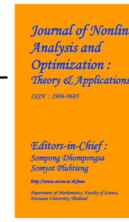

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LOCAL CONVERGENCE OF A MULTI-POINT JARRATT-TYPE METHOD IN BANACH SPACE UNDER WEAK CONDITIONS

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ABSTRACT. We present a local convergence analysis of a multi-point Jarratt-type method of high convergence order in order to approximate a solution of a nonlinear equation in a Banach space. Our sufficient convergence conditions involve only hypotheses on the first Fréchet-derivative of the operator involved. In contrast to earlier studies using hypotheses up to the third Fréchet-derivative [26]. Numerical examples are also provided in this study.

KEYWORDS : Jarratt-type methods, Banach space, Local Convergence, Fréchet-derivative.
AMS Subject Classification:65H10, 65D99.

1. Introduction

In this study we are concerned with the problem of approximating a solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a subset D of a Banach space X with values in a Banach space Y.

Many problems in computational sciences and other disciplines can be brought in a form like (1.1) using mathematical modelling [3]. The solutions of equation (1.1) can rarely be found in closed form. That is why solutions of equation (1.1) are approximated by iterative methods. In particular, the practice of Numerical Functional Analysis for finding such solution is essentially connected to Newton-like methods [1]-[27]. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the

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radii of convergence balls. There exist many studies which deal with the local and semilocal convergence analysis of Newton-like methods such as [1]-[27].

We present a local convergence analysis for the multi-point Jarratt-type method defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} u_n &= x_n - F'(x_n)^{-1}F(x_n), \\ y_n &= u_n + \frac{1}{3}F'(x_n)^{-1}F(x_n), \\ z_n &= u_n + B_nF(x_n), \\ x_{n+1} &= z_n - A_nF(z_n), \end{aligned} \tag{1.2}$$

where $x_0 \in D$ is an initial point, $J_n = (6F'(y_n) - 2F'(x_n))^{-1}(3F'(y_n) + F'(x_n))$, $B_n = (I - J(x_n))F'(x_n)^{-1}$, and $A_n = \frac{3}{2}J_nF'(y_n)^{-1} + (I - \frac{3}{2}J_n)F'(x_n)^{-1}$.

A semilocal convergence analysis was given in [26] but the operator A_n was defined by

$$\overline{A_n} = \frac{3}{2}F'(y_n)^{-1}J_n + F'(x_n)^{-1}(I - \frac{3}{2}J_n).$$

That is, their method is defined by

$$\begin{aligned} u_n &= x_n - F'(x_n)^{-1}F(x_n), \\ y_n &= u_n + \frac{1}{3}F'(x_n)^{-1}F(x_n), \\ z_n &= u_n + B_nF(x_n), \\ x_{n+1} &= z_n - \overline{A_n}F(z_n). \end{aligned} \tag{1.3}$$

Notice that for two linear operators Q_1 and Q_2 we have that $Q_1Q_2 \neq Q_2Q_1$, so $A_n \neq \overline{A_n}$ in general. The fifth order of convergence of method (1.3) was established in [26]. These results were given in a non-affine invariant form. However, the results obtained in our paper are given in affine invariant form. The sufficient semilocal convergence conditions (given in affine invariant form) used in [26] are ($\mathcal{C}$):

($\mathcal{C}_1$): There exists $F'(x_0)^{-1} \in L(Y, X)$ and $\|F'(x_0)^{-1}\| \leq \beta$;

($\mathcal{C}_2$):
$$\|F'(x_0)^{-1}F(x_0)\| \leq \beta_1;$$

($\mathcal{C}_3$):
$$\|F'(x_0)^{-1}F''(x)\| \leq \beta_2 \quad \text{for each } x \in D;$$

($\mathcal{C}_4$):
$$\|F'(x_0)^{-1}F'''(x)\| \leq \beta_3 \quad \text{for each } x \in D;$$

($\mathcal{C}_5$):
$$\|F'(x_0)^{-1}(F'''(x) - F'''(y))\| \leq w(\|x - y\|) \quad \text{for each } x, y \in D$$

where $w(s)$ is a nondecreasing continuous real function for $s > 0$ with $w(0) \geq 0$.

($\mathcal{C}_6$): there exists a non-negative real function $\phi \in C[0, 1]$, with $\phi(t) \leq 1$, such that $w(ts) \leq \phi(t)w(s)$ for each $t \in [0, 1]$, $s \in (0, \infty)$.

Similar conditions have been used by other authors [1]-[27], on other high convergence order methods. The corresponding conditions for the local convergence analysis are given by simply replacing x_0 by x^* in the preceding ($\mathcal{C}$) conditions.

These conditions however are very restrictive. As an academic example, let us define function F on $X = [-\frac{5}{2}, \frac{1}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Then, obviously, e.g, function F cannot satisfy condition ($\mathcal{C}_4$) since $F'''(x) = 6lnx^2 + 60x^2 - 24x + 22$ is unbounded on D. In the present paper we only use hypotheses on the first Fréchet derivative (see conditions (2.9)-(2.12)). This way we expand the applicability of method (1.2).

The paper is organized as follows. The local convergence of method (1.2) is given in Section 2, whereas the numerical examples are given in the concluding Section 3.

2. Local convergence analysis

We present the local convergence analysis of method (1.2) in this section. Denote by $U(v,\rho), \bar{U}(v,\rho)$ the open and closed balls, respectively, in X of center v and radius $\rho > 0$.

Let $L_0 > 0$, $L > 0$ and $M > 0$ be given parameters. It is convenient for the local convergence analysis of method (1.2) that follows to define functions on the interval $[0, \frac{1}{L_0})$ by

$$\begin{aligned}
g_1(r) &= \frac{Lr}{2(1-L_0r)}, \\
g_2(r) &= g_1(r) + \frac{M}{3(1-L_0r)}, \\
g_3(r) &= \frac{3}{2}\frac{L_0(1+g_2(r))r}{1-L_0r}, \\
g_4(r) &= g_1(r) + \frac{3}{4}\frac{ML_0(1+g_2(r))r}{(1-g_3(r))(1-L_0r)^2}, \\
g_5(r) &= [1 + \frac{3}{4}(1+\frac{1}{1-g_3(r)})\frac{M}{1-L_0g_2(r)r} \\
&\quad +(1+\frac{3}{4}(1+\frac{1}{1-g_3(r)}))\frac{M}{1-L_0r}]g_4(r).
\end{aligned}$$

We have that

$$\begin{aligned}
g_2(r) &= g_1(r) + \frac{M}{3(1-L_0r)} \\
&= \frac{Lr}{2(1-L_0r)} + \frac{M}{3(1-L_0r)}.
\end{aligned}$$

Hence, if

$$r_2 = \frac{3-M}{3(\frac{L}{2}+L_0)}$$

and

$$M < 3, \tag{2.1}$$

then, we have that

$$0 < g_2(r) < 1 \text{ and } 0 < g_1(r) < 1 \text{ for each } r \in [0, r_2). \tag{2.2}$$

Notice that

$$r_2 < r_R := \frac{2}{3L} < r_A := \frac{1}{\frac{L}{2}+L_0} \text{ for } L_0 < L \tag{2.3}$$

and

$$r_2 < r_A = r_R \text{ for } L_0 = L. \tag{2.4}$$

Function g_3 can be written as

$$\begin{aligned} g_3(r) &= \frac{3}{2}\frac{L_0 r}{1 - L_0 r}\left(1 + \frac{Lr}{2(1 - L_0 r)} + \frac{M}{3(1 - L_0 r)}\right) \\ &= \frac{L_0 r}{4(1 - L_0 r)^2}(6(1 - L_0 r) + 3Lr + 2M). \end{aligned}$$

Define polynomial p_3 by

$$p_3(r) = L_0 r(6(1 - L_0 r) + 3Lr + 2M) - 4(1 - L_0 r)^2.$$

We have that $p_3(0) = -4 < 0$ and $p_3(\frac{1}{L_0}) = \frac{3L}{L_0} + 2M > 0$. It then follows from the intermediate value theorem that polynomial p_3 has roots in the interval $(0, \frac{1}{L_0})$. Let us denote by r_3 the smallest such root. Then, we have that

$$0 < p_3(r) < 1 \text{ and } 0 < g_3(r) < 1 \text{ for each } r \in [0, r_3). \tag{2.5}$$

By some algebraic manipulation we see that function g_5 can be written as

$$g_5(r) = \frac{N(r)}{D(r)},$$

where

$$\begin{aligned} N(r) &= [4(1 - L_0 g_2(r)r)(1 - g_3(r))(1 - L_0 r) + 3M(1 - g_3(r))(1 - L_0 r) \\ &\quad + 3M(1 - L_0 r) + 7M(1 - L_0 g_2(r)r)(1 - g_3(r)) + 3M(1 - L_0 g_2(r)r)] \\ &\quad \times [4(1 - g_3(r))(1 - L_0 r)^2 g_1(r) + 3ML_0(1 + g_2(r)r)] \end{aligned}$$

and

$$D(r) = 16(1 - g_3(r))^2(1 - L_0 r)^3(1 - L_0 g_2(r)r).$$

Moreover, define function g_6 on the interval $[0, \frac{1}{L_0})$ by

$$g_6(r) = N(r) - D(r) \tag{2.6}$$

we have that $g_6(0) = -16(1 - \frac{L_0 M r}{3}) < 0$, since $L_0 r < 1$ and $M < 3$ and $g_6(r) \to \infty$ as $r \to (\frac{1}{L_0})^-$. It follows that function g_6 has zeros in the interval $(0, \frac{1}{L_0})$ (i.e., function g_5 has zeros in the interval $(0\frac{1}{L_0})$). Denote by r_5 the smallest such zero. Then, we have that

$$0 < g_5(r) < 1 \text{ and } 0 < g_4(r) < 1 \text{ for each } r \in [0, r_5). \tag{2.7}$$

Finally, set

$$r^* = \min\{r_2, r_3, r_5\}. \tag{2.8}$$

Then, clearly (2.2), (2.5) and (2.7) hold for each $r \in [0, r^*)$ (provided that (2.1) and (2.6) also hold).

Next, we present the local convergence analysis of method (1.2).

THEOREM 2.1. Let $F : D \subseteq X \to Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, parameters $L_0 > 0, L > 0$ and $0 < M < 3$ such that for all $x \in D$

$$F(x^*) = 0, \; F'(x^*)^{-1} \in L(Y, X), \tag{2.9}$$

$$\|F'(x^*)^{-1}(F(x) - F(x^*))\| \le L_0\|x - x^*\|, \tag{2.10}$$

$$\|F'(x^*)^{-1}(F(x) - F(x^*) - F'(x)(x - x^*)\| \le \frac{L}{2}\|x - x^*\|^2, \tag{2.11}$$

$$\|F'(x^*)^{-1}F'(x)\| \le M \tag{2.12}$$

and

$$\bar{U}(x^*, r^*) \subseteq D,$$

where r^* is given by (2.8). Then, sequence $\{x_n\}$ generated by method (1.2) for $x_0 \in U(x^*, r^*)$ is well defined, remains in $U(x^*, r^*)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold for each $n = 0, 1, 2, \ldots,$

$$\|u_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r^*, \tag{2.13}$$

$$\|y_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.14}$$

$$\|z_n - x^*\| \leq g_4(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.15}$$

$$\frac{3}{2}\|F'(x_n)^{-1}(F'(y_n) - F'(x_n))\| \leq g_3(\|x_n - x^*\|) < 1 \tag{2.16}$$

and

$$\|x_{n+1} - x^*\| \leq g_5(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.17}$$

where the "g" functions are defined above Theorem 2.1.

Proof. Using (2.10), the definition of r^* and the hypothesis $x_0 \in U(x^*, r^*)$ we get that

$$\|F'(x^*)^{-1}(F(x_0) - F(x^*))\| \leq L_0\|x_0 - x^*\| < L_0 r^* < 1. \tag{2.18}$$

It follows from (2.18) and the Banach Lemma on invertible operators [3] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|} < \frac{1}{1 - L_0 r^*}. \tag{2.19}$$

Hence, u_0, y_0 are well defined. Using method (1.2) for $n = 0$, (2.19),(2.2), (2.17), (2.11), (2.12) (for $x = x_0$) and the definition of functions g_1 and g_2, we obtain in turn that

$$\begin{aligned}
u_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0)\\
&= F'(x_0)^{-1}[-F(x_0) + F'(x_0)(x_0 - x^*)]\\
&= -F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^*)]
\end{aligned}$$

so,

$$\begin{aligned}
\|u_0 - x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^*)]\|\\
&\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)}\\
&= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r^*,
\end{aligned}$$

which shows (2.13) for $n = 0$. Consequently from

$$y_0 - x^* = u_0 - x^* + \frac{1}{3}F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}F(x_0),$$

we get that

$$\begin{aligned}
\|y_0 - x^*\| &\leq \|u_0 - x^*\| + \frac{1}{3}\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\|\\
&\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| + \frac{1}{3}\frac{\|\int_0^1 F'(x^*)^{-1}\int_0^1 F'(x^* + t(x^* - x_0)dt\|\|x_0 - x^*\|}{1 - L_0\|x_0 - x^*\|}\\
&\leq [g_1(\|x_0 - x^*\|) + \frac{M}{3(1 - L_0\|x_0 - x^*\|)}]\|x_0 - x^*\|\\
&= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r^*,
\end{aligned}$$

which shows (2.14) for $n=0$. We need a norm estimate on B_0. Let us start with an estimate on

$$\begin{aligned}
& \frac{3}{2}\|F'(x_0)^{-1}(F'(y_0)-F'(x_0))\| \\
\leq\ & \frac{3}{2}\|F'(x_0)^{-1}F'(x^*)\|(\|F'(x^*)^{-1}(F'(y_0)-F'(x^*))\| \\
& +\|F'(x^*)^{-1}(F'(x_0)-F'(x^*))\| \\
\leq\ & \frac{3}{2}\frac{L_0(\|y_0-x^*\|+\|x_0-x^*\|}{1-L_0\|x_0-x^*\|} \\
\leq\ & \frac{3}{2}\frac{L_0(1+g_2(\|x_0-x^*\|)\|x_0-x^*\|}{1-L_0\|x_0-x^*\|} \\
=\ & g_3(\|x_0-x^*\|)<g_3(r^*)<1,
\end{aligned}$$

(by (2.5)) which shows (2.16) for $n=0$, $(I+\frac{3}{2}F'(x_0)^{-1}(F'(y_0)-F'(x_0)))^{-1}\in L(Y,X)$ and

$$\|(I+\frac{3}{2}F'(x_0)^{-1}(F'(y_0)-F'(x_0)))^{-1}\|\leq\frac{1}{1-g_3(\|x_0-x^*\|)}.$$

Then, we have the estimate

$$\begin{aligned}
& \frac{1}{2}(I+\frac{3}{2}F'(x_0)^{-1}(F'(y_0)-F'(x_0)))^{-1}F'(x_0)^{-1} \\
=\ & [2F'(x_0)(I+\frac{3}{2}F'(x_0)^{-1}(F'(y_0)-F'(x_0)))^{-1} \\
=\ & [3(F'(y_0)-F'(x_0))+2F'(x_0)]^{-1} \\
=\ & (3F'(y_0)-F'(x_0))^{-1}.
\end{aligned}$$

Then, B_0 can now be written as

$$B_0=\frac{3}{4}(I+\frac{3}{2}F'(x_0)^{-1}(F'(y_0)-F'(x_0)))^{-1}F'(x_0)^{-1}(F'(y_0)-F'(x_0))F'(x_0)^{-1},$$

so,

$$\begin{aligned}
\|B_0\| \leq\ & \frac{3}{4}\frac{1}{1-g_3(\|x_0-x^*\|)} \\
& \times\frac{L_0(\|x_0-x^*\|+\|y_0-x^*\|)}{1-L_0\|x_0-x^*\|}\frac{M\|x_0-x^*\|}{1-L_0\|x_0-x^*\|} \\
\leq\ & \frac{3}{4}\frac{ML_0(1+g_2(\|x_0-x^*\|))\|x_0-x^*\|^2}{(1-g_3(\|x_0-x^*\|))(1-L_0\|x_0-x^*\|)^2}. \qquad (2.20)
\end{aligned}$$

Moreover from (2.20) and (2.7) we get that

$$\begin{aligned}
\|z_0-x^*\| \leq\ & \|u_0-x^*\|+\|B_0F(x_0)\| \\
\leq\ & [g_1(\|x_0-x^*\|)+\frac{3}{4}\frac{ML_0(1+g_2(\|x_0-x^*\|))\|x_0-x^*\|}{(1-g_3(\|x_0-x^*\|))(1-L_0\|x_0-x^*\|)^2}]\|x_0-x^*\| \\
=\ & g_4(\|x_0-x^*\|)\|x_0-x^*\|<\|x_0-x^*\|<r^*,
\end{aligned}$$

which shows (2.15) for $n=0$.

Next, we need an estimate on J_0 and A_0. We have that

$$\begin{aligned}
J_0 =\ & \frac{1}{2}(3F'(y_0)-F'(x_0))^{-1}(3F'(y_0)-F'(x_0)+2F'(x_0)) \\
=\ & \frac{1}{2}(I+2(3F'(y_0)-F'(x_0))^{-1}F'(x_0))
\end{aligned}$$

so,

$$\|J_0\| \leq \frac{1}{2}(1+\frac{1}{1-g_3(\|x_0-x^*\|)})$$

and

$$\begin{aligned}\|A_0F(z_0)\| &\leq [\frac{3}{2}\|J_0\|\|F'(y_0)^{-1}F'(x^*)\| \\ &+\|I-\frac{3}{2}J_0\|\|F'(x_0)^{-1}F'(x^*)\|]\|F'(x^*)^{-1}F(z_0)\| \\ &\leq \left\{\frac{3}{4}(1+\frac{1}{1-g_3(\|x_0-x^*\|)})\frac{M}{1-L_0g_2(\|x_0-x^*\|)\|x_0-x^*\|}\right. \\ &\left.+(1+\frac{3}{4}(1+\frac{1}{1-g_3(\|x_0-x^*\|)}))\frac{M}{1-L_0g_2(\|x_0-x^*\|)\|x_0-x^*\|}\right\}\|z_0-x^*\|,\end{aligned}$$

so, from the preceding estimate,

$$\begin{aligned}\|x_1-x^*\| &\leq \|z_0-x^*\|+\|A_0F(z_0)\| \\ &\leq g_4(\|x_0-x^*\|)\|x_0-x^*\|+\|A_0F(z_0)\| \\ &\leq \left\{1+\frac{3}{4}(1+\frac{1}{1-g_3(\|x_0-x^*\|)})\frac{M}{1-L_0g_2(\|x_0-x^*\|)\|x_0-x^*\|}\right. \\ &\left.+(1+\frac{3}{4}(1+\frac{1}{1-g_3(\|x_0-x^*\|}))\frac{M}{1-L_0\|x_0-x^*\|}\right\}g_4(\|x_0-x^*\|)\|x_0-x^*\| \\ &= g_5(\|x_0-x^*\|)\|x_0-x^*\| \\ &< \|x_0-x^*\|<r^*,\end{aligned}$$

(by (2.7)) which shows (2.17) for $n=0$. By simply replacing u_0,y_0,z_0,x_1 by u_k,y_k,z_k,x_{k+1} in the preceding estimates we arrive at estimates (2.13)-(2.17). Finally, from the estimates $\|x_{k+1}-x^*\|<\|x_k-x^*\|$ we obtain $\lim_{k\to\infty}x_k=x^*$.

□

REMARK 2.2. 1. In view of (2.10) and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x)-F'(x^*))+I\| \\ &\leq 1+\|F'(x^*)^{-1}(F'(x)-F'(x^*))\|\leq 1+L_0\|x-x^*\|\end{aligned}$$

condition (2.12) can be dropped and M can be replaced by

$$M(t)=1+L_0t.$$

Moreover, condition (2.11) can be replaced by the popular but stronger conditions

$$\|F'(x^*)^{-1}(F'(x)-F'(y))\|\leq L\|x-y\| \text{ for each } x,y\in D \tag{2.21}$$

or

$$\|F'(x^*)^{-1}(F'(x^*+t(x-x^*))-F'(x))\|\leq L(1-t)\|x-x^*\| \text{ for each } x,y\in D \text{ and } t\in[0,1].$$

2. The results obtained here can be used for operators F satisfying autonomous differential equations [3] of the form

$$F'(x)=P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*)=P(F(x^*))=P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x)=e^x-1$. Then, we can choose: $P(x)=x+1$.

3. The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method(GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [3, 4].
4. The radius r_A given in (2.3) was shown by us to be the convergence radius of Newton's method [3, 4]
$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \ldots \tag{2.22}$$
under the conditions (2.10) and (2.21). It follows from (2.3) and (2.8) that the convergence radius r^* of the three-step method (1.2) cannot be larger than the convergence radius r_A of the second order Newton's method (2.22). As already noted in [3, 4] r_A is at least as large as the convergence ball given by Rheinboldt [3, 4]
$$r_R = \frac{2}{3L}.$$
In particular, for $L_0 < L$ we have that
$$r_R < r_A$$
and
$$\frac{r_R}{r_A} \to \frac{1}{3} \;\; as \;\; \frac{L_0}{L} \to 0.$$
That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [3, 4].
5. It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of the stronger $(\mathcal{C})$ conditions used in [26]. Moreover, we can compute the computational order of convergence (COC) defined by
$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$
or the approximate computational order of convergence
$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$
This way we obtain in practice the order of convergence in a way that avoids the bounds given in [26] involving estimates up to the third Fréchet derivative of operator F.

3. Numerical Examples

We present numerical examples in this section.

EXAMPLE 3.1. Let $X = Y = \mathbb{R}^2, D = \bar{U}(0, 1), x^* = (0, 0)^T$ and define function F on D by
$$F(x) = (\sin x, \frac{1}{3}(e^x + 2x - 1))^T.$$
Then, using (2.10)-(2.12), we get $L_0 = L = 1$, $M = \frac{1}{3}(e + 2)$. Then, by (2.8) we obtain
$$r^* = 0.0270 < r_R = r_A = 0.6667$$

EXAMPLE 3.2. Let $X = Y = \mathbb{R}^3,\ D = \overline{U}(0,1)$. Define F on D for $v = (x, y, z)^T$ by

$$F(v) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $x^* = (0,0,0)^T$, $F'(x^*) = F'(x^*)^{-1} = diag\{1,1,1\}$, $L_0 = e - 1 < L = e$, $M = e$. Then, by (2.8) we obtain

$$r^* = 0.0045 < r_R = 0.2453 < r_A = 0.3249.$$

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