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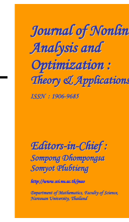

CONVERGENCE THEOREMS FOR LIPSCHITZ PSEUDOCONTRACTIVE NON-SELF MAPPINGS IN BANACH SPACES

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ABSTRACT. In this paper, we introduce an iterative process and prove strong convergence result for finding the fixed point of Lipschitz pseudocontractive non-self mapping in Banach spaces more general than Hilbert spaces. In addition, strong and weak convergence of Mann type sequence to a fixed point of λ-strictly pseudocontractive non-self mapping is investigated. Moreover, a numerical example which shows the conclusion of our result is presented. Our results improve and generalize many known results in the current literature.

KEYWORDS : Fixed points, nonexpansive non-self mappings, pseudocontractive mappings, uniformly Gâteaux differentiable norm.
AMS Subject Classification: 37C20, 47H09, 47H10, 47J05.

1. INTRODUCTION

Let C be a nonempty subset of a real Banach space E with dual E^*. A mapping $T : C \longrightarrow E$ is called *L-Lipschitz* if there exits $L \geq 0$ such that

$$||Tx - Ty| \leq L||x - y||, \forall x, y \in C.$$

If $L = 1$ then T is called *nonexpansive* mapping. T is called *λ-strictly pseudocontractive mapping* if there exist $\lambda \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq ||x - y||^2 - \lambda||(I - T)x - (I - T)y||^2, \forall x, y \in C,$$

and T is called *pseudocontractive* if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq ||x - y||^2, \text{ for all } x, y \in C, \tag{1.1}$$

where $J : E \to 2^{E^*}$ is the *normalized duality* mapping given by $Jx := \{f^* \in E^* : \langle x, f^*\rangle = ||x||^2 = ||f^*||^2\}$, where $\langle ., .\rangle$ denotes the generalized duality pairing. It is well known that J is single-valued whenever E is *smooth*. J is said to be weakly

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sequentially continuous if it is single valued and weak-to-weak* continuous; that is, if $x_n \rightharpoonup x$ in E, then $J(x_n) \rightharpoonup^* J(x)$ in E^*.

Due to Kato [11] inequality (1.1) is equivalent to the following inequality for all $t \geq 0$.

$$||x-y|| \leq ||x-y+t[(I-T)x-(I-T)y]||, \text{ for all } x,y \in C.$$

We note that every nonexpansive and every λ-strictly pseudocontractive mappings are Lipschitz with constants $L=1$ and $L=\frac{1+\lambda}{\lambda}$, respectively. Moreover, we observe that the class of Lipschitz pseudocontractive mappings includes the class of nonexpansive and the class of λ- strictly pseudocontractive mappings.

Pseudocontractive mappings are also related to the important class of nonlinear operators known as *accretive* mappings. A mapping $A : C \longrightarrow E$ is called *accretive* if there exists $j(x-y) \in J(x-y)$ such that $\langle Ax-Ay, j(x-y)\rangle \geq 0$ for all $x,y \in C$.

A mapping A is accretive if and only if $T := I - A$ is pseudocontractive and thus the zero set of A, $N(A) = \{x \in C : Ax = 0\}$, is the fixed point set of T, $F(T) = \{x \in C : Tx = x\}$. It is also known that the equilibrium points of some evolution systems are the solutions of the equation $Ax = 0$, when A is accretive mapping (see e.g. [31]). Consequently, several authors have studied iterative methods for approximating fixed points of a nonexpansive or pseudocontractive mapping T (see for example [2, 4, 13, 18, 22, 32] and the references contained therein).

In 1953, Mann [13] introduced the following iteration:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \tag{1.2}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0,1]$. The sequence $\{x_n\}$ generated by (1.2) is called *Mann iteration sequence*. The Mann iteration has been extensively investigated for nonexpansive mappings (see, e.g., [21]). In an infinite-dimensional Hilbert space, the Mann iteration can provide only weak convergence [8].

Attempts to modify the Mann iterative method, so that strong convergence is guaranteed, have been made. In 1974, Ishikawa [9] introduced an iterative process, which in some sense is more general than that of Mann and which converges to a fixed point of a Lipschitz pseudocontractive self-mapping T of C. The following theorem is proved.

Theorem IS ([9]). If C is a compact convex subset of a Hilbert space H, $T : C \longrightarrow C$ is a Lipschitz pseudocontractive mapping and x_0 is any point of C, then the sequence $\{x_n\}$ converges strongly to a fixed point of T, where $\{x_n\}$ is defined iteratively for each integer $n \geq 0$ by

$$y_n = \beta_n x_n + (1-\beta_n)Tx_n, \; x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n, \; n \geq 0, \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers satisfying the conditions:
(i) $0 \leq \alpha_n \leq \beta_n \leq 1$; (ii) $\lim\limits_{n\longrightarrow\infty} \beta_n = 0$; (iii) $\sum \alpha_n\beta_n = \infty$.

The iterative method of Theorem IS, which is now referred to as the Ishikawa iterative method has been studied extensively by various authors. But it is still an open question whether or not this method can be employed to approximate fixed points of Lipschitz pseudocontractive mappings without the compactness assumption on C or T (see, e.g., [3, 19, 20]).

In 2003, Chidume and Zegeye [4] constructed an iterative scheme which provides the conclusion of Theorem IS for an important class of Lipschitz pseudocontractive self-mapping without the requirement that C or T is compact. They proved the following Theorem.

Theorem CZ ([4]). Let C be nonempty closed convex subset of a reflexive real Banach space E with a uniformly Gâteaux differentiable norm. Let $T : C \to C$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$ and $F(T) \neq \emptyset$. Suppose that every closed, convex and bounded subset of C has the fixed point property for nonexpansive self-mappings. Let a sequence $\{x_n\}$ be generated from arbitrary $x_1 \in C$ by

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - x_1), \ for\ all\ n \geq 0, \tag{1.4}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are real sequences in $(0, 1]$ satisfying certain conditions. Then, $\{x_n\}$ converges strongly to a fixed point of T.

Remark 1.1. We remark that Theorem CZ has extended the results of Ishikawa [9] and other related results to Banach spaces more general than Hilbert spaces without compactness assumption on C. However, in all the above results, the operator T remains a self-mapping of a nonempty closed convex subset C of a Banach space E. If, however, the domain of T, C, is a proper subset of E (and this is the case in several applications), and T maps C into E, then the iterative processes (1.2), (1.3) and (1.4) studied by these authors may fail to be well defined. Many researchers have made significant progress to overcome this problem by employing the concept of sunny nonexpansive mappings.

Let C be a nonempty closed convex subset of a Banach space E and let D be a nonempty subset of C. A mapping $P : C \to D$ is said to be *retraction*, if $Px = x$ for all $x \in D$. A retraction $P : C \to D$ is sunny if it satisfies the property: $P(Px + t(x - Px)) = Px$ for $x \in C$ and $t > 0$, whenever $Px + t(x - Px) \in C$. P is said to be *sunny nonexpansive* if it is both sunny and nonexpansive mapping. A subset D of C is also called *sunny nonexpansive* retract of C if there exists a sunny nonexpansive retraction of C onto D.

It is well known [1] that in a smooth Banach space E, a retraction mapping P is sunny nonexpansive if and only if the following inequality holds:

$$\langle x - Px, J(y - Px)\rangle \leq 0, x \in C, y \in D. \tag{1.5}$$

Recall that, if $E = H$, a real Hilbert space, then the nearest point metric projection $P_C : H \longrightarrow C$ is characterized by the inequality $\langle x - P_C x, y - P_C x\rangle \leq 0$, for all $y \in C$. Hence, the metric projection, P_C, is a sunny nonexpansive retraction in the setting of Hilbert spaces. However, this fact characterizes Hilbert spaces and it is not available in more general Banach spaces.

For the approximation of fixed points of non-self mappings, Matsushita and Takahashi [12] have studied and proved the following theorem.

Theorem MT ([12]). Let E be a uniformly convex Banach space whose norm is uniformly Gateâux differentiable, let C be a nonempty closed convex subset of E and let T be a nonexpansive mapping from C into E with $F(T) \neq \emptyset$. Suppose that C is a sunny nonexpansive retract of E. Let $\{\alpha_n\}$ be a sequence such that $0 \leq \alpha_n \leq 1$, $\lim_{n \longrightarrow \infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$ and $\sum |\alpha_{n+1} - \alpha_n| < \infty$. Let u and x_0 be elements of C. Suppose that $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)PTx_n, for\ n = 0, 1, 2, ..., \tag{1.6}$$

where P is a sunny nonexpansive retraction from E onto C. Then $\{x_n\}$ converges strongly to $z \in F(T)$.

Several authors have studied implicit and explicit iterative schemes of the type (1.6) (see, e.g., [5, 14, 23, 24, 25, 28, 29, 33] and the references therein) for non-self mappings. However, as it has been indicated by Colao and Marino [7], calculating P is time consuming process, even in Hilbert spaces when P is a metric projection, and it may require an approximating algorithm for itself. To avoid the necessity of using an auxiliary mapping P, Colao and Marino [7] introduced a new search strategy for the coefficient α_n which makes the Krasnoselskii-Mann algorithm well defined in the Hilbert space setting. They obtained the following weak and strong convergence of the algorithm for nonexpansive non-self mappings. We shall need the following definitions.

A set $C \subset E$ is said to be *strictly convex* if it is convex and with the property that $x, y \in \partial C$ and $t \in (0, 1)$ implies that $tx + (1 - t)y \in \mathring{C}$, where ∂C and $\mathring{C}$ denotes boundary and interior of C respectively. In other words, if the boundary of C does not contain any segment.

A mapping $T : C \longrightarrow E$ is said to satisfy the *inward* condition if, for any $x \in C$, we have $Tx \in I_C(x) := \{x + c(u - x) : c \geq 1, u \in C\}$. T is said to satisfy the *weakly inward* condition if for each $x \in C$, $Tx \in \overline{I_C(x)}$, where $\overline{I_C(x)}$ is the closure of $I_C(x)$. Note that $I_C(x)$ is convex whenever C is convex.

Theorem CM ([7]). Let C be a convex, closed and nonempty subset of a Hilbert space H and let $T : C \longrightarrow H$ be a mapping. Then the algorithm

$$\begin{cases} x_0 \in C, \\ \alpha_0 = max\{\frac{1}{2}, h(x_0)\}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ \alpha_{n+1} = max\{\alpha_n, h(x_{n+1})\}, \end{cases} \tag{1.7}$$

where $h : C \longrightarrow \mathbb{R}$ is defined by $h(x) = \inf\{\lambda \geq 0 : \lambda x + (1 - \lambda)Tx \in C\}$, is well defined. Further, if C is strictly convex and T is nonexpansive mapping which satisfies the inward condition and such that $F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to a point $p \in F(T)$. Moreover, if $\sum \alpha_n < \infty$, then the convergence is strong.

We remark that *Theorem CM* is applicable for approximating fixed points of nonexpansive non-self mappings.

Our concern now is the following: *can we construct an iterative scheme which converges strongly to a fixed point of pseudocontractive non-self mappings, without using projection mapping, which is more general than nonexpansive mappings in*

Banach spaces?

It is our purpose in this paper to construct an iterative scheme which converges strongly to a fixed point of Lipschitz pseudocontractive non-self mappings in Banach spaces more general than Hilbert spaces. Our results provide an affirmative answer to our concern. Our results extend *Theorem CM*, *Theorem CZ* and the references therein to the more general class of Lipschitz pseudocontractive non-self mappings.

2. PRELIMINARIES

Let E be a real Banach space and let $S := \{x \in E : ||x|| = 1\}$ denote the unit sphere of E. E is said to have *Gâteaux differentiable* norm if the limit

$$\lim_{t\to 0} \frac{||x+ty|| - ||x||}{t} \tag{2.1}$$

exists for each $x, y \in S$. Such E is called *smooth*. The space E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$, the limit (2.1) is attained uniformly for $x \in S$.

The *modulus of smoothness* of E is a function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \sup\Big\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\Big\}.$$

A Banach space E is called q*-uniformly smooth* if there exist a constant $c > 0$ and a real number $q \in (1, \infty)$ such that $\rho_E(\tau) \le c\tau^q$. E is called *uniformly smooth* if $\lim\limits_{\tau\to 0} \dfrac{\rho_E(\tau)}{\tau} = 0$. The Lebesgue L_p, the sequence ℓ_p and the Sobolev W_p^m spaces, for $p \in (1, \infty)$, are examples of uniformly smooth Banach spaces.

It is well known that every uniformly smooth space has uniformly Gâteaux differentiable norm (see, e.g., [6]).

Let C be a nonempty subset of E. A sequence $\{x_n\} \subset C$ is said to be *Fejèr-monotone* with respect to a set $D \subset C$ if, for any element $x \in D$, $||x_{n+1} - x|| \le ||x_n - x||, \forall n \in \mathbb{N}$.

In the sequel, we shall make use of the following lemmas.

Lemma 2.1. [16] *Let E be a real normed linear space and J be the normalized duality mapping on E. Then for any given $x, y \in E$, the following inequality holds:*

$$||x + y||^2 \le ||x||^2 + 2\langle y, j(x + y)\rangle, \forall j(x + y) \in J(x + y).$$

Lemma 2.2. [15] *Let $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative numbers satisfying the conditions: $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_1^\infty \alpha_n = \infty$, and $\frac{\gamma_n}{\alpha_n} \to 0$, as $n \to \infty$. Let the recursive inequality*

$$\lambda_{n+1}^2 \le \lambda_n^2 - \alpha_n\psi(\lambda_{n+1}) + \gamma_n, n = 1, 2, ..., \tag{2.2}$$

be given where $\psi : [0, \infty) \to [0, \infty)$ is a strictly increasing function such that it is positive on $(0, \infty)$ and $\psi(0) = 0$. Then $\lambda_n \to 0$ as $n \to \infty$.

Lemma 2.3. [16] *Let E be a Banach space. Suppose C is a nonempty closed convex subset of E and $T : C \to E$ be a continuous pseudocontractive mapping* *satisfying the weakly inward condition. Then for* $u \in C$*, there exists a unique path* $t \to y_t \in C,\ t \in [0,1)$*, satisfying the following condition:*

$$y_t = tTy_t + (1-t)u.$$

We note that in Lemma 2.3 if, in addition, $F(T) \neq \emptyset$ then $\{y_t\}$ is bounded. Furthermore, if E is assumed to be a reflexive Banach space with uniformly Gâteaux differentiable norm and every closed convex and bounded subset of C has the fixed point property for nonexpansive self-mappings, then as $t \to 1^-$, the path converges strongly to a fixed point x^* of T, which is the unique solution of the variational inequality (see [17]):

$$\langle x^* - u, J(x^* - w)\rangle \leq 0, \forall w \in F(T).$$

Lemma 2.4. [27] *Let* E *be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, J(x)\rangle + 2||Ky||^2,\ \forall x, y \in E.$$

Lemma 2.5. [30] *Let* E *be a reflexive Banach space and let C be a closed convex subset of* E*. Let* f *be a proper convex lower semi-continuous function of* C *into* $(-\infty, \infty]$ *and suppose that* $f(x_n) \to \infty$ *as* $||x_n|| \to \infty$*. Then, there exists* $x_0 \in C$ *such that* $f(x_0) = \inf\{f(x) : x \in C\}$*.*

Lemma 2.6. [34] *Let* C *be a non empty subset of a real 2-uniformly smooth Banach space* E *with the best smooth constant* $K > 0$ *and let* $T : C \longrightarrow E$ *be a* $\lambda-$*strictly pseudocontractive mapping. For* $\alpha \in (0,1) \cap (0, \frac{\lambda}{K^2}]$*, we define* $T_\alpha : C \longrightarrow E$ *by* $T_\alpha x = (1-\alpha)x + \alpha Tx$*. Then* T_α *is nonexpansive and* $F(T_\alpha) = F(T)$*.*

Remark 2.7. If C is convex and T satisfies the inward condition then T_α satisfies the inward condition.

Lemma 2.8. [34] *Let* C *be a non empty subset of a real 2-uniformly smooth Banach space* E*. Suppose that the normalized duality mapping* $J : E \longrightarrow E^*$ *is weakly sequentially continuous at zero. Let* $T : C \longrightarrow E$ *be a* $\lambda-$*strictly pseudocotractive mapping for* $0 < \lambda < 1$*. Then, for any* $\{x_n\} \subset C$*, if* $x_n \rightharpoonup x$*, and* $x_n - Tx_n \longrightarrow y \in E$ *then* $x - Tx = y$*.*

3. MAIN RESULTS

Lemma 3.1. *Let* C *be a nonempty, closed and convex subset of a real Banach space* E *and* $T : C \longrightarrow E$ *be a mapping. For any given element* u *in* C *and any arbitrarily fixed* μ *in* $[0,1)$ *define* $f : C \longrightarrow \mathbb{R}$ *by* $f(x) = \inf\{\lambda \geq 0 : \lambda\big(\mu u + (1-\mu)x\big) + (1-\lambda)Tx \in C\}$*. Then the following hold:*

1) *For any* $x \in C$*,* $f(x) \in [0,1]$ *and* $f(x) = 0$ *if and only if* $Tx \in C$*;*
2) *For any* $x \in C$ *and* $\beta \in [f(x), 1]$*,* $\beta\big(\mu u + (1-\mu)x\big) + (1-\beta)Tx \in C$*;*
3) *If* T *satisfies the inward condition, then* $f(x) < 1$*, for all* $x \in C$*;*
4) *If* $Tx \notin C$*, then* $f(x)\big(\mu u + (1-\mu)x\big) + \big(1 - f(x)\big)Tx \in \partial C$*.*

Proof. 1) Clearly $f(x) \geq 0$ for all $x \in C$. If $\lambda = 1$, by convexity of C, we have $\mu u + (1-\mu)x \in C$. Therefore, $f(x) \leq 1$ and hence $f(x) \in [0,1]$. One can also easily show that $f(x) = 0$ if and only if $Tx \in C$.
2) The proof of (2) follows directly from the definition of $f(x)$.
3) We first show that $I_C(x) \subset I_C(z)$, where $z = \mu u + (1-\mu)x$. Let $y \in I_C(x)$. Then $y = x + c(v - x)$, for some $c \geq 1$ and $v \in C$. Since $\mu < 1$, we can choose a real number $k > 1$ such that $\mu < 1 - \frac{1}{k}$. Then we have:

$$y = cv + (1-c)x$$

$$\begin{aligned}
&= z + cv + (1-c)x - z \\
&= z + kc\big[\frac{1}{k}v + \frac{(1-c)}{kc}x - \frac{1}{kc}z + z - z\big] \\
&= z + kc\big[\frac{1}{k}v + \frac{(1-c)}{kc}x + (1-\frac{1}{kc})z - z\big] \\
&= z + kc\big[\frac{1}{k}v + \frac{(1-c)}{kc}x + (1-\frac{1}{kc})(\mu u + (1-\mu)x) - z\big] \\
&= z + kc\big[\frac{1}{k}v + \frac{(\mu + (k-1)c - kc\mu)}{kc}x + \mu(1-\frac{1}{kc})u - z\big].
\end{aligned}$$

It is easy to verify that $\frac{\mu+(k-1)c-kc\mu}{kc} \in (0,1)$. Then, since C is convex, $w := \frac{1}{k}v + \frac{(\mu+(k-1)c-kc\mu)}{kc}x + \mu(1-\frac{1}{kc})u \in C$. Then, $y \in I_C(z)$ and hence $I_C(x) \subset I_C(z)$. Now if T satisfies the inward condition, then $Tx \in I_C(x) \subset I_C(z)$. Thus, $Tx = z + b(w' - z)$ for some $b \geq 1$ and $w' \in C$ which gives

$$\frac{1}{b}Tx + (1-\frac{1}{b})z = w' \in C.$$

This implies that

$$f(x) = \inf\{\lambda \geq 0 : \lambda(\mu u + (1-\mu)x) + (1-\lambda)Tx \in C\} \leq 1 - \frac{1}{b} < 1.$$

4) Let $\{\beta_n\} \subset (0, f(x))$ be a real sequence such that $\beta_n \longrightarrow f(x)$. By the definition of f, we have $z_n := \beta_n\big(\mu u + (1-\mu)x\big) + (1-\beta_n)Tx \notin C$.
Now, since $\beta_n \longrightarrow f(x)$, we have:

$$\begin{aligned}
&||z_n - \big[f(x)\big(\mu u + (1-\mu)x\big) + (1-f(x))Tx\big]|| \\
&= ||(\beta_n - f(x))\big[\mu u + (1-\mu)x - Tx\big]|| \\
&\leq |\beta_n - f(x)|\big[||\mu u + (1-\mu)x|| + ||Tx||\big] \longrightarrow 0 \; as \; n \to \infty.
\end{aligned}$$

Thus, $z_n \longrightarrow f(x)\big(\mu u + (1-\mu)x\big) + (1-f(x))Tx \in C$.
But since $z_n = \beta_n\big(\mu u + (1-\mu)x\big) + (1-\beta_n)Tx \notin C$, for all $n \geq 1$, we have

$$f(x)\big(\mu u + (1-\mu)x\big) + (1-f(x))Tx \in \partial C.$$

3.1. **Convergence Theorem for Pseudocontractive Mappings.**

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Banach space E and $T : C \to E$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$ and $F(T) \neq \emptyset$. Suppose that T satisfies the inward condition. Let $\{\mu_n\} \subset (0,1)$, u be any point in C and $\{x_n\}$ be a sequence generated from arbitrary $x_1 \in C$ by:*

$$\begin{cases} \alpha_1 = max\{\frac{1}{2}, f(x_1)\}, \\ x_{n+1} = \alpha_n\big(\mu_n u + (1-\mu_n)x_n\big) + (1-\alpha_n)Tx_n, \\ \alpha_{n+1} \in [max\{\alpha_n, f(x_{n+1})\}, 1), \; n \geq 1, \end{cases} \tag{3.1}$$

where $f(x_n) := \inf\{\lambda \geq 0 : \lambda\big(\mu_n u + (1-\mu_n)x_n\big) + (1-\lambda)Tx_n \in C\}$. Let the pair (α_n, μ_n) satisfies the following conditions:

(i) $\lim_{n\to\infty} \frac{\mu_n\alpha_n}{1-\alpha_n} = 0$; *(ii)* $\sum_{n=1}^{\infty} \alpha_n\mu_n = \infty$;

(iii) $\lim_{n\to\infty} \frac{(1-\alpha_n)^2}{\alpha_n\mu_n} = 0$; *(iv)* $\lim_{n\to\infty} \frac{\big(\frac{(1-\alpha_n)\alpha_{n-1}\mu_{n-1}}{(1-\alpha_{n-1})\alpha_n\mu_n} - 1\big)}{\alpha_n\mu_n} = 0.$

Then, algorithm (3.1) is well defined and $||x_n - Tx_n|| \to 0$ as $n \to \infty$.

Proof. Since T satisfies the inward condition by Lemma 3.1(3) we have that $\alpha_n \in [f(x_n), 1)$ for all $n \in \mathbb{N}$. Hence by Lemma 3.1(2), we have:

$$x_{n+1} = \mu_n\alpha_n u + (1-\mu_n)\alpha_n x_n + (1-\alpha_n)Tx_n \in C.$$

Thus, algorithm (3.1) is well defined. To prove the second assertion we proceed as follows:

Since $\lim_{n\to\infty} \dfrac{\mu_n\alpha_n}{1-\alpha_n} = 0$ and $\lim_{n\to\infty} \dfrac{(1-\alpha_n)^2}{\mu_n\alpha_n} = 0$, for $\epsilon := \frac{1}{2(\frac{5}{2}+L)(2+L)}$ there exists $N_0 > 0$ such that $\frac{\mu_n\alpha_n}{1-\alpha_n} \le 1$ and $\frac{(1-\alpha_n)^2}{\mu_n\alpha_n} \le \epsilon, \forall n \ge N_0$. Let $x^* \in F(T)$ and $r > 0$ be sufficiently large such that $x_{N_0} \in B_r(x^*)$ and $u \in B_{\frac{r}{2}}(x^*)$.

We first show by mathematical induction that $\{x_n\}$ is bounded. To this end, it suffices to show that $x_n \in B_r(x^*)$ for all $n \ge N_0$. By construction $x_{N_0} \in B_r(x^*)$. Now, assume that $x_n \in B_r(x^*)$ for any $n \ge N_0$. we need to show that $x_{n+1} \in B_r(x^*)$ for all $n \ge N_0$. For contradiction, suppose $x_{n+1} \notin B_r(x^*)$. Then $||x_{n+1} - x^*|| > r$. From (3.1) and Lemma 2.1, we have:

$$\begin{aligned}
||x_{n+1} - x^*||^2 &= ||\alpha_n\big(\mu_n u + (1-\mu_n)x_n\big) + (1-\alpha_n)Tx_n - x^*||^2 \\
&= ||\mu_n\alpha_n u + (1-\mu_n)\alpha_n x_n + x_n - x_n + (1-\alpha_n)Tx_n - x^*||^2 \\
&= ||x_n - x^* - (1-\alpha_n)\big[\frac{\mu_n\alpha_n}{1-\alpha_n}(x_n - u) + x_n - Tx_n\big]||^2 \\
&\le ||x_n - x^*||^2 - 2(1-\alpha_n)\langle (x_n - Tx_n) + \frac{\mu_n\alpha_n}{1-\alpha_n}(x_n - u), \\
&\quad j(x_{n+1} - x^*)\rangle \\
&= ||x_n - x^*||^2 - 2(1-\alpha_n)\langle \frac{\mu_n\alpha_n}{1-\alpha_n}(x_{n+1} - x^*) - \frac{\mu_n\alpha_n}{1-\alpha_n} \\
&\quad \times(x_{n+1} - x^*) + x_n - Tx_n + \frac{\mu_n\alpha_n}{1-\alpha_n}(x_n - u), j(x_{n+1} - x^*)\rangle \\
&= ||x_n - x^*||^2 - 2\mu_n\alpha_n||x_{n+1} - x^*||^2 \\
&\quad +2(1-\alpha_n)\langle \frac{\mu_n\alpha_n}{1-\alpha_n}(x_{n+1} - x_n) - (x_n - Tx_n) + \frac{\mu_n\alpha_n}{1-\alpha_n}(u - x^*) \\
&\quad +(x_{n+1} - Tx_{n+1}) - (x_{n+1} - Tx_{n+1}), j(x_{n+1} - x^*)\rangle. \qquad (3.2)
\end{aligned}$$

Since T is pseudocontractive, we have $\langle x_{n+1} - Tx_{n+1}, j(x_{n+1} - x^*)\rangle \ge 0$. Thus, from (3.2) and the fact that $x_n \in B_r(x^*)$, $u \in B_{\frac{r}{2}}(x^*)$ and $\frac{\mu_n\alpha_n}{1-\alpha_n} \le 1, \forall n \ge N_0$, we obtain:

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\le ||x_n - x^*||^2 - 2\mu_n\alpha_n||x_{n+1} - x^*||^2 \\
&\quad +2(1-\alpha_n)\langle \frac{\mu_n\alpha_n}{1-\alpha_n}(x_{n+1} - x_n) + \frac{\mu_n\alpha_n}{1-\alpha_n}(u - x^*) \\
&\quad +(x_{n+1} - Tx_{n+1}) - (x_n - Tx_n), j(x_{n+1} - x^*)\rangle \\
&\le ||x_n - x^*||^2 - 2\mu_n\alpha_n||x_{n+1} - x^*||^2 + 2(1-\alpha_n) \\
&\quad \times\Big[(2+L)||x_{n+1} - x_n|| + \frac{\mu_n\alpha_n}{1-\alpha_n}||u - x^*||\Big]||x_{n+1} - x^*|| \\
&= ||x_n - x^*||^2 - 2\mu_n\alpha_n||x_{n+1} - x^*||^2 \\
&\quad +2(1-\alpha_n)\Big[(2+L)(1-\alpha_n)||(x_n - Tx_n) + \frac{\mu_n\alpha_n}{1-\alpha_n}(x_n - u)|| \\
&\quad +\frac{\mu_n\alpha_n}{1-\alpha_n}||u - x^*||\Big]||x_{n+1} - x^*||
\end{aligned}$$

$$
\begin{aligned}
&\leq ||x_n - x^*||^2 - 2\mu_n\alpha_n||x_{n+1} - x^*||^2 + 2(1-\alpha_n)\Big[(2+L)(1-\alpha_n)\\
&\quad\times\big(||x_n - x^*|| + L||x^* - x_n|| + \frac{\mu_n\alpha_n}{1-\alpha_n}(||x_n - x^*|| + ||x^* - u||)\big)\\
&\quad+\frac{\mu_n\alpha_n}{1-\alpha_n}||u - x^*||\Big]||x_{n+1} - x^*||\\
&\leq ||x_n - x^*||^2 - 2\mu_n\alpha_n||x_{n+1} - x^*||^2 + 2(1-\alpha_n)\\
&\quad\times\Big[(2+L)(1-\alpha_n)\Big((2+L)||x_n - x^*|| + \frac{\mu_n\alpha_n}{1-\alpha_n}||u - x^*||\Big)\\
&\quad+\frac{\mu_n\alpha_n}{1-\alpha_n}||u - x^*||\Big]||x_{n+1} - x^*||\\
&\leq ||x_n - x^*||^2 - 2\mu_n\alpha_n||x_{n+1} - x^*||^2\\
&\quad+2(1-\alpha_n)\Big[(2+L)(1-\alpha_n)\Big((2+L)r + \frac{r}{2}\Big) + \frac{1}{2}r\Big]||x_{n+1} - x^*||\\
&= ||x_n - x^*||^2 - 2\mu_n\alpha_n||x_{n+1} - x^*||^2\\
&\quad+2(1-\alpha_n)\Big[(1-\alpha_n)(2+L)(\frac{5}{2} + L)r + \frac{1}{2}r\Big]||x_{n+1} - x^*||. \qquad (3.3)
\end{aligned}
$$

But since $||x_{n+1} - x^*|| > ||x_n - x^*||$, (3.3) implies that

$$
\begin{aligned}
2\mu_n\alpha_n||x_{n+1} - x^*||^2 &\leq 2(1-\alpha_n)\Big[(1-\alpha_n)(2+L)(\frac{5}{2} + L)r + \frac{1}{2}r\Big]\\
&\quad\times||x_{n+1} - x^*||.
\end{aligned}
$$

Then the fact that $\alpha_n \in [\frac{1}{2}, 1)$, $\mu_n \in (0,1)$ and $\frac{(1-\alpha_n)^2}{\mu_n\alpha_n} \leq \frac{1}{2(\frac{5}{2}+L)(2+L)}, \forall n \geq N_0$ implies

$$
\begin{aligned}
||x_{n+1} - x^*|| &\leq \frac{(1-\alpha_n)^2}{\mu_n\alpha_n}(2+L)(\frac{5}{2} + L)r + \frac{r}{2}\\
&\leq r,\ \forall n \geq N_0,
\end{aligned}
$$

which is a contradiction. Therefore, $x_{n+1} \in B_r(x^*)$ for all positive integers $n \geq N_0$ and hence the sequence $\{x_n\}$ is bounded.

Next we show that $||x_n - y_n|| \to 0$ as $n \to \infty$, where $y_n := y_{t_n} = t_nTy_{t_n} + (1-t_n)u,\ t_n := \frac{1}{1+\frac{\alpha_n\mu_n}{1-\alpha_n}},\ \forall n \in \mathbb{N}$. From (3.1) and Lemma 2.1, we have:

$$
\begin{aligned}
||x_{n+1} - y_n||^2 &\leq ||x_n - y_n||^2 - 2(1-\alpha_n)\langle x_n - Tx_n + \frac{\mu_n\alpha_n}{1-\alpha_n}(x_n - u),\\
&\quad j(x_{n+1} - y_n)\rangle\\
&= ||x_n - y_n||^2 - 2(1-\alpha_n)\langle\frac{\mu_n\alpha_n}{1-\alpha_n}x_{n+1} - \frac{\mu_n\alpha_n}{1-\alpha_n}y_n + x_n - Tx_n\\
&\quad-\frac{\mu_n\alpha_n}{1-\alpha_n}x_{n+1} + \frac{\mu_n\alpha_n}{1-\alpha_n}y_n + \frac{\mu_n\alpha_n}{1-\alpha_n}(x_n - u), j(x_{n+1} - y_n)\rangle\\
&= ||x_n - y_n||^2 - 2\mu_n\alpha_n||x_{n+1} - y_n||^2 + 2(1-\alpha_n)\langle Tx_n - x_n +\\
&\quad\frac{\mu_n\alpha_n}{1-\alpha_n}(x_{n+1} - x_n) + \frac{\mu_n\alpha_n}{1-\alpha_n}(u - y_n), j(x_{n+1} - y_n)\rangle\\
&= ||x_n - y_n||^2 - 2\mu_n\alpha_n||x_{n+1} - y_n||^2 + 2(1-\alpha_n)\langle\frac{\mu_n\alpha_n}{1-\alpha_n}\\
&\quad\times(x_{n+1} - x_n) + [\frac{\mu_n\alpha_n}{1-\alpha_n}(u - y_n) - (y_n - Ty_n)]\\
&\quad-[(x_{n+1} - Tx_{n+1}) - (y_n - Ty_n)] + [(x_{n+1} - Tx_{n+1})\\
&\quad-(x_n - Tx_n)], j(x_{n+1} - y_n)\rangle. \qquad (3.4)
\end{aligned}
$$

On the other hand, the property of y_n implies

$$\begin{aligned} y_n - Ty_n &= t_nTy_n + (1-t_n)u - Ty_n = (1-t_n)(u - Ty_n) \\ &= \frac{\mu_n\alpha_n}{1-\alpha_n+\mu_n\alpha_n}(u - Ty_n) \\ &= \frac{\mu_n\alpha_n}{1-\alpha_n+\mu_n\alpha_n}\big[u - \big(\frac{1-\alpha_n+\mu_n\alpha_n}{1-\alpha_n}y_n - \frac{\mu_n\alpha_n}{1-\alpha_n}u\big)\big] \\ &= \frac{\mu_n\alpha_n}{1-\alpha_n}(u - y_n). \end{aligned}$$

Thus, we get that

$$\frac{\mu_n\alpha_n}{1-\alpha_n}(u - y_n) - (y_n - Ty_n) = 0. \tag{3.5}$$

Then from (3.4), (3.5) and the pseudocontractivity of T, we obtain:

$$\begin{aligned} ||x_{n+1} - y_n||^2 &\le ||x_n - y_n||^2 - 2\mu_n\alpha_n||x_{n+1} - y_n||^2 + 2(1-\alpha_n)\langle\frac{\mu_n\alpha_n}{1-\alpha_n} \\ &\quad \times(x_{n+1} - x_n) + (x_{n+1} - Tx_{n+1}) - (x_n - Tx_n), j(x_{n+1} - y_n)\rangle \\ &\le ||x_n - y_n||^2 - 2\mu_n\alpha_n||x_{n+1} - y_n||^2 + 2(1-\alpha_n)\big[\frac{\mu_n\alpha_n}{1-\alpha_n} \\ &\quad \times||x_{n+1} - x_n|| + ||x_{n+1} - x_n|| + ||Tx_{n+1} - Tx_n||\big]||x_{n+1} - y_n|| \\ &\le ||x_n - y_n||^2 - 2\mu_n\alpha_n||x_{n+1} - y_n||^2 \\ &\quad +2(1-\alpha_n)(2+L)||x_{n+1} - x_n|| \times ||x_{n+1} - y_n|| \\ &= ||x_n - y_n||^2 - 2\mu_n\alpha_n||x_{n+1} - y_n||^2 + 2(1-\alpha_n)^2(2+L) \\ &\quad \times||x_n - Tx_n + \frac{\mu_n\alpha_n}{1-\alpha_n}(x_n - u)|| \times ||x_{n+1} - y_n||. \end{aligned}$$

But since $F(T) \neq \emptyset$, by Proposition 2 of [16] we have that $\{y_n\}$ is bounded. Then there exists $M_1 > 0$ such that

$$\begin{aligned} ||x_{n+1} - y_n||^2 &\le ||x_n - y_n||^2 - 2\mu_n\alpha_n||x_{n+1} - y_n||^2 \\ &\quad +2(1-\alpha_n)^2(2+L)M_1. \end{aligned} \tag{3.6}$$

Furthermore, since T is pseudocontractive, we have that

$$\begin{aligned} ||y_{n-1} - y_n|| &\le ||y_{n-1} - y_n + \frac{1-\alpha_n}{\mu_n\alpha_n}(y_{n-1} - Ty_{n-1} - (y_n - Ty_n))|| \\ &= ||\frac{1-\alpha_n+\mu_n\alpha_n}{\mu_n\alpha_n}(y_{n-1} - y_n) + \frac{1-\alpha_n}{\mu_n\alpha_n}(Ty_n - Ty_{n-1})|| \\ &= ||\frac{1-\alpha_n+\mu_n\alpha_n}{\mu_n\alpha_n}(y_{n-1} - (1-t_n)u) \\ &\quad -\frac{1-\alpha_n}{\mu_n\alpha_n}\big(\frac{1+\mu_{n-1}\alpha_{n-1}-\alpha_{n-1}}{1-\alpha_{n-1}}\big)(y_{n-1} - (1-t_{n-1})u)|| \\ &= ||\big[1 - \frac{(1-\alpha_n)}{\mu_n\alpha_n}\big(\frac{\mu_{n-1}\alpha_{n-1}}{1-\alpha_{n-1}}\big)\big](y_{n-1} - u)|| \\ &\le |1 - \frac{(1-\alpha_n)\mu_{n-1}\alpha_{n-1}}{\mu_n\alpha_n(1-\alpha_{n-1})}|(||y_{n-1}|| + ||u||). \end{aligned} \tag{3.7}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded from (3.7), we have:

$$\begin{aligned} ||x_n - y_n||^2 &= ||x_n - y_{n-1} + y_{n-1} - y_n||^2 \\ &\le (||x_n - y_{n-1}|| + ||y_{n-1} - y_n||)^2 \\ &\le ||x_n - y_{n-1}||^2 + ||y_{n-1} - y_n||\big[2||x_n - y_{n-1}|| + ||y_{n-1} - y_n||\big] \end{aligned}$$

$$\leq \quad ||x_n - y_{n-1}||^2 + |1 - \frac{(1-\alpha_n)\mu_{n-1}\alpha_{n-1}}{\mu_n\alpha_n(1-\alpha_{n-1})}|M_2, \tag{3.8}$$

for some positive real number M_2.

Now, from (3.6) and (3.8) we obtain:

$$\begin{aligned} ||x_{n+1} - y_n||^2 \quad &\leq \quad ||x_n - y_{n-1}||^2 - 2\mu_n\alpha_n||x_{n+1} - y_n||^2 \\ &\quad + |1 - \frac{(1-\alpha_n)\mu_{n-1}\alpha_{n-1}}{\mu_n\alpha_n(1-\alpha_{n-1})}|M + 2(1-\alpha_n)^2(2+L)M, \end{aligned} \tag{3.9}$$

where $M = max\{M_1, M_2\}$. Thus, by (3.9) and Lemma 2.2, we get $x_{n+1} - y_n \to 0$ as $n \to \infty$. Consequently, $||x_n - y_n|| \to 0$ as $n \to \infty$.

Finally, we show that $||x_n - Tx_n|| \to 0$ as $n \to \infty$. Since $\{y_n\}$ (and hence $\{Ty_n\}$) is bounded and $t_n \to 1^-$ as $n \to \infty$, we have $||y_n - Ty_n|| \leq (1-t_n)(||Ty_n|| + ||u||) \to 0$ as $n \to \infty$. Hence, we have

$$\begin{aligned} ||x_n - Tx_n|| \quad &\leq \quad ||x_n - y_n|| + ||y_n - Ty_n|| + ||Ty_n - Tx_n|| \\ &\leq \quad (1+L)||x_n - y_n|| + ||y_n - Ty_n|| \to 0 \text{ as } n \to \infty. \end{aligned}$$

The proof is complete. □

Theorem 3.2. *Let C be a nonempty, closed and convex subset of a reflexive real Banach space E with a uniformly Gâteaux differentiable norm. Let $T : C \to E$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$ and $F(T) \neq \emptyset$. Suppose that T satisfies the inward condition and every closed convex and bounded subset of C has the fixed point property for nonexpansive self-mappings. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to the fixed point x^* of T, which is the unique solution of the variational inequality:*

$$\langle x^* - u, J(x^* - w)\rangle \leq 0, \forall w \in F(T).$$

Proof. As in the proof of Theorem 3.1, we have that $||x_n - y_n|| \to 0$. Then, by Theorem 2 of [17], we have $y_n \to x^* \in F(T)$, which is the unique solution of the variational inequality :

$$\langle x^* - u, J(x^* - w)\rangle \leq 0, \forall w \in F(T).$$

Consequently, $\{x_n\}$ converges strongly to x^*.

If, in Theorem 3.2, we assume that E is uniformly smooth Banach space, then E has uniformly Gâteaux differentiable norm and every closed, bounded and convex subset of C has the fixed point property for nonexpansive self-mappings (see e.g., [26]). Hence we have the following corollary.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real uniformly smooth Banach space E. Let $T : C \to E$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$ and $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to the fixed point x^* of T, which is the unique solution of the variational inequality:*

$$\langle x^* - u, J(x^* - w)\rangle \leq 0, \forall w \in F(T).$$

If, in Theorem 3.2, we assume that T is λ-strictly pseudocontractive, then it is Lipschitz with Lipschitz constant $\frac{1+\lambda}{\lambda}$ and hence we have the following corollary.

Corollary 3.3. *Let C be a nonempty closed convex subset of a reflexive real Banach space E with a uniformly Gâteaux differentiable norm. Let $T : C \to E$ be a λ-strictly pseudocontractive mappings. Suppose that T satisfies the inward condition, $F(T) \neq \emptyset$ and every closed convex and bounded subset of C has the fixed point property for nonexpansive self-mappings. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to the fixed point x^* of T, which is the unique solution of the variational inequality:*

$$\langle x^* - u, J(x^* - w)\rangle \leq 0, \forall w \in F(T).$$

In Theorem 3.2, if $E = H$, a real Hilbert space, then we have the following corollary.

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$ and $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to the fixed point x^* of T nearest to u.*

We note that the method of proof of Theorem 3.2 provides the following theorem for approximating the minimum-norm point of fixed points of Lipschitz pseudocontractive non-self mappings in the Hilbert space settings.

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$ and $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to the minimum-norm point x^* of $F(T)$.*

In, Theorem 3.2, if we consider an accretive mapping $A : C \to E$, then we obtain the following corollary.

Corollary 3.5. *Let C be a nonempty closed convex subset of a reflexive real Banach space E with a uniformly Gâteaux differentiable norm. Let $A : C \to E$ be a Lipschitz accretive mapping with Lipschitz constant $L' \geq 0$ and $N(A) \neq \emptyset$. Suppose that $I - A$ satisfies the inward condition and every closed convex and bounded subset of C has the fixed point property for nonexpansive self-mappings. Let $\{\mu_n\} \subset (0, 1)$, u be any point in C and $\{x_n\}$ be a sequence generated from arbitrary $x_1 \in C$ by:*

$$\begin{cases} \alpha_1 := max\{\frac{1}{2}, f(x_1)\}, \\ x_{n+1} := x_n + \alpha_n\mu_n(u - x_n)) - (1 - \alpha_n)Ax_n, \\ \alpha_{n+1} \in [max\{\alpha_n, f(x_{n+1})\}, 1), \ n \geq 1, \end{cases} \tag{3.10}$$

where $f(x_n) := \inf\{\lambda \geq 0 : \lambda\mu_n(u - x_n) + x_n - (1 - \lambda)Ax_n \in C\}$. If the pair (μ_n, α_n) satisfies conditions (i)-(iv) of Theorem 3.1, then $\{x_n\}$ converges strongly to the solution of the equation $Ax = 0$, which is the unique solution of the variational inequality:

$$\langle x^* - u, J(x^* - w)\rangle \leq 0, \forall w \in N(A).$$

Proof. Since $T := (I - A)$ is a Lipschitz pseudocontractive mapping with Lipschitz constant $L := (L' + 1)$ and the fixed point of T is the solution of the equation $Ax = 0$, the conclusion follows from Theorem 3.2. □

Remark 3.6. In Theorem 3.1, if in addition, C is bounded, then the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Therefore, the condition that $F(T) \neq \emptyset$ is not required in the proof. Hence, we have the conclusions of Theorem 3.2 and Corollary 3.2 without the assumption that $F(T) \neq \emptyset$.

For λ-strictly pseudocontractive mappings, we have also the following Krasnoselskii-Mann type method in 2-uniformly smooth Banach spaces.

3.2. **Krasnoselskii-Mann Type Algorithm for λ-strictly Pseudocontractive Mapping.**

We first prove the following lemma.

Lemma 3.7. *Let C be a closed convex nonempty subset of a 2-uniformly smooth Banach space E. Suppose that E has a weakly sequentially continuous normalized duality mapping and $\{x_n\} \subset E$ is Fejèr-monotone with respect to C. Then $\{x_n\}$ converges weakly to a point in C if C contains all weak limit points of $\{x_n\}$.*

Proof. For each $x \in C$, let $p(x) = \lim_{n\to\infty} ||x_n - x||$. Then $p(x)$ is well defined for all $x \in C$. Moreover, p is proper lower semi-continuous convex function on C and $\{x_n\}$ is bounded. Then, by Lemma 2.5, p assumes its minimum at some point $x^* \in C$. It suffices to show that $\{x_n\}$ converges weakly to x^*. Suppose that $x_{n_j} \rightharpoonup z$ as $j \longrightarrow \infty$. Let $t \in (0, 1)$. Then since C is convex and E is 2-uniformly smooth, by Lemma 2.4, we have

$$||x_{n_j} - x^*||^2 + 2t\langle x^* - z, J(x_{n_j} - x^*)\rangle + 2(Kt)^2||x^* - z||^2 \geq ||x_{n_j} - (1-t)x^* - tz||^2,$$

where K is the best smooth constant of E. Since J is weakly sequentially continuous, taking the limit as $j \longrightarrow \infty$ on both sides of the above inequality, we obtain:

$$(p(x^*))^2 + 2t\langle x^* - z, J(z - x^*)\rangle + 2(Kt)^2||x^* - z||^2 \geq (p(x^*))^2,$$

which implies that

$$(p(x^*))^2 - 2t||x^* - z||^2 + 2(Kt)^2||x^* - z||^2 \geq (p(x^*))^2,$$

or

$$K^2t||x^* - z||^2 \geq ||x^* - z||^2.$$

Now, taking the limit as $t \longrightarrow 0^+$, we get that $x^* = z$. Hence, $x_n \rightharpoonup x^*$, since $\{x_{n_j}\}$ is arbitrary subsequence of $\{x_n\}$.

Now we introduce and prove Krasnoselskii-Mann type algorithm for $\lambda-$strictly pseudocontractive mappings in a 2-uniformly smooth Banach space.

Theorem 3.4. *Let C be a closed strictly convex and nonempty subset of a 2-uniformly smooth Banach space E with the best smooth constant K. Let $T : C \longrightarrow E$ be $\lambda-$strictly pseudocontractive mapping satisfying the inward condition and $F(T) \neq \emptyset$. Suppose that E has a weakly sequentially continuous normalized duality mapping and $\{x_n\}$ be a sequence generated from arbitrary x_1 in C by:*

$$\begin{cases} x_1 \in C, \\ \alpha_1 = max\{\frac{1}{2}, f(x_1)\}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)[(1-\alpha)x_n + \alpha T x_n], \\ \alpha_{n+1} = max\{\alpha_n, f(x_{n+1})\}, \end{cases} \tag{3.11}$$

where $f(x) := \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)[(1-\alpha)x + \alpha Tx] \in C\}$ and $\alpha \in (0, \frac{\lambda}{K^2})$. Then, $\{x_n\}$ converges weakly to a point in $F(T)$. Moreover, if $\sum(1-\alpha_n) < \infty$, then the convergence is strong.

Proof. Define $T_\alpha : C \longrightarrow E$ by $T_\alpha x = (1-\alpha)x + \alpha Tx$. Then by Lemma 2.6 and Remark 2.7, T_α is nonexpansive and satisfies the inward condition with $F(T) = F(T_\alpha)$. Furthermore, algorithm (3.11) could be rewritten as:

$$\begin{cases} x_1 \in C, \\ \alpha_1 = max\{\frac{1}{2}, f(x_1)\}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_\alpha x_n, \\ \alpha_{n+1} = max\{\alpha_n, f(x_{n+1})\}, \end{cases} \tag{3.12}$$

where $f(x) := \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)T_\alpha \in C\}$. Then, as in the proof of *Theorem CM* we get that algorithm (3.12) (and hence algorithm (3.11)) is well defined.

For convergence analysis, we consider two cases. We first assume that $\sum(1-\alpha_n) = \infty$. Then by Lemma 2 of Ishikawa [10] we have that $||x_n - T_\alpha x_n|| \to 0$ as $n \to \infty$. This together with Lemma 2.8 implies that $F(T_\alpha)$ contains all weak limit points of $\{x_n\}$. Then, by Lemma 3.7, $\{x_n\}$ converges weakly to a point in $F(T_\alpha) = F(T)$.

Now, we assume that $\sum(1-\alpha_n) < \infty$. Then following the method of the proof of Theorem 1 of Colao and Marino [7], we obtain that $\{x_n\}$ converges strongly to a point in $F(T_\alpha) = F(T)$.

Remark 3.8. In all the above results, it is not difficult to observe that the same results hold true if $\frac{1}{2}$ is replaced by arbitrarily fixed point $b \in (0,1)$.

Remark 3.9. Based on Algorithm 3.1 and the nature of the mapping T, we may use different ways of choosing the parameters $\{\mu_n\}$ and $\{\alpha_n\}$. In general, we first take $\{\mu_n\} \subset (0,1)$ which goes to zero as $n \longrightarrow \infty$. Then we choose an increasing sequence $\{\alpha_n\} \subset (0,1)$ satisfying conditions $(i)-(iv)$ of Theorem 3.1 and $\alpha_n(\mu_n u + (1-\mu_n)x_n) + (1-\alpha_n)Tx_n \in C$ for each $n \geq 1$, where the last condition shows that $\alpha_{n+1} \in \big[\max\{\alpha_n, f(x_{n+1})\}, 1\big)$ for each $n \geq 1$. Hence, the pair (α_n, μ_n) satisfies all conditions of the theorem.

Remark 3.10. In case, we have a set $D \subseteq C$, convex, such that $Tx \in D$ for all $x \in D$, we may take $u \in D$ and $x_1 \in C$, arbitrary and $\{\mu_n\} \subset (0,1)$ which goes to zero as $n \longrightarrow \infty$. Then we choose an increasing sequence $\{\alpha_n\} \subset (0,1)$ satisfying conditions $(i)-(iv)$ of Theorem 3.1 such that $\alpha_1(\mu_1 u + (1-\mu_1)x_1) + (1-\alpha_1)Tx_1 \in D$. Thus, we obtain that for each $n \geq 1$, we have $\alpha_n(\mu_n u + (1-\mu_n)x_n) + (1-\alpha_n)Tx_n \in D \subseteq C$ and hence the pair (α_n, μ_n) satisfies all conditions of the theorem.

Remark 3.11. Theorem 3.1 improves *Theorem CZ* in the sense that it extends the class of Lipschitz pseudocontractive self-mappings to the class of Lipschitz pseudocontractive non-self mappings.

Remark 3.12. Theorem 3.2 extends *Theorem CM* in the sense that it provides a convergent scheme for approximating fixed points of Lipschitz pseudocontractive non-self mappings more general than nonexpansive non-self mappings in Banach spaces more general than Hilbert spaces.

4. Numerical example

Now, we give an example of a Lipschitz pseudocontractive mapping that satisfies the conditions of Theorem 3.2 and some numerical experiment results to explain the conclusion of the theorem as follows:

Example 4.1. Let $H = \mathbb{R}$ with Euclidean norm. Let $C = [-1, 2]$ and $T : C \longrightarrow \mathbb{R}$ be defined by

$$Tx = \begin{cases} -3x, & x \in [-1, 0), \\ x, & x \in [0, 1), \\ x - (x-1)^2, & x \in [1, 2]. \end{cases} \tag{4.1}$$

Then we observe that T satisfies the inward condition and $F(T) = [0, 1]$. Moreover, $\langle x - Tx - (y - Ty), x - y\rangle \geq 0$ for all $x, y \in C$. Hence, T is pseudocontractive mapping. To show that T is a Lipschitz mapping, we consider the following cases.

Case 1: Let $x, y \in [-1, 0)$. Then we have:

$$|Tx - Ty| = | - 3x + 3y| = 3|x - y|.$$

Case 2: Let $x, y \in [0, 1)$. Then we have:

$$|Tx - Ty| = |x - y|.$$

Case 3: Let $x, y \in [1, 2]$. Then we have:

$$\begin{aligned} |Tx - Ty| &= |x - (x-1)^2 - y + (y-1)^2| = |3x - 3y + y^2 - x^2| \\ &\leq 3|x - y| + 4|x - y| = 7|x - y|. \end{aligned}$$

Case 4: Let $x \in [-1, 0)$ and $y \in [0, 1)$. Then we have:

$$\begin{aligned} |Tx - Ty| = | - 3x - y| &= |3x + y| \\ &= |x - y + 2x + 2y| \leq |x - y| + 2|x + y| \\ &\leq |x - y| + 2|x - y| = 3|x - y|. \end{aligned}$$

Case 5: Let $x \in [-1, 0)$ and $y \in [1, 2]$. Then we have:

$$\begin{aligned} |Tx - Ty| = | - 3x - y + (y-1)^2| &\leq |3x + y| + (y-1)^2 \\ &\leq |x - y + 2x + 2y| + |x - y| \\ &\leq 2|x + y| + |x - y| + |x - y| \leq 4|x - y|. \end{aligned}$$

Case 6: Let $x \in [0, 1)$ and $y \in [1, 2]$. Then we have:

$$\begin{aligned} |Tx - Ty| = |x - y + (y-1)^2| &\leq |x - y| + (y-1)^2 \\ &\leq |x - y| + |y - x| = 2|x - y|. \end{aligned}$$

From the above Cases, we conclude that T is Lipschitz with Lipschitz constant $L = 7$.

Now, for $u = -0.8$ and $x_1 = 2$ in $C = [-1, 2]$, following Remark 3.9, we take $\mu_n = \frac{1}{(n+5)^{0.3}[(n+5)^{0.6}-1]}$ and choose $\alpha_n = 1 - \frac{1}{(n+5)^{0.6}}$. Then we see that the pair (μ_n, α_n) satisfies (i)-(iv) of the conditions of Theorem 3.2 and $\alpha_n(\mu_n u + (1 - \mu_n)x_n) + (1 - \alpha_n)Tx_n \in C$, for each $n \geq 1$. Thus, Algorithm (3.1) converges strongly to $0 = P_{F(T)}(-0.8)$ in $F(T)$ (see, Figure 1).

On the other hand, in this particular example, note that T is a self mapping on $D = [0, 2] \subset C = [-1, 2]$. Now, if we consider $u = 0.6 \in D$ and $x_1 = -1 \in C$, following Remark 3.10, we may take $\mu_n = \frac{1}{(n+5)^{0.3}[(n+5)^{0.6}-1]}$, and choose $\alpha_n = 1 - \frac{1}{(n+5)^{0.6}}$. Then we obtain that the pair (μ_n, α_n) satisfies (i)-(iv) of the conditions of Theorem 3.2 and $\alpha_1(\mu_1 u + (1 - \mu_1)x_1) + (1 - \alpha_1)Tx_1 \in D$ and hence we get $\alpha_n(\mu_n u + (1 - \mu_n)x_n) + (1 - \alpha_n)Tx_n \in D \subset C$ for all $n \geq 1$. Therefore, Algorithm (3.1) converges strongly to $0.6 = P_{F(T)}(0.6)$ in $F(T)$ (see, Figure 1). The following graph is obtained using MATLAB version 7.5.0.342(R2007b).

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