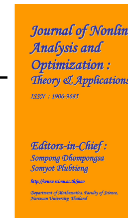
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol.6, No.2, (2015), 53-65
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



GENERALISED CESÀRO-ORLICZ DOUBLE SEQUENCE SPACES OVER N-NORMED SPACES

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ABSTRACT. This article is an attempt to highlight wide-ranging Cesàro-Orlicz double difference sequence spaces over n-normed spaces. The aim here lies in analyzing some topological properties and inclusion relations between these spaces.

KEYWORDS : Double sequence; Orlicz function; Difference sequence; Paranormed space; n-Normed spaces; Cesàro sequence space.
AMS Subject Classification: 40A05, 40A25, 46A30

1. INTRODUCTION, PRELIMINARIES AND NOTATIONS

Let $\mathbb{N}$, $\mathbb{R}$, w and w^2 denote the sets of positive integers, real numbers, single real sequences and double real sequence respectively in the entire paper. For $1 \le p < \infty$, the *Cesàro sequence space* Ces_p is defined by

$$Ces_p = \Big\{ x \in w : \sum_{j=1}^{\infty} \Big(\frac{1}{j} \sum_{i=1}^{j} |x_i| \Big)^p < \infty \Big\},$$

equipped with the norm

$$\|x\| = \Big(\sum_{j=1}^{\infty} \Big(\frac{1}{j} \sum_{i=1}^{j} |x_i| \Big)^p \Big)^{\frac{1}{p}}.$$

Beginning with the first premise of Shiue [26], the concept of space played a very significant role in the theory of matrix operators and others. In the advent, Sanhan and Suantai studied a generalized Cesàro sequence space Ces_p, where $p = (p_j)$ symbolized a bounded sequence of positive real numbers (see [25]). Later, this spaces was studied by many authors in ([8], [10], [15]).
A *double sequence* on a normed linear space X is a function x from $\mathbb{N} \times \mathbb{N}$ into X and briefly denoted by $x = (x_{kl})$. A double sequence (x_{kl}) is said to converge (in

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Article history : Received June 16, 2015. Accepted June 20, 2016.

terms of Pringsheim) to $a \in X$ [19], if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\|x_{kl} - a\|_X < \varepsilon$ whenever $k, l > n_\varepsilon$.

A double series $\sum\limits_{k,l=1}^{\infty} x_{kl}$ is *convergent* if and only if its sequence of partial sums s_{nm} is convergent (see [1], [2]), where $s_{nm} = \sum\limits_{k=1}^{n}\sum\limits_{l=1}^{m} x_{kl}$ for all $m, n \in \mathbb{N}$.

A double sequence $x = (x_{kl})$ is said to be *bounded* if $\|x\|_{(\infty,2)} = \sup_{k,l} |x_{kl}| < \infty$. The space of all bounded double sequences is denoted by l^2_∞.

Initially introduced by Kızmaz [9], the notion of *difference sequence spaces* was conceptualized as $l_\infty(\Delta)$, $c(\Delta)$ and $c_o(\Delta)$. Further, the notion was generalized by Et and Çolak [3] as they familiarized the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_o(\Delta^n)$. Let m, n be non-negative integers, then for Z a given sequence space, we have

$$Z(\triangle^n_m) = \{x = (x_k) \in w : (\triangle^n_m x_k) \in Z\}$$

for $Z = c, c_0$ and l_∞ where $\triangle^n_m x = (\triangle^n_m x_k) = (\triangle^{n-1}_m x_k - \triangle^{n-1}_m x_{k+m})$ and $\triangle^0_m = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^n_m x_k = \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k+mv}.$$

If $m = 1$, we get the spaces $l_\infty(\triangle^n), c(\triangle^n)$ and $c_0(\triangle^n)$ studied by Et and Çolak [3].

If $m = n = 1$, we get the spaces $l_\infty(\triangle), c(\triangle)$ and $c_0(\triangle)$ introduced and studied by Kizmaz [9]. Likewise, the difference operators on double sequence spaces can be examined as:

$$\begin{aligned} \Delta x_{k,l} &= (x_{k,l} - x_{k,l+1}) - (x_{k+1,l} - x_{k+1,l+1}) \\ &= x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1}, \end{aligned}$$

$$\Delta^n x_{k,l} = \Delta^{n-1} x_{k,l} - \Delta^{n-1} x_{k,l+1} - \Delta^{n-1} x_{k+1,l} + \Delta^{n-1} x_{k+1,l+1}$$

and

$$\Delta^n_m x_{k,l} = \Delta^{n-1}_m x_{k,l} - \Delta^{n-1}_m x_{k,l+1} - \Delta^{n-1}_m x_{k+1,l} + \Delta^{n-1}_m x_{k+1,l+1}.$$

For further details about sequence spaces one can refer to ([16], [17], [20], [21], [22], [24]) and references therein.

An *Orlicz function* $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. The function is said to be *modulus function* if the convexity of Orlicz function is substituted by $M(x + y) \le M(x) + M(y)$. Lindenstrauss and Tzafriri [11] used the conception of Orlicz function to describe the following sequence space,

$$\ell_M = \Big\{x = (x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

termed as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \le 1\Big\}.$$

Prior, [11] indicates that every Orlicz sequence space ℓ_M comprises of a subspace isomorphic to $\ell_p (p \ge 1)$. An Orlicz function M can always be imputed in the following integral form

$$M(x) = \int_0^x \eta(t)dt,$$

where η is known as the kernel of M, is a right differentiable for $t \geq 0,\ \eta(0) = 0,\ \eta(t) > 0,\ \eta$ is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a *Musielak-Orlicz function* (see [12, 13]). Complementary function where $\mathcal{N} = (N_k)$, defined as

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1, 2, \cdots$$

is derived from the Musielak-Orlicz function $\mathcal{M}$.

The sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ for a given Musielak-Orlicz function $\mathcal{M}$, can be specified as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for some } \ c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for all } \ c > 0\Big\},$$

where $I_{\mathcal{M}}$ as a convex modular can be described as

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k),\ x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

A Musielak-Orlicz function $\mathcal{M} = (M_k)$ is said to be Δ_2-*condition* if there exist constants $a,\ K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in l_+^1$ (the positive cone of l^1) such that the inequality

$$M_k(2u) \leq KM_k(u) + c_k$$

holds for all $k \in \mathbb{N}$ and $u \in \mathbb{R}^+$, whenever $M_k(u) \leq a$.

2. The Spaces of Double Sequences over $n-$ Normed Spaces

This section brings to limelight Cesàro-Orlicz double difference sequence spaces over n-normed spaces with the help of Musielak-Orlicz functions. Before proceeding futher, first we recall the notion of paranormed space as follows:

A linear topological space X over the real field $\mathbb{R}$ (the set of real numbers) is said to be a *paranormed space* if there is a subadditive function $g : X \longrightarrow \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \longrightarrow 0$ and $g(x_n - x) \longrightarrow 0$ imply $g(\alpha_n x_n - \alpha x) \longrightarrow 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X. A paranorm g for which $g(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, g) is called a total paranormed space. The metric of any linear metric space is given by some total paranorm (see [27], Theorem 10.4.2, pp. 183).

In the mid of 1960's, Gähler [4] introduced the concept of 2-normed spaces while Misiak [14] propounded the n-normed spaces. This concept was further surveyed by critics like Gunawan ([5], [6]) and Gunawan and Mashadi [7] who studied it and obtained various results. Let $n \in \mathbb{N}$ and X be a linear space over the field of real numbers $\mathbb{R}$ of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n substantiates the following four conditions:

(i) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X,
(ii) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation,
(iii) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{R}$, and
(iv) $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called an n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is said to be n-*normed space* over the field $\mathbb{R}$.
For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$.
Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} as defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

is called an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to *converge* to some $L \in X$ if

$$\lim_{k \to \infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \ \text{ for every } \ z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be *Cauchy* if

$$\lim_{k,p \to \infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \ \text{ for every } \ z_1, \cdots, z_{n-1} \in X.$$

X is said to be *complete* with respect to the n-norm if every Cauchy sequence in X converges to some $L \in X$. Thereby, any complete n-normed space is said to be n-*Banach space*.

Suppose $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space and $w(n-X)$ denotes the space of X-valued double sequences. Let $\mathcal{M} = (M_{nm})$ be a Musielak-Orlicz function, that is, $\mathcal{M}$ is a sequence of Orlicz functions, $p = (p_{nm})$ be a bounded double sequence of positive real numbers and $u = (u_{nm})$ be a double sequence of strictly positive real numbers. In this paper we have analysed the following sequence spaces:

$$Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|] =$$

$$\Big\{x \in w(n-X) : \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \infty, \text{ for some } \rho > 0\Big\}.$$

Let us consider a few special cases of the above sequence spaces:

(i) If $\mathcal{M} = M_{nm}(x) = I$ for all $n, m \in \mathbb{N}$, then we have
$Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|] = Ces^{(2)}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$.

(ii) If $u = (u_{nm}) = 1$, for all $n, m \in \mathbb{N}$ then we have
$Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|] = Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, p, \|\cdot, \cdots, \cdot\|]$.

If we take $u = (u_{nm}) = 1$, $M_{nm}(x) = M(x)$ for all $n, m \in \mathbb{N}$, $\Delta^m_n = \Delta$ and X

is a normed space, then we get the spaces $Ces_M^{(2)}[\Delta, p]$ which were introduced and studied by Oğur and Duyar [18].
The following inequality will be used throughout the paper. If $0 \leq p_{nm} \leq \sup p_{nm} = H, K = \max(1, 2^{H-1})$ then

$$|a_{nm} + b_{nm}|^{p_{nm}} \leq K\{|a_{nm}|^{p_{nm}} + |b_{nm}|^{p_{nm}}\} \tag{2.1}$$

for all n, m and $a_{nm}, b_{nm} \in \mathbb{C}$. Also $|a|^{p_{nm}} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.
The paper is an endeavor to introduce the new sequence spaces $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$. The focus here is on some topological properties and inclusion relations between these sequence spaces.

3. MAIN RESULTS

Theorem 3.1. *In order to prove the double sequence $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$ is a linear space over the real field $\mathbb{R}$, let us suppose $\mathcal{M} = (M_{nm})$ be a Musielak-Orlicz function, $p = (p_{nm})$ be a bounded double sequence of positive real numbers and $u = (u_{nm})$ be a double sequence of strictly positive real numbers.*

Proof. Suppose $x = (x_{ij})$ and $y = (y_{ij}) \in Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$ and $\alpha, \beta \in \mathbb{R}$. Then based on the presumption there exist positive numbers ρ_1, ρ_2 such that

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \infty, \text{ for some } \rho_1 > 0,$$

and

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m y_{ij}}{\rho_2}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \infty, \text{ for some } \rho_2 > 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_{nm})$ is a non-decreasing and convex so by using inequality (2.1), we have

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\alpha\Delta_n^m x_{ij} + \beta\Delta_n^m y_{ij}}{\rho_3}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}$$

$$\leq \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{|\alpha|}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho_3}, z_1, \cdots, z_{n-1}\Big\| + \frac{|\beta|}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m y_{ij}}{\rho_3}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}$$

$$\leq \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{2nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho_1}, z_1, \cdots, z_{n-1}\Big\| + \frac{1}{2nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m y_{ij}}{\rho_2}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}$$

$$\leq K\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}$$

$$+ K\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m y_{ij}}{\rho_2}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}$$

$$< \infty.$$

Thus $\alpha x+\beta y \in Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$. This proves that $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$ is a linear space. ∎

Theorem 3.2. *Let $\mathcal{M} = (M_{nm})$ be a Musielak-Orlicz function, $p = (p_{nm})$ be a bounded double sequence of positive real numbers and $u = (u_{nm})$ be a double sequence of strictly positive real numbers. Then the double sequence $Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$ is a paranormed space with the paranorm*

$$g(x) = \inf\Big\{\rho^{\frac{pqr}{R}} > 0 : \Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \le 1;\quad q, r \in \mathbb{N}\Big\}$$

where $0 < p_{nm} \le \sup p_{nm} = H < \infty$ *and* $R = \max(1, H)$.

Proof. (i) Clearly $g(x) \ge 0$ for $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$. Since $M_{nm}(0) = 0$, we get $g(0) = 0$.
(ii) $g(-x) = g(x)$
(iii) Let $x = (x_{ij}), y = (y_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$ there exist positive numbers ρ_1 and ρ_2 such that

$$\Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \le 1$$

and

$$\Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n y_{ij}}{\rho_2}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \le 1.$$

Let $\rho_3 = 2^{\frac{R}{h}}(\rho_1 + \rho_2)$, where $h = \inf p_{nm} > 0$. Since $\mathcal{M}$ ia a non-decreasing convex function, we have

$$\begin{aligned}
&\Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij} + \Delta^m_n y_{ij}}{\rho_3}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \\
&\le \Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{2^{\frac{R}{h}}(\rho_1+\rho_2)}, z_1, \cdots, z_{n-1}\Big\| \\
&+ \frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n y_{ij}}{2^{\frac{R}{h}}(\rho_1+\rho_2)}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \\
&\le \Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[\frac{\rho_1}{2^{\frac{R}{h}}(\rho_1+\rho_2)}M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho_1}, z_1, \cdots, z_{n-1}\Big\| \\
&+ \frac{\rho_2}{2^{\frac{R}{h}}(\rho_1+\rho_2)}M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n y_{ij}}{\rho_2}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \\
&\le \Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[\frac{1}{2^{\frac{R}{h}}}M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \\
&+ \Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[\frac{1}{2^{\frac{R}{h}}}M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n y_{ij}}{\rho_2}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \\
&= \frac{1}{2}\Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \\
&+ \frac{1}{2}\Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n y_{ij}}{\rho_2}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}}
\end{aligned}$$

$\leq \quad 1.$

Since ρ_1, ρ_2 and ρ_3 are positive real numbers, we get

$$g(x+y)$$

$$\inf\Big\{\rho_3^{\frac{p_{qr}}{R}} > 0 : \Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}+\Delta_n^m x_{ij}}{\rho_3}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \leq 1;\ \ q,r\in\mathbb{N}\Big\}$$

$$\leq \inf\Big\{\rho_1^{\frac{p_{qr}}{R}} > 0 : \Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho_1}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \leq 1;\ \ q,r\in\mathbb{N}\Big\}$$

$$+\inf\Big\{\rho_2^{\frac{p_{qr}}{R}} > 0 : \Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m y_{ij}}{\rho_2}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \leq 1;\ \ q,r\in\mathbb{N}\Big\}$$

$$= g(x)+g(y).$$

Let $(x^n) = \{x_{ij}^n\}$ be any sequence in the space $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot,\cdots,\cdot\|]$ such that $g(x^n - x) \to 0$, as $n \to \infty$ and (λ_n) is a sequence of reals with $\lambda_n \to \lambda$, as $n \to \infty$. Then, since the inequality

$$g(x^n) \leq g(x) + g(x^n - x)$$

holds by subadditivity of the function g, $\{g(x^n)\}$ is bounded. Taking into account this fact we therefore derive the inequality

$$g(\lambda_n x^n - \lambda x) \leq |\lambda_n - \lambda| g(x^n) + |\lambda| g(x^n - x)$$

which tends to zero as $n \to \infty$. Hence, the scalar multiplication is continuous follows from the above inequality and thus proving the theorem. □

Theorem 3.3. *Let $\mathcal{M} = (M_{nm})$ be a Musielak-Orlicz function, $p = (p_{nm})$ be a bounded double sequence of positive real numbers and $u = (u_{nm})$ be a double sequence of strictly positive real numbers. Then the space $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot,\cdots,\cdot\|]$ is complete with respect to its paranorm.*

Proof. Let $(x^s) = \{x_{ij}^s\}$ be any Cauchy sequence in the space $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot,\cdots,\cdot\|]$. Since (x^s) is a Cauchy sequence, we have $g(x^s - x^t) \to 0$ as $s,t \to \infty$. Then, we have

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}^s - \Delta_n^m x_{ij}^t}{\rho}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \to 0$$

as $s,t \to \infty$ for all $i,j \in \mathbb{N}$. Then, we have $\{x_{ij}^s\}$ is a Cauchy sequence in $\mathbb{R}$ for each fixed $i,j \in \mathbb{N}$. Since $\mathbb{R}$ is complete as $t \to \infty$, we have $x_{ij}^s \to x_{ij}$ as $s \to \infty$ for each (i,j) and $\mathcal{M} = (M_{nm})$ is continuous. For $\epsilon > 0$, there exists a natural number N such that

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\sum_{s,t>N}\Big\|\frac{\Delta_n^m x_{ij}^s - \Delta_n^m x_{ij}^t}{\rho}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \varepsilon.$$

Since for any fixed natural number M, we have

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j\leq M \quad s,t>N}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}^s-\Delta_n^m x_{ij}^t}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{nm}}<\varepsilon,$$

by letting $t\to\infty$ in the above expression we obtain

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j\leq M \quad s,t>N}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}^s-\Delta_n^m x_{ij}}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{nm}}<\varepsilon.$$

Since M is arbitrary, by letting $M\to\infty$ we obtain

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}^s-\Delta_n^m x_{ij}}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{nm}}<\varepsilon.$$

Then $g(x^s-x)\to 0$ as $t\to\infty$. Since $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m,u,p,\|\cdot,\cdots,\cdot\|]$ is linear space, we get $x=\{x_{ij}\}\in Ces_{\mathcal{M}}^{(2)}[\Delta_n^m,u,p,\|\cdot,\cdots,\cdot\|]$. This completes the proof. □

Theorem 3.4. *If* $0<p_{nm}\leq q_{nm}<\infty$ *for each* n *and* m*, then we have*

$$Ces_{\mathcal{M}}^{(2)}[\Delta_n^m,u,p,\|\cdot,\cdots,\cdot\|]\subset Ces_{\mathcal{M}}^{(2)}[\Delta_n^m,u,q,\|\cdot,\cdots,\cdot\|].$$

Proof. Let $x\in Ces_{\mathcal{M}}^{(2)}[\Delta_n^m,u,p,\|\cdot,\cdots,\cdot\|]$. Then there exists $\rho>0$ such that

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{nm}}<\infty.$$

This implies that

$$u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{nm}}<1,$$

for sufficiently large values of n and m. Since M_{nm} is non-decreasing, we get

$$\begin{aligned}\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{q_{nm}}\\ \leq \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\\ <\infty.\end{aligned}$$

Thus $x\in Ces_{\mathcal{M}}^{(2)}[\Delta_n^m,u,q,\|\cdot,\cdots,\cdot\|]$. This completes the proof. □

Theorem 3.5. *Suppose* $\mathcal{M}=(M_{mn})$ *be a Musielak-Orlicz function,* $p=(p_{mn})$ *be a bounded double sequence of positive real numbers and* $u=(u_{mn})$ *be a double sequence of strictly positive real numbers. Then*

(a) If $0<\inf p_{mn}<p_{mn}\leq 1$. *Then* $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m,u,p,\|\cdot,\cdots,\cdot\|]\subset Ces_{\mathcal{M}}^{(2)}[\Delta_n^m,u,\|\cdot,\cdots,\cdot\|]$.

(b) If $1\leq p_{mn}\leq\sup p_{mn}<\infty$. *Then* $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m,u,\|\cdot,\cdots,\cdot\|]\subset Ces_{\mathcal{M}}^{(2)}[\Delta_n^m,u,p,\|\cdot,\cdots,\cdot\|]$

Proof. (a) Let $x=(x_{ij})\in Ces_{\mathcal{M}}^{(2)}[\Delta_n^m,u,p,\|\cdot,\cdots,\cdot\|]$. Since $0<\inf p_{mn}\leq 1$, we obtain the following

$$\begin{aligned}\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]\\ \leq \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\end{aligned}$$

$$< \quad \infty.$$

and hence $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, \|\cdot, \cdots, \cdot\|]$.

(b) Let $p_{nm} \geq 1$ for each n and m and $\sup p_{nm} < \infty$. Let $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, \|\cdot, \cdots, \cdot\|]$. Then for each $0 < \epsilon < 1$ there exists a positive integer N such that

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big] \leq \epsilon < 1 \text{ for all } n, m \geq N.$$

This implies that

$$\begin{aligned}\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \\ \leq \quad \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big] \\ < \quad \infty.\end{aligned}$$

Therefore, $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$. This completes the proof. □

Theorem 3.6. *Let $0 < p_{nm} \leq q_{nm}$ for all $n, m \in \mathbb{N}$ and $(\frac{q_{nm}}{p_{nm}})$ be bounded. Then we have $Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, q, \|\cdot, \cdots, \cdot\|] \subset Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$.*

Proof. Let $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$. Then

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{q_{nm}} < \infty, \quad \text{for some } \rho > 0.$$

Let $s_{nm} = \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{q_{nm}}$ and $\lambda_{nm} = \frac{p_{nm}}{q_{nm}}$.
Since $p_{nm} \leq q_{nm}$, we have $0 \leq \lambda_{nm} \leq 1$. Take $0 < \lambda < \lambda_{nm}$.

Define

$$u_{nm} = \begin{cases} s_{nm} & \text{if } s_{nm} \geq 1 \\ 0 & \text{if } s_{nm} < 1 \end{cases}$$

and

$$v_{nm} = \begin{cases} 0 & \text{if } s_{nm} \geq 1 \\ s_{nm} & \text{if } s_{nm} < 1 \end{cases}$$

$s_{nm} = u_{nm} + v_{nm}$, $\quad s^{\lambda_{nm}}_{nm} = u^{\lambda_{nm}}_{nm} + v^{\lambda_{nm}}_{nm}$. It follows that $u^{\lambda_{nm}}_{nm} \leq u_{nm} \leq s_{nm}$, $v^{\lambda_{nm}}_{nm} \leq v^{\lambda}_{nm}$. Since $s^{\lambda_{nm}}_{nm} = u^{\lambda_{nm}}_{nm} + v^{\lambda_{nm}}_{nm}$, then $s^{\lambda_{nm}}_{nm} \leq s_{nm} + v^{\lambda}_{nm}$

$$\begin{aligned}\sum_{n,m=1}^{\infty} u_{nm}\Big[\Big(M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big)^{q_{nm}}\Big]^{\lambda_{nm}} \\ \leq \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{q_{nm}}\end{aligned}$$

$$\Longrightarrow \sum_{n,m=1}^{\infty} u_{nm}\Big[\Big(M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big)^{q_{nm}}\Big]^{p_{nm}/q_{nm}}$$

$$\leq \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{q_{nm}}$$

$$\Longrightarrow \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}$$
$$\leq \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{q_{nm}},$$

but

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{q_{nm}} < \infty \text{ for some } \rho > 0.$$

Therefore,

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \infty \text{ for some } \rho > 0.$$

Hence $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$. Thus, we get $Ces^{(2)}_{\mathcal{M}}[\Delta_n^m, u, q, \|\cdot, \cdots, \cdot\|] \subset Ces^{(2)}_{\mathcal{M}}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$. □

Theorem 3.7. *Let $\mathcal{M}' = (M'_{nm})$ and $\mathcal{M}'' = (M''_{nm})$ be two Musielak-Orlicz functions satisfying Δ_2-condition. Then*
(a) $Ces^{(2)}_{\mathcal{M}'}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|] \subset Ces^{(2)}_{\mathcal{M}''\circ\mathcal{M}'}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$,

(b) $Ces^{(2)}_{\mathcal{M}'}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|] \cap Ces^{(2)}_{\mathcal{M}''}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|] \subset Ces^{(2)}_{\mathcal{M}'+\mathcal{M}''}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$.

Proof. (a) Let $x \in Ces^{(2)}_{\mathcal{M}'}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$. Then there exists $\rho > 0$ such that

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M'_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \infty.$$

Since $\mathcal{M}' = (M'_{nm})$ is a continuous function, we can find a real number δ with $0 < \delta < 1$ such that $M'_{nm}(t) < \varepsilon$. Let $y_{nm} = M'_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)$.

Hence we write

$$\sum_{n,m=1}^{\infty} u_{nm}[M''_{nm}(y_{nm})]^{p_{nm}} = \sum_{y_{nm}\leq\delta}^{\infty} u_{nm}[M''_{nm}(y_{nm})]^{p_{nm}} + \sum_{y_{nm}>\delta}^{\infty} u_{nm}[M''_{nm}(y_{nm})]^{p_{nm}}$$

so we have

$$\sum_{y_{nm}\leq\delta}^{\infty} u_{nm}[M''_{nm}(y_{nm})]^{p_{nm}} \leq \max\{1, M''_{nm}(1)^H\}\sum_{y_{nm}\leq\delta}^{\infty} u_{nm}[y_{nm}]^{p_{nm}} \tag{3.1}$$

For $y_{mn} > \delta$, we use the fact $y_{mn} < \frac{y_{mn}}{\delta} < 1 + \frac{y_{mn}}{\delta}$. Since $\mathcal{M}'' = (M''_{nm})$ is non-decreasing and convex it follows that

$$M''_{nm}(y_{nm}) < M''_{nm}(1 + \frac{y_{nm}}{\delta}) < \frac{1}{2}M''_{nm}(2) + \frac{1}{2}\Big(\frac{2y_{nm}}{\delta}\Big).$$

Since $\mathcal{M}'' = (M''_{nm})$ satisfying the Δ_2-condition and $\frac{y_{mn}}{\delta} > 1$, there exists $T > 0$ such that

$$M''_{nm}(y_{nm}) < \frac{1}{2}T\frac{y_{mn}}{\delta}M''_{nm}(2) + \frac{1}{2}T\frac{y_{mn}}{\delta}M''_{nm}(2) = T\frac{y_{mn}}{\delta}M''_{nm}(2).$$

Therefore, we have

$$\sum_{y_{nm}>\delta}^{\infty} u_{nm}[M''_{nm}(y_{nm})]^{p_{nm}} \leq \max\left\{1, \left(T\frac{M''_{nm}(2)}{\delta}\right)^H\right\} \sum_{y_{nm}>\delta}^{\infty} u_{nm}[y_{nm}]^{p_{nm}} \quad (3.2)$$

Hence by the equation (3.1) and (3.2), we have

$$\begin{aligned}\sum_{n,m=1}^{\infty} u_{nm}\Big[(M''_{nm}\circ M'_{nm})\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\\ = \sum_{n,m=1}^{\infty} u_{nm}[M''_{nm}y_{nm}]^{p_{nm}}\\ \leq D\sum_{y_{nm}\leq\delta}^{\infty} u_{nm}[(y_{nm})]^{p_{nm}}\\ +G\sum_{y_{nm}>\delta}^{\infty} u_{nm}[y_{nm}]^{p_{nm}}\end{aligned}$$

where $D = \max\{1, M''_{nm}(1)^H\}$ and $G = \max\left\{1, \left(T\frac{M''_{nm}(2)}{\delta}\right)^H\right\}$.
Hence $Ces^{(2)}_{\mathcal{M}'}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|] \subset Ces^{(2)}_{\mathcal{M}''\circ\mathcal{M}'}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$.

(b) Let $x \in Ces^{(2)}_{\mathcal{M}'}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|] \cap Ces^{(2)}_{\mathcal{M}''}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$. Then

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M'_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \infty, \text{ for some } \rho > 0$$

and

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M''_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \infty, \text{ for some } \rho > 0.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. The result follows from the inequality

$$\begin{aligned}&\sum_{n,m=1}^{\infty} u_{nm}\Big[(M'_{nm}+M''_{nm})\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\\ &= \sum_{n,m=1}^{\infty} u_{nm}\Big[M'_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\\ &+ \sum_{n,m=1}^{\infty} u_{nm}\Big[M''_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\\ &\leq K\sum_{n,m=1}^{\infty} u_{nm}\Big[M'_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\\ &+ K\sum_{n,m=1}^{\infty} u_{nm}\Big[M''_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\\ &< \infty,\end{aligned}$$

where $K = \{\max 1, 2^{H-1}\}$. Therefore, $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}'+\mathcal{M}''}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$. □

Theorem 3.8. *Let $\mathcal{M} = (M_{nm})$ be a Musielak-Orlicz function and Suppose that $\beta = \lim_{t\to\infty} \frac{M_{nm}(t)}{t} < \infty$. Then $Ces^{(2)}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|] = Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$.*

Proof. In order to prove that $Ces^{(2)}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|] = Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$. It is adequate to show that $Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|] \subset Ces^{(2)}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$. Now, let $\beta > 0$. By definition of β, we have $M_{nm}(t) \geq \beta t$ for all $t \geq 0$. Since $\beta > 0$, we have $t \leq \frac{1}{\beta} M_{nm}(t)$ for all $t \geq 0$. Let $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$. Thus, we have

$$\begin{aligned}
&\sum_{n,m=1}^{\infty} u_{nm}\Big[\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \\
&\leq \frac{1}{\beta}\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \\
&< \infty,
\end{aligned}$$

which implies that $x = (x_{ij}) \in Ces^{(2)}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$. This completes the proof. □

Theorem 3.9. *The double sequence space $Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$ is solid.*

Proof. Suppose $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \infty, \text{ for some } \rho > 0.$$

Let (α_{ij}) be a double sequence of scalars such that $|\alpha_{ij}| \leq 1$ for all $i, j \in \mathbb{N}$. Then we get

$$\begin{aligned}
&\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n \alpha_{ij} x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \\
&\leq \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \\
&< \infty.
\end{aligned}$$

This completes the proof. □

Theorem 3.10. *The double sequence space $Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$ is monotone.*

Proof. The proof is insignificant so we exclude it. □

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