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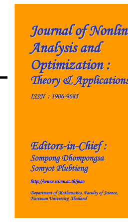

FIXED POINTS VIA wt-DISTANCE IN b-METRIC SPACES ENDOWED WITH A GRAPH

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ABSTRACT. We prove some fixed point theorems by using the concept of wt-distance in a b-metric space endowed with a graph. Our results will improve and supplement several well known comparable results in the existing literature.

KEYWORDS : b-metric; wt-distance; reflexive digraph; fixed point.
AMS Subject Classification: 54H25, 47H10

1. INTRODUCTION

Fixed point theory plays an important role in applications of many branches of mathematics such as variational and linear inequalities, linear algebra, mathematical models, optimization and the like. In 1989, Bakhtin [4] introduced the concept of b-metric spaces as a generalization of metric spaces and generalized the famous Banach contraction principle in metric spaces to b-metric spaces. Afterwards, several articles have been dedicated to the improvement of fixed point theory for single valued and multivalued mappings in b-metric spaces. In [15], Hussain et. al. introduced the concept of wt-distance on b-metric spaces and obtained some fixed point theorems in partially ordered b-metric spaces. In recent investigations, the study of fixed point theory endowed with a graph occupies a prominent place in many aspects. In 2005, Echenique [12] studied fixed point theory by using graphs. Espinola and Kirk [13] applied fixed point results in graph theory. Motivated by the idea given in some recent work on metric spaces with a graph (see [2, 3, 5, 7, 8, 17]), we reformulated some important fixed point results in metric spaces to b-metric spaces endowed with a graph by using wt-distance. As some consequences of our results, we obtain Banach Contraction Principle, Kannan fixed point theorem, Theorem 4[18] and some fixed point theorems in partially ordered b-metric spaces. Finally, some examples are provided to illustrate our results.

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2. SOME BASIC CONCEPTS

In this section, we recall some standard notations, definitions, and necessary results in b-metric spaces.

Definition 2.1. [10] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq s\,(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

It seems important to note that if $s = 1$, then the triangle inequality in a metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of b-metric spaces is effectively larger than that of the ordinary metric spaces. The following example illustrates the above remarks.

Example 2.2. [19] Let $X = \{-1, 0, 1\}$. Define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = 0$, $x \in X$ and $d(-1, 0) = 3$, $d(-1, 1) = d(0, 1) = 1$. Then (X, d) is a b-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that

$$d(-1, 1) + d(1, 0) = 1 + 1 = 2 < 3 = d(-1, 0).$$

It is easy to verify that $s = \frac{3}{2}$.

Example 2.3. [20] Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.

Definition 2.4. [6] Let (X, d) be a b-metric space, $x \in X$ and (x_n) be a sequence in X. Then

(i) (x_n) converges to x if and only if $\lim\limits_{n\to\infty} d(x_n, x) = 0$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$.
(ii) (x_n) is Cauchy if and only if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.
(iii) (X, d) is complete if and only if every Cauchy sequence in X is convergent.

Remark 2.5. [6] In a b-metric space (X, d), the following assertions hold:

(i) A convergent sequence has a unique limit.
(ii) Each convergent sequence is Cauchy.
(iii) In general, a b-metric is not continuous.

Theorem 2.6. [1] *Let (X, d) be a b-metric space and suppose that (x_n) and (y_n) converge to $x, y \in X$, respectively. Then, we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to\infty}\, d(x_n, y_n) \leq \limsup_{n\to\infty}\, d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$.

Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x, z) \leq \liminf_{n\to\infty}\, d(x_n, z) \leq \limsup_{n\to\infty}\, d(x_n, z) \leq s d(x, z).$$

Definition 2.7. [15] Let (X, d) be a b-metric space with constant $s \geq 1$. Then a function $p : X \times X \to [0, \infty)$ is called a wt-distance on X if the following conditions are satisfied:

(i) $p(x,z) \leq s\,(p(x,y) + p(y,z))$ for any $x, y, z \in X$;
(ii) for any $x \in X$, $p(x,.) : X \to [0,\infty)$ is s-lower semi-continuous;
(iii) for any $\epsilon > 0$ there exists $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \epsilon$.

Let us recall that a real valued function f defined on a b-metric space X is said to be s-lower semi-continuous at a point x_0 in X if $\liminf_{x_n \to x_0} f(x_n) = \infty$ or $f(x_0) \leq \liminf_{x_n \to x_0} sf(x_n)$, whenever $x_n \in X$ for each $n \in \mathbb{N}$ and $x_n \to x_0$[16].

We now present some examples of wt-distance.

Example 2.8. [15] Let (X,d) be a b-metric space. Then d is a wt-distance on X.

Example 2.9. Let (X,d) be a b-metric space. Then a function $p : X \times X \to [0,\infty)$ defined by $p(x,y) = c$ for every $x, y \in X$ is a wt-distance on X, where c is a positive real number.

Example 2.10. Let $X = [0,\infty)$ and $d(x,y) = (x-y)^2$ be a b-metric on X with constant $s = 2$. Then a function $p : X \times X \to [0,\infty)$ defined by $p(x,y) = x^2 + y^2$ for every $x, y \in X$ is a wt-distance on X.

Example 2.11. Let $X = [0,\infty)$ and $d(x,y) = (x-y)^2$ be a b-metric on X with constant $s = 2$. Then a function $p : X \times X \to [0,\infty)$ defined by $p(x,y) = y^2$ for every $x, y \in X$ is a wt-distance on X.

On wt-distance, we have the following important remark:

Remark 2.12. If p is a wt-distance on X, then

(i) $p(x,y) = p(y,x)$ does not necessarily hold for all x, $y \in X$;
(ii) $p(x,y) = 0$ if and only if $x = y$ does not necessarily hold for all x, $y \in X$.

Lemma 2.13. [15] *Let (X,d) be a b-metric space with constant $s \geq 1$ and let p be a wt-distance on X. Let (x_n) and (y_n) be sequences in X, let (α_n) and (β_n) be sequences in $[0,\infty)$ converging to 0, and let $x, y, z \in X$. Then the following hold:*

(i) *If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$;*
(ii) *if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converges to z;*
(iii) *if $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence;*
(iv) *if $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence.*

We next review some basic notions in graph theory.

Let (X,d) be a b-metric space. We assume that G is a reflexive digraph with the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains no parallel edges. So we can identify G with the pair $(V(G), E(G))$. G may be considered as a weighted graph by assigning to each edge the distance between its vertices. By G^{-1} we denote the graph obtained from G by reversing the direction of edges i.e., $E(G^{-1}) = \{(x,y) \in X \times X : (y,x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a digraph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Our graph theory notations and terminology are standard and can be found in all graph theory books, like [9, 11, 14]. If x, y are vertices of the digraph G, then a path in G from x to y of length n $(n \in \mathbb{N})$ is a sequence $(x_i)_{i=0}^n$ of $n+1$ vertices such that $x_0 = x,\ x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \cdots, n$. A graph G is connected if there is a path between any two vertices of G. G is weakly connected if $\tilde{G}$ is connected.

3. MAIN RESULTS

In this section we assume that (X, d) is a b-metric space with the coefficient $s \geq 1$, and G is a reflexive digraph such that $V(G) = X$ and G has no parallel edges. For any mapping $f : X \to X$, C_f is the set of all elements x of X such that $(f^n x, f^m x) \in E(\tilde{G})$ for $m,\ n = 0, 1, 2, \cdots$.

Theorem 3.1. *Let (X, d) be a complete b-metric space endowed with a graph G, p a wt-distance on X and the mapping $f : X \to X$ be such that*

$$p(fx, fy) \leq k\, p(x, y) \tag{3.1}$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$, where $k \in [0, \frac{1}{s})$ is a constant. Suppose the triple (X, d, G) has the following property:

$(*)$ If (x_n) is a sequence in X such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that $(x_{n_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.
Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, f has a unique fixed point in X if the graph G has the following property:

$(**)$ If $x,\ y$ are fixed points of f in X, then $(x, y) \in E(\tilde{G})$.

Furthermore, if $u = fu$, then $p(u, u) = 0$.

Proof. Suppose that $C_f \neq \emptyset$. We choose an $x_0 \in C_f$ and keep it fixed. We can construct a sequence (x_n) such that $x_n = fx_{n-1} = f^n x_0,\ n = 1, 2, 3, \cdots$. Evidently, $(x_n, x_m) \in E(\tilde{G})$ for $m,\ n = 0, 1, 2, \cdots$.

We now show that (x_n) is Cauchy in (X, d).
For any natural number n, we have by using condition (3.1) that

$$p(x_n, x_{n+1}) = p(fx_{n-1}, fx_n) \leq k\, p(x_{n-1}, x_n). \tag{3.2}$$

By repeated use of condition (3.2), we get

$$p(x_n, x_{n+1}) \leq k^n\, p(x_0, x_1) \tag{3.3}$$

for all $n \in \mathbb{N}$.
For $m, n \in \mathbb{N}$ with $m > n$, using condition (3.3), we have

$$\begin{aligned}
p(x_n, x_m) &\leq sp(x_n, x_{n+1}) + s^2 p(x_{n+1}, x_{n+2}) \\
&\quad + \cdots + s^{m-n-1} p(x_{m-2}, x_{m-1}) + s^{m-n-1} p(x_{m-1}, x_m) \\
&\leq \left[sk^n + s^2 k^{n+1} + \cdots + s^{m-n-1} k^{m-2} + s^{m-n-1} k^{m-1}\right] p(x_0, x_1) \\
&\leq sk^n \left[1 + sk + (sk)^2 + \cdots + (sk)^{m-n-2} + (sk)^{m-n-1}\right] p(x_0, x_1) \\
&\leq \frac{sk^n}{1 - sk}\, p(x_0, x_1).
\end{aligned}$$

By Lemma 2.13 (iii), it follows that (x_n) is a Cauchy sequence in X. Since (X, d) is complete, there exists $u \in X$ such that $x_n \to u$.

Let $n \in \mathbb{N}$ be an arbitrary but fixed. Since $x_m \to u$ and $p(x_n, \cdot)$ is s-lower semi-continuous, we have

$$p(x_n, u) \leq \liminf_{m\to\infty} sp(x_n, x_m) \leq \frac{s^2k^n}{1-sk}p(x_0, x_1). \tag{3.4}$$

By property $(*)$, there exists a subsequence (x_{n_i}) of (x_n) such that $(x_{n_i}, u) \in E(\tilde{G})$ for all $i \geq 1$.

Again, using condition (3.1), we have

$$p(x_{n_i+1}, fu) = p(fx_{n_i}, fu) \leq k\, p(x_{n_i}, u). \tag{3.5}$$

Thus, it follows from conditions (3.4) and (3.5) that
$p(x_{n_i+1}, u) \leq \frac{s^2k^{n_i+1}}{1-sk}p(x_0, x_1) \to 0$ and $p(x_{n_i+1}, fu) \leq k\, p(x_{n_i}, u) \to 0$ as $i \to \infty$. By Lemma 2.13(i), we obtain $fu = u$. Therefore, u is a fixed point of f.

The next is to show that the fixed point is unique. Assume that there is another fixed point v of f in X. By property $(**)$, we have $(u, v) \in E(\tilde{G})$. Then,

$$p(u, v) = p(fu, fv) \leq kp(u, v)$$

which gives that, $p(u, v) = 0$.

Again,

$$p(u, u) = p(fu, fu) \leq kp(u, u)$$

which gives that, $p(u, u) = 0$.
Now, $p(u, v) = 0$ and $p(u, u) = 0$ imply that $u = v$. Therefore, f has a unique fixed point u in X. Moreover, if $u = fu$, then $p(u, u) = 0$.

□

Corollary 3.2. *Let (X, d) be a complete b-metric space endowed with a wt-distance p and the mapping $f : X \to X$ be such that*

$$p(fx, fy) \leq k\, p(x, y)$$

for all $x, y \in X$, where $k \in [0, \frac{1}{s})$ is a constant. Then f has a unique fixed point in X. Moreover, if $u = fu$, then $p(u, u) = 0$.

Proof. The proof follows from Theorem 3.1 by taking $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. □

Remark 3.3. Theorem 3.1 is a generalization of Banach contraction theorem in metric spaces to b-metric spaces.

Corollary 3.4. *Let (X, d) be a complete b-metric space endowed with a partial ordering $\preceq$, p a wt-distance on X and the mapping $f : X \to X$ be such that*

$$p(fx, fy) \leq k\, p(x, y)$$

for all $x, y \in X$ with $x \preceq y$ or, $y \preceq x$, where $k \in [0, \frac{1}{s})$ is a constant. Suppose the triple $(X, d, \preceq)$ has the following property:

($\dagger$) *If (x_n) is a sequence in X such that $x_n \to x$ and x_n, x_{n+1} are comparable for all $n \geq 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that x_{n_i}, x are comparable for all $i \geq 1$.*

If there exists $x_0 \in X$ such that $f^n x_0, f^m x_0$ are comparable for $m,\ n = 0, 1, 2, \cdots$, then f has a fixed point in X. Moreover, f has a unique fixed point in X if the following property holds:

($\dagger\dagger$) *If $x,\ y$ are fixed points of f in X, then x, y are comparable.*

Furthermore, if $u = fu$, then $p(u, u) = 0$.

Proof. The proof can be obtained from Theorem 3.1 by taking $G = G_2$, where the graph G_2 is defined by $E(G_2) = \{(x, y) \in X \times X : x \preceq y \ or \ y \preceq x\}$. □

Theorem 3.5. *Let (X, d) be a complete b-metric space endowed with a graph G, p a wt-distance on X and the mapping $f : X \to X$ be such that*

$$p(fx, f^2x) \leq k\, p(x, fx) \tag{3.6}$$

for all $x \in X$ with $(x, fx) \in E(\tilde{G})$, where $k \in [0, \frac{1}{s})$ is a constant.

Suppose that

$$inf\{p(x, y) + p(x, fx) : x \in X \ with \ (x, fx) \in E(\tilde{G})\} > 0 \tag{3.7}$$

for every $y \in X$ with $y \neq fy$. Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, if $u = fu$, then $p(u, u) = 0$.

Proof. As in the proof of Theorem 3.1, we can construct a sequence (x_n) such that $x_n = fx_{n-1},\ n = 1,\ 2,\ 3, \cdots$ with $(x_n, x_m) \in E(\tilde{G})$ for $m,\ n = 0, 1, 2, \cdots$ and

$$p(x_n, x_{n+1}) \leq k^n\, p(x_0, x_1) \tag{3.8}$$

for all $n \in \mathbb{N}$.

By an argument similar to that used in Theorem 3.1, it follows that (x_n) is Cauchy in (X, d). As (X, d) is complete, there exists $u \in X$ such that $x_n \to u$. Proceeding similarly to that of Theorem 3.1, we obtain

$$p(x_n, u) \leq \frac{s^2k^n}{1 - sk}\, p(x_0, x_1).$$

Assume that $u \neq fu$. Then by using conditions (3.7) and (3.8), we have

$$\begin{aligned} 0 &< inf\{p(x_n, u) + p(x_n, x_{n+1}) : n \in \mathbb{N}\} \\ &\leq inf\{\frac{s^2k^n}{1 - sk}\, p(x_0, x_1) + k^n p(x_0, x_1) : n \in \mathbb{N}\} \\ &= 0 \end{aligned}$$

which is a contradiction. Therefore, $u = fu$ i.e., u is a fixed point of f in X.

Moreover,

$$p(u, u) = p(fu, fu) \leq kp(u, u)$$

implies that, $p(u, u) = 0$. □

Corollary 3.6. *Let (X, d) be a complete b-metric space endowed with a wt-distance p and the mapping $f : X \to X$ be such that*

$$p(fx, f^2x) \leq k\, p(x, fx)$$

for all $x \in X$, where $k \in [0, \frac{1}{s})$ is a constant.

Suppose that

$$inf\{p(x,y)+p(x,fx) : x\in X\} > 0$$

for every $y \in X$ with $y \neq fy$. Then f has a fixed point in X. Moreover, if $u = fu$, then $p(u,u) = 0$.

Proof. It can be obtained from Theorem 3.5 by taking $G = G_0$. □

Remark 3.7. Theorem 3.5 is a generalization of Theorem 4[18] in metric spaces to b-metric spaces.

Corollary 3.8. *Let (X,d) be a complete b-metric space endowed with a partial ordering $\preceq$, p a wt-distance on X and the mapping $f : X \to X$ be such that*

$$p(fx, f^2x) \leq k\, p(x, fx)$$

for all $x \in X$ with $x \preceq fx$ or, $fx \preceq x$, where $k \in [0, \frac{1}{s})$ is a constant.

Suppose that

$$inf\{p(x,y)+p(x,fx) : x\in X \ with \ x \preceq fx \ or \ fx \preceq x\} > 0$$

for every $y \in X$ with $y \neq fy$. If there exists $x_0 \in X$ such that $f^n x_0, f^m x_0$ are comparable for $m, n = 0, 1, 2, \cdots$, then f has a fixed point in X. Moreover, if $u = fu$, then $p(u,u) = 0$.

Proof. The proof can be obtained from Theorem 3.5 by taking $G = G_2$. □

As an application of Theorem 3.5 we obtain the following results.

Theorem 3.9. *Let (X,d) be a complete b-metric space with constant $s \geq 1$ and let $f : X \to X$ be such that*

$$d(fx, fy) \leq \alpha\, d(x,y) + \beta\, d(x,fx) + \gamma\, d(y,fy) \tag{3.9}$$

for every $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ with $\alpha+\beta+\gamma < \frac{1}{s}$. Then f has a unique fixed point in X.

Proof. We consider the b-metric d as a wt-distance on X. We also consider $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. Then, $C_f \neq \emptyset$ and $(x, fx) \in E(G)$ for all $x \in X$. From (3.9), we have

$$d(fx, f^2x) \leq \alpha d(x,fx) + \beta d(x,fx) + \gamma d(fx, f^2x)$$

which gives that

$$d(fx, f^2x) \leq \frac{\alpha+\beta}{1-\gamma} d(x,fx). \tag{3.10}$$

Let us put $k = \frac{\alpha+\beta}{1-\gamma}$. Then $k \in [0, \frac{1}{s})$ since $s(\alpha+\beta)+\gamma \leq s(\alpha+\beta+\gamma) < 1$.

Therefore, (3.10) becomes

$$d(fx, f^2x) \leq k\, d(x, fx)$$

for every $x \in X$ with $(x, fx) \in E(\tilde{G})$.

Suppose there exists $y \in X$ with $y \neq fy$ and

$$inf\ \left\{d(x,y)+d(x,fx) : x \in X \ with \ (x,fx) \in E(\tilde{G})\right\} = 0.$$

Then there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty}\{d(x_n, y) + d(x_n, fx_n)\} = 0.$$

So, we get $d(x_n, y) \to 0$ and $d(x_n, fx_n) \to 0$. By Lemma 2.13, it follows that $fx_n \to y$. We also have

$$\begin{aligned} d(y, fy) &\leq s\,[d(y, fx_n) + d(fx_n, fy)] \\ &\leq s\,[d(y, fx_n) + \alpha d(x_n, y) + \beta d(x_n, fx_n) + \gamma d(y, fy)] \end{aligned}$$

for any $n \in \mathbb{N}$ and hence

$$d(y, fy) \leq s\gamma\, d(y, fy).$$

Therefore, $d(y, fy) = 0$ i.e., $y = fy$. This is a contradiction. Hence, if $y \neq fy$, then

$$inf\,\left\{d(x, y) + d(x, fx) : x \in X \ with \ (x, fx) \in E(\tilde{G})\right\} > 0.$$

Now applying Theorem 3.5, we obtain a fixed point of f in X. Clearly, f has a unique fixed point in X. □

Theorem 3.10. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and let $f : X \to X$ be such that*

$$d(fx, fy) \leq \alpha\, d(x, fy) + \beta\, d(y, fx) \tag{3.11}$$

for every $x, y \in X$, where $\alpha, \beta \geq 0$ with $\alpha s < \frac{1}{1+s}$ or $\beta s < \frac{1}{1+s}$. Then f has a fixed point in X. Moreover, if $\alpha + \beta < 1$, then f has a unique fixed point in X.

Proof. We consider the b-metric d as a wt-distance on X. We also consider $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. Then, $C_f \neq \emptyset$ and $(x, fx) \in E(G)$ for all $x \in X$. From (3.11), we have

$$d(fx, f^2x) \leq \alpha d(x, f^2x) + \beta d(fx, fx) \leq \alpha s[d(x, fx) + d(fx, f^2x)]$$

which gives that

$$d(fx, f^2x) \leq \frac{\alpha s}{1 - \alpha s} d(x, fx). \tag{3.12}$$

Let us put $k = \frac{\alpha s}{1-\alpha s}$. Then $k \in [0, \frac{1}{s})$.
Therefore, (3.12) becomes

$$d(fx, f^2x) \leq k\, d(x, fx)$$

for every $x \in X$ with $(x, fx) \in E(\tilde{G})$.
Suppose there exists $y \in X$ with $y \neq fy$ and

$$inf\,\left\{d(x, y) + d(x, fx) : x \in X \ with \ (x, fx) \in E(\tilde{G})\right\} = 0.$$

Then there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty}\{d(x_n, y) + d(x_n, fx_n)\} = 0.$$

So, we get $d(x_n, y) \to 0$ and $d(x_n, fx_n) \to 0$. By Lemma 2.13, it follows that $fx_n \to y$. We also have

$$\begin{aligned} d(y, fy) &\leq s\,[d(y, fx_n) + d(fx_n, fy)] \\ &\leq s\,[d(y, fx_n) + \alpha d(x_n, fy) + \beta d(y, fx_n)] \\ &\leq s\,[d(y, fx_n) + \alpha s d(x_n, y) + \alpha s d(y, fy) + \beta d(y, fx_n)] \end{aligned}$$

for any $n \in \mathbb{N}$ and hence

$$d(y, fy) \leq s^2\alpha\, d(y, fy).$$

Therefore, $d(y, fy) = 0$ i.e., $y = fy$. This is a contradiction. Hence, if $y \neq fy$, then

$$inf \left\{ d(x, y) + d(x, fx) : x \in X \ with \ (x, fx) \in E(\tilde{G}) \right\} > 0.$$

By applying Theorem 3.5, we obtain a fixed point of f in X.
Now suppose that $\alpha + \beta < 1$. Assume that there are $u, v \in X$ such that $fu = u$ and $fv = v$. Then

$$d(u, v) = d(fu, fv) \leq \alpha\, d(u, v) + \beta\, d(v, u) = (\alpha + \beta) d(u, v).$$

This shows that $d(u, v) = 0$ i.e., $u = v$. Therefore, f has a unique fixed point in X. □

We furnish some examples in favour of our results.

Example 3.11. Let $X = [0, \infty)$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = \mid x - y \mid^2$ for all $x,\, y \in X$. Then (X, d) is a complete b-metric space with the coefficient $s = 2$. Let $p : X \times X \to [0, \infty)$ defined by $p(x, y) = x^2 + y^2$ for all $x, y \in X$ be a wt-distance on X. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(0, \frac{1}{7^n}) : n = 0,\, 1,\, 2,\, \cdots\}$.

Let $f : X \to X$ be defined by

$$fx = \frac{x}{7}, \ for \ all \ \ x \in X.$$

It is easy to check that

$$p(fx, fy) = k\, p(x, y)$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$, where $k = \frac{1}{49} \in [0, \frac{1}{s})$ is a constant. Obviously, $0 \in C_f$.
Also, any sequence (x_n) with the property $(x_n, x_{n+1}) \in E(\tilde{G})$ must be either a constant sequence or a sequence of the following form

$$\begin{aligned} x_n &= 0, \ \ if \ \ n \ is \ odd \\ &= \frac{1}{7^n}, \ \ if \ \ n \ is \ even \end{aligned}$$

where the words 'odd' and 'even' are interchangeable. Consequently it follows that property $(*)$ holds. Thus, we have all the conditions of Theorem 3.1 and 0 is the unique fixed point of f in X with $p(0, 0) = 0$.

We now examine that the condition $C_f \neq \emptyset$ in Theorem 3.5 can neither be relaxed.

Example 3.12. Let $X = \{0\} \cup \{\frac{1}{5^n} : n \geq 1\}$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = \mid x - y \mid^2$ for all $x,\, y \in X$. Then (X, d) is a complete b-metric space with the coefficient $s = 2$. Let $p : X \times X \to [0, \infty)$ defined by $p(x, y) = y^2$ for all $x, y \in X$ be a wt-distance on X. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(0, \frac{1}{5^n}) : n = 1,\, 2,\, 3,\, \cdots\}$.

Let $f : X \to X$ be defined by $f(0) = \frac{1}{5}$ and $f(\frac{1}{5^n}) = \frac{1}{5^{n+1}}$ for $n \geq 1$. Then, $(x, fx) \in E(\tilde{G})$ only for $x = 0$ and it is easy to check that

$$p(fx, f^2x) = k\, p(x, fx)$$

for all $x \in X$ with $(x, fx) \in E(\tilde{G})$, where $k = \frac{1}{25} \in [0, \frac{1}{s})$ is a constant.

Moreover, $y \neq fy$ for every $y \in X$. Let $y \in X$ be arbitrary and kept it fixed. Then,

$$inf\{p(x, y) + p(x, fx) : x \in X \ with \ (x, fx) \in E(\tilde{G})\}$$

$$\begin{aligned} &= inf\,\{p(x,y)+p(x,fx) : x = 0\} \\ &= p(0,y)+p(0,\frac{1}{5}) \\ &= y^2+\frac{1}{25} \\ &> 0. \end{aligned}$$

Thus,

$$inf\{p(x,y)+p(x,fx) : x \in X \ with \ (x,fx) \in E(\tilde{G})\} > 0$$

for every $y \in X$ with $y \neq fy$. Obviously, $C_f = \emptyset$.
Thus, we have all the conditions of Theorem 3.5 except $C_f \neq \emptyset$ and f possesses no fixed point in X.

We now supplement Theorem 3.5 by examination of conditions (3.6) and (3.7) in respect of their independence. We furnish Examples 3.13 and 3.14 below to show that these two conditions are independent in the sense that Theorem 3.5 shall fall through by dropping one in favour of the other.

Example 3.13. Let $X = \{0\} \cup \{\frac{1}{3^n} : n \geq 1\}$ and $d(x,y) = \mid x-y \mid^2$ for all $x,y \in X$. Then (X,d) is a complete b-metric space with the coefficient $s = 2$. Let $p : X \times X \to [0,\infty)$ defined by $p(x,y) = y^2$ for all $x,y \in X$ be a wt-distance on X. Let G be a digraph such that $V(G) = X$ and $E(G) = \{(0,0)\} \cup \{(\frac{1}{3^n}, \frac{1}{3^m}) : n,\ m = 1,\ 2,\ 3, \cdots\}$. Let $f : X \to X$ be defined by $f(0) = \frac{1}{3}$ and $f(\frac{1}{3^n}) = \frac{1}{3^{n+1}}$ for $n \geq 1$. Then, $(x,fx) \in E(\tilde{G})$ for all $x \in X \setminus \{0\}$, and it is easy to verify that

$$p(fx,f^2x) = k\,p(x,fx)$$

for all $x \in X$ with $(x,fx) \in E(\tilde{G})$, where $k = \frac{1}{9} \in [0,\frac{1}{s})$ is a constant.
Therefore, f satisfies condition (3.6).

On the other hand, $y \neq fy$ for $y = 0$. But, for $y = 0$ we have

$$inf\{p(x,y)+p(x,fx) : x \in X \ with \ (x,fx) \in E(\tilde{G})\}$$

$$\begin{aligned} &= inf\,\{p(x,0)+p(x,fx) : x \in X \setminus \{0\}\} \\ &= inf\,\{(fx)^2 : x \in X \setminus \{0\}\} \\ &= inf\left\{\frac{1}{3^{2n+2}} : n \geq 1\right\} \\ &= 0. \end{aligned}$$

Thus, condition (3.7) does not hold. Clearly, $\frac{1}{3} \in C_f$ but f possesses no fixed point in X. We note that Theorem 3.5 does not hold without condition (3.7).

Example 3.14. Let $X = [4,\infty) \cup \{2,3\}$ and $d(x,y) = \mid x-y \mid^2$ for all $x,y \in X$. Then (X,d) is a complete b-metric space with the coefficient $s = 2$. Let $p : X \times X \to [0,\infty)$ defined by $p(x,y) = x^2+y^2$ for all $x,y \in X$ be a wt-distance on X. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(2,3)\}$.

Define $f : X \to X$ where

$$\begin{aligned} fx &= 2,\ for\, x \in (X \setminus \{2\}) \\ &= 3,\ for\, x = 2. \end{aligned}$$

Then, $(x,fx) \in E(\tilde{G})$ for $x = 2,\ 3$.

We note that $y \neq fy$ for every $y \in X$. Let $y \in X$ be arbitrary and kept it fixed. Then,

$$\begin{aligned} & inf\{p(x,y)+p(x,fx) : x \in X \ with \ (x,fx) \in E(\tilde{G})\} \\ &= inf\ \{p(x,y)+p(x,fx) : x = 2,\, 3\} \\ &= inf\ \{p(x,y)+13 : x = 2,\, 3\} \\ &> 0. \end{aligned}$$

Therefore,

$$inf\{p(x,y)+p(x,fx) : x \in X \ with \ (x,fx) \in E(\tilde{G})\} > 0$$

for every $y \in X$ with $y \neq fy$. Thus, condition (3.7) is satisfied. However, for $x = 2$, we find that $p(fx, f^2x) = p(3,2) = 13 > k\,p(x,fx)$ for any $k \in [0, \frac{1}{s})$. So, condition (3.6) does not hold. Clearly, $2,\ 3 \in C_f$ but f possesses no fixed point in X. In this case we observe that Theorem 3.5 is invalid without condition (3.6).

We now examine validity of Theorem 3.5.

Example 3.15. Let $X = [0,\infty)$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) =\mid x-y \mid^2$ for all $x,\ y \in X$. Then (X,d) is a complete b-metric space with the coefficient $s = 2$. Let $p : X \times X \to [0,\infty)$ defined by $p(x,y) = y^2$ for all $x, y \in X$ be a wt-distance on X. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(1, \frac{1}{2})\}$.

Let $f : X \to X$ be defined by

$$fx = \frac{x}{2},\ for\ all\ \ x \in X.$$

Then, $(x,fx) \in E(\tilde{G})$ for $x = 0,\ 1$.

It is easy to verify that

$$p(fx, f^2x) = k\,p(x,fx)$$

for all $x \in X$ with $(x,fx) \in E(\tilde{G})$, where $k = \frac{1}{4} \in [0, \frac{1}{s})$ is a constant.

Obviously, $0 \in C_f$.

We note that $y \neq fy$ for every $y \in X \setminus \{0\}$. Let $y \in X \setminus \{0\}$ be arbitrary and kept it fixed. Then,

$$\begin{aligned} & inf\{p(x,y)+p(x,fx) : x \in X \ with \ (x,fx) \in E(\tilde{G})\} \\ &= inf\ \{p(x,y)+p(x,fx) : x = 0,\, 1\} \\ &= inf\ \left\{y^2+\frac{x^2}{4} : x = 0,\, 1\right\} \\ &> 0. \end{aligned}$$

Therefore,

$$inf\{p(x,y)+p(x,fx) : x \in X \ with \ (x,fx) \in E(\tilde{G})\} > 0$$

for every $y \in X$ with $y \neq fy$. Thus, we have all the conditions of Theorem 3.5 and 0 is the unique fixed point of f in X with $p(0,0) = 0$.

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