$, $B(t,s)=\frac{(\ln t)^{q-1}}{\Lambda}\cdot\sum_{j=1}^{n}\frac{\mu_j}{\Gamma(q+\beta_j)}(\ln\frac{e}{s})^{q+\beta_j-1}\frac{1}{s}$, $C_j(t,s)=-\frac{(\ln t)^{q-1}}{\Lambda}\frac{\mu_j}{\Gamma(q+\beta_j)}(\ln\frac{\xi_j}{s})^{q+\beta_j-1}\frac{1}{s}\chi_{[1,\xi_j]}(s)$, $j=\overline{1,n}$, $D_i(t,s)=-\frac{(\ln t)^{q-1}}{\Lambda}\frac{\lambda_i}{\Gamma(q+\alpha_i)}(\ln\frac{\eta_i}{s})^{q+\alpha_i-1}\frac{1}{s}\chi_{[1,\eta_i]}(s)$, $i=\overline{1,m}$, and $G(t,s)=A(t,s)+B(t,s)+\sum_{j=1}^{n}C_j(t,s)+\sum_{i=1}^{m}D_i(t,s)$, where $\chi_S(\cdot)$ is the characteristic function of the set S, then the solution $x(\cdot)$ in Lemma 2.1 may be written as

$$x(t)=\int_1^e G(t,s)f(s)ds. \tag{2.1}$$

Using the fact that, for fixed t, the function $g(s)=(\ln\frac{t}{s})^{q-1}\frac{1}{s}$ is decreasing and $g(1)=(\ln t)^{q-1}$ we deduce that, for any $t,s\in I$,

$$|A(t,s)|\leq\frac{1}{\Gamma(q)}(\ln t)^{q-1}\leq\frac{1}{\Gamma(q)},$$

$$|B(t,s)|\leq\sum_{j=1}^{n}\frac{|\mu_j|}{|\Lambda|\Gamma(q+\beta_j)}(\ln t)^{q-1}\leq\sum_{j=1}^{n}\frac{|\mu_j|}{|\Lambda|\Gamma(q+\beta_j)},$$

$$|C_j(t,s)|\leq\frac{(\ln t)^{q-1}}{|\Lambda|}\frac{|\mu_j|}{\Gamma(q+\beta_j)}(\ln\xi_j)^{q+\beta_j-1}\leq\frac{|\mu_j|}{|\Lambda|\Gamma(q+\beta_j)}(\ln\xi_j)^{q+\beta_j-1},$$

$$|D_i(t,s)|\leq\frac{(\ln t)^{q-1}}{|\Lambda|}\frac{|\lambda_i|}{\Gamma(q+\alpha_i)}(\ln\eta_i)^{q+\alpha_i-1}\leq\frac{|\lambda_i|}{|\Lambda|\Gamma(q+\alpha_i)}(\ln\eta_i)^{q+\alpha_i-1},$$

and therefore,

$$|G(t,s)|\leq\frac{1}{\Gamma(q)}+\sum_{j=1}^{n}\frac{|\mu_j|}{|\Lambda|\Gamma(q+\beta_j)}(1+(\ln\xi_j)^{q+\beta_j-1})+\sum_{i=1}^{m}\frac{|\lambda_i|}{|\Lambda|\Gamma(q+\alpha_i)}(\ln\eta_i)^{q+\alpha_i-1}=:M_1\quad\forall\, t,s\in I.$$

Definition 2.3. A function $x(.)\in C(I,\mathbf{R})$ with its Hadamard derivative of order q existing on $[1,e]$ is a solution of problem (1.1)-(1.2) if there exists a function $f(.)\in L^1(I,\mathbf{R})$ that satisfies $f(t)\in F(t,x(t),I^\gamma x(t))$ *a.e.* (I), $D^qx(t)=f(t)$ a.e. (I) and conditions (1.2) are satisfied.

3. THE MAIN RESULT

First we recall a selection result ([3]) which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem.

Lemma 3.1. *Consider X a separable Banach space, B is the closed unit ball in X, $H:I\longrightarrow\mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g:I\longrightarrow X, L:I\longrightarrow\mathbf{R}_+$ are measurable functions. If*

$$H(t)\cap(g(t)+L(t)B)\neq\emptyset\quad a.e.(I),$$

then the set-valued map $t\longrightarrow H(t)\cap(g(t)+L(t)B)$ has a measurable selection.

In order to prove our results we need the following hypotheses.

Hypothesis H1. i) $F(.,.):I\times\mathbf{R}\times\mathbf{R}\longrightarrow\mathcal{P}(\mathbf{R})$ *has nonempty closed values and is* $\mathcal{L}(I)\otimes\mathcal{B}(\mathbf{R}\times\mathbf{R})$ *measurable.*

ii) *There exists $L(.)\in L^1(I,(0,\infty))$ such that, for almost all $t\in I$, $F(t,.,.)$ is $L(t)$-Lipschitz in the sense that*

$$d_H(F(t,x_1,y_1),F(t,x_2,y_2))\leq L(t)(|x_1-x_2|+|y_1-y_2|)\;\forall\, x_1,x_2,y_1,y_2\in\mathbf{R}.$$

We use next the following notations

$$M(t) := L(t)(1 + \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln \frac{t}{s} \right)^{\gamma-1} \frac{1}{s} ds) = L(t)(1 + \frac{(\ln t)^\gamma}{\Gamma(\gamma+1)}), \tag{3.1}$$

$$M_0 = \int_1^e M(t)dt. \tag{3.2}$$

Theorem 3.1. *Assume that Hypothesis H1 is satisfied and* $M_1 M_0 < 1$. *Consider* $y(.) \in C(I, \mathbf{R})$ *with its Hadamard derivative of order* q *existing on* $[1, e]$ *such that* $y(1) = 0$, $\sum_{i=1}^m \lambda_i I^{\alpha_i} y(\eta_i) = \sum_{j=1}^n \mu_i (I^{\beta_j} y(e) - I^{\beta_j} y(\xi_j))$ *and there exists* $p(.) \in L^1(I, \mathbf{R}_+)$ *verifying* $d(D^q y(t), F(t, y(t), I^\gamma y(t))) \leq p(t)$ *a.e.* (I).

Then there exists $x(.)$ *a solution of problem (1.1)-(1.2) satisfying for all* $t \in I$

$$|x(t) - y(t)| \leq \frac{M_1}{1 - M_1 M_0} \int_1^e p(t)dt. \tag{3.3}$$

Proof. The set-valued map $t \longrightarrow F(t, y(t), I^\gamma y(t))$ is measurable with closed values and

$$F(t, y(t), I^\gamma y(t)) \cap \{D^q y(t) + p(t)[-1, 1]\} \neq \emptyset \quad a.e.\ (I).$$

It follows from Lemma 3.1 that there exists a measurable selection $f_1(t) \in F(t, y(t), I^\gamma y(t))$ $a.e.$ (I) such that

$$|f_1(t) - D^q y(t)| \leq p(t) \quad a.e.\ (I) \tag{3.4}$$

Define $x_1(t) = \int_1^e G(t, s) f_1(s) ds$ and one has

$$|x_1(t) - y(t)| \leq M_1 \int_1^e 1p(t)dt.$$

We claim that it is enough to construct the sequences $x_n(.) \in C(I, \mathbf{R})$, $f_n(.) \in L^1(I, \mathbf{R})$, $n \geq 1$ with the following properties

$$x_n(t) = \int_1^e G(t, s) f_n(s) ds, \quad t \in I, \tag{3.5}$$

$$f_n(t) \in F(t, x_{n-1}(t), I^\gamma x_{n-1}(t)) \quad a.e.\ (I), \tag{3.6}$$

$$|f_{n+1}(t) - f_n(t)| \leq L(t)(|x_n(t) - x_{n-1}(t)| + \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln \frac{t}{s} \right)^{\gamma-1} \frac{1}{s} |x_n(s) - x_{n-1}(s)| ds) \tag{3.7}$$

for almost all $t \in I$.

If this construction is realized then from (3.4)-(3.7) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \leq M_1 (M_1 M_0)^n \int_1^e p(t)dt \quad \forall n \in \mathbf{N}.$$

Indeed, assume that the last inequality is true for $n - 1$ and we prove it for n. One has

$$|x_{n+1}(t) - x_n(t)| \leq \int_1^e |G(t, t_1)|.|f_{n+1}(t_1) - f_n(t_1)| dt_1 \leq$$

$$M_1 \int_1^e L(t_1)[|x_n(t_1) - x_{n-1}(t_1)| + \frac{1}{\Gamma(\gamma)} \int_1^{t_1} \left(\ln \frac{t_1}{s} \right)^{\gamma-1} \frac{1}{s} |x_n(s) - x_{n-1}(s)| ds)$$

$$\leq M_1 \int_0^1 L(t_1)(1 + \frac{1}{\Gamma(\gamma)} \int_1^{t_1} \left(\ln \frac{t_1}{s} \right)^{\gamma-1} \frac{1}{s} ds) dt_1 . M_1^n M_0^{n-1} \int_1^e p(t)dt =$$

$$= M_1 (M_1 M_0)^n \int_1^e p(t)dt$$

Therefore $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I,\mathbf{R})$, hence converging uniformly to some $x(.)\in C(I,\mathbf{R})$. Therefore, by (3.7), for almost all $t\in I$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbf{R}$. Let $f(.)$ be the pointwise limit of $f_n(.)$. Moreover, one has

$$\begin{aligned}&|x_n(t)-y(t)|\leq|x_1(t)-y(t)|+\sum_{i=1}^{n-1}|x_{i+1}(t)-x_i(t)|\leq\\&M_1\int_1^e p(t)dt+\sum_{i=1}^{n-1}(M_1\int_1^e p(t)dt)(M_1M_0)^i=\tfrac{M_1\int_1^e p(t)dt}{1-M_1M_0}.\end{aligned}\tag{3.8}$$

On the other hand, from (3.4), (3.7) and (3.8) we obtain for almost all $t\in I$

$$\begin{aligned}&|f_n(t)-D^qy(t)|\leq\\&\sum_{i=1}^{n-1}|f_{i+1}(t)-f_i(t)|++|f_1(t)-D^qy(t)|\leq L(t)\tfrac{M_1\int_1^e p(t)dt}{1-M_1M_0}+p(t)\end{aligned}$$

Hence the sequence $f_n(.)$ is integrably bounded and therefore $f(.)\in L^1(I,\mathbf{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (3.5), (3.6) we deduce that $x(.)$ is a solution of (1.1). Finally, passing to the limit in (3.8) we obtained the desired estimate on $x(.)$.

It remains to construct the sequences $x_n(.), f_n(.)$ with the properties in (3.5)-(3.7). The construction will be done by induction.

Since the first step is already realized, assume that for some $N\geq 1$ we already constructed $x_n(.)\in C(I,\mathbf{R})$ and $f_n(.)\in L^1(I,\mathbf{R})$, $n=1,2,...N$ satisfying (3.5), (3.7) for $n=1,2,...N$ and (3.6) for $n=1,2,...N-1$. The set-valued map $t\longrightarrow F(t,x_N(t),I^\gamma x_N(t))$ is measurable. Moreover, the map $t\longrightarrow L(t)(|x_N(t)-x_{N-1}(t)|+\frac{1}{\Gamma(\gamma)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-1}\frac{1}{s}|x_N(s)-x_{N-1}(s)|ds)$ is measurable. By the lipschitzianity of $F(t,.,.)$ we have that for almost all $t\in I$

$$\begin{aligned}&F(t,x_N(t),I^\gamma x_N(t))\cap\{f_N(t)+L(t)(|x_N(t)-x_{N-1}(t)|+\\&\tfrac{1}{\Gamma(\gamma)}\int_1^t\left(\ln\tfrac{t}{s}\right)^{\gamma-1}\tfrac{1}{s}|x_N(s)-x_{N-1}(s)|ds)[-1,1]\}\neq\emptyset.\end{aligned}$$

Lemma 3.1 yields that there exists a measurable selection $f_{N+1}(.)$ of $F(.,x_N(.),I^\gamma x_N(.))$ such that for almost all $t\in I$

$$\begin{aligned}&|f_{N+1}(t)-f_N(t)|\leq\\&L(t)(|x_N(t)-x_{N-1}(t)|+\tfrac{1}{\Gamma(\gamma)}\int_1^t\left(\ln\tfrac{t}{s}\right)^{\gamma-1}\tfrac{1}{s}|x_N(s)-x_{N-1}(s)|ds).\end{aligned}$$

We define $x_{N+1}(.)$ as in (3.5) with $n=N+1$. Thus $f_{N+1}(.)$ satisfies (3.6) and (3.7) and the proof is complete. □

The assumption in Theorem 3.1 is satisfied, in particular, for $y(.)=0$ and therefore with $p(.)=L(.)$. We obtain the following consequence of Theorem 3.1.

Corollary 3.2. *Assume that Hypothesis H1 is satisfied, $d(0,F(t,0,0)\leq L(t)$ a.e. (I) and $M_1M_0<1$. Then there exists $x(.)$ a solution of problem (1.1)-(1.2) satisfying for all $t\in I$*

$$|x(t)|\leq\frac{M_1}{1-M_1M_0}\int_1^e L(t)dt.$$

If F does not depend on the last variable, Hypothesis H1 became

Hypothesis H2. i) $F(.,.):I\times\mathbf{R}\longrightarrow\mathcal{P}(\mathbf{R})$ *has nonempty closed values and is $\mathcal{L}(I)\otimes\mathcal{B}(\mathbf{R})$ measurable.*

ii) *There exists* $L(.) \in L^1(I,(0,\infty))$ *such that, for almost all* $t \in I$, $F(t,.)$ *is* $L(t)$*-Lipschitz in the sense that*

$$d_H(F(t,x_1),F(t,x_2)) \le L(t)|x_1-x_2| \quad \forall\, x_1,x_2 \in \mathbf{R}.$$

Denote $L_0 = \int_1^e L(t)dt$. and consider the fractional differential inclusion

$$D^q x(t) \in F(t,x(t)) \quad a.e.\ ([1,e]), \tag{3.9}$$

Corollary 3.3. *Assume that Hypothesis H2 is satisfied,* $d(0,F(t,0) \le L(t)$ *a.e.* (I) *and* $M_1L_0 < 1$. *Then there exists* $x(.)$ *a solution of problem (3.9)-(1.2) satisfying for all* $t \in I$

$$|x(t)| \le \frac{M_1L_0}{1-M_1L_0}.$$

Remark 3.4. If in (1.2) $\lambda_i = 0$, $i = \overline{1,m}$, $j = 1$, $\mu_1 = 1$, then Theorem 3.1 yields Theorem 3.1 in [4].

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