Jiraporn Limprayoon, J. Nonlinear Anal. Optim. Vol. 14(1) (2023), 21-31

Journal of Nonlinear Analysis and Optimization
Volume 14(1) (2023)
http://www.math.sci.nu.ac.th
ISSN : 1906-9685

J. Nonlinear Anal. Optim.

THEOREMS OF THE MINIMIZATION PROBLEM AND FIXED POINT PROBLEM OF TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

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ABSTRACT. In this paper, we propose an iterative algorithm designed to address the minimization and fixed point problems associated with total asymptotically nonexpansive mappings in CAT(0) spaces. We establish strong convergence theorems and $\triangle$-convergence theorems for solving these problems. Furthermore, we apply the key findings to solve the equilibrium problem in CAT(0) spaces.

KEYWORDS minimization problem; fixed point problem; total asymptotically nonexpansive mapping; convergence theorem; CAT(0) space
AMS Subject Classification: 47H09; 47J25

1. Introduction

Let K be a nonempty subset of a CAT(0) space (X, d), and consider the mapping $T : K \to K$. We denote the set of fixed points of T by $F(T) = u \in K : u = Tu$. The study of fixed point theory in CAT(0) spaces was initiated by Kirk [14] in 2003. Kirk demonstrated the existence of a fixed point for a nonexpansive mapping defined on a bounded, closed, and convex subset of a CAT(0) space. Subsequently, numerous authors proposed various iterative schemes to approximate fixed points of nonexpansive mappings in CAT(0) spaces. One such algorithm is the Mann iterative algorithm introduced by He et al. [21] in CAT(κ) spaces, defined as follows:

$$\begin{cases} u_1 \in X, \\ u_{n+1} = \alpha_n u_n \oplus (1-\alpha_n) T u_n, \quad \forall_n \geq 1, \end{cases} \tag{1.1}$$

where α_n is a sequence in $[0, 1]$, and they proved some $\triangle$-convergence theorems of nonexpansive mappings in CAT(κ) spaces for $\kappa \geq 0$. Other iterative algorithms have

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Article history : Received 31 March 2023: Accepted 8 June 2023.

also been proposed to solve this problem, such as the Ishikawa iteration method, S-iteration method, and hybrid-CR three steps iteration methods. For further details, refer to [1, 22, 23, 24, 25, 31, 32, 33, 34, 35].

The proximal point algorithm (PPA), introduced by Martinet [2] in 1970, has attracted significant attention from researchers. Rockafellar further utilized the PPA to solve convex minimization problems in Hilbert spaces. Nevanlinna investigated the minimization problem in Banach spaces using the PPA under suitable conditions [15]. More information on PPA in Hilbert or Banach spaces can be found in the works of Solodov [16], Kamimura [17], Shehu [18], and others.

Recently, many PPA convergence results have been extended from linear to nonlinear spaces. Bačák introduced the PPA in CAT(0) spaces to solve the minimization problem in 2013, which is defined as follows:

$$\begin{cases} u_1 \in X, \\ u_{n+1} = \arg\min_{q \in X} \left[g(q) + \frac{1}{2\lambda_n} d^2(q, u_n) \right] \quad \forall n \geq 1, \end{cases} \tag{1.2}$$

where $\lambda_n > 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$.

Cholamjiak et al. [20] proposed the following iteration method in 2015 to solve the minimization and fixed point problems of nonexpansive mappings in CAT(0) spaces:

$$\begin{cases} p_n = \arg\min_{q \in X} \left[g(q) + \frac{1}{2\lambda_n} d^2(q, u_n) \right], \\ y_n = (1 - \beta_n) u_n \oplus \beta_n T_1 p_n, \\ u_{n+1} = (1 - \alpha_n) T_1 u_n \oplus \alpha_n T_2 y_n, \quad \forall_n \geq 1. \end{cases} \tag{1.3}$$

where $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b, \lambda_n \geq \lambda > 0$, f is a proper convex lower semi-continuous function. They obtained a $\triangle$-convergence theorem.

Chang, Yao, Wang, and Qin [4] introduced the iteration method described below in 2016 to solve the minimization and fixed point problems of asymptotically nonexpansive mappings in CAT(0) spaces:

$$\begin{cases} p_n = \arg\min_{q \in K} \left[g(q) + \frac{1}{2\lambda_n} d^2(q, u_n) \right], \\ y_n = \alpha_n u_n \oplus \beta_n T_1^n u_n \oplus \gamma_n T_2^n p_n, \\ x_{n+1} = \delta_n T_2^n u_n \oplus \eta_n S_1^n u_n \oplus \xi_n S_2^n y_n, \quad n \geq 1. \end{cases} \tag{1.4}$$

where $0 < a \leq \alpha_n, \beta_n, \gamma_n, \delta_n, \eta_n, \xi_n < 1, a \in (0, 1)$ is a positive constant, $\lambda_n \geq \lambda > 0$, g is a proper convex lower semi-continuous function. They obtained a $\triangle$-convergence result, and when one of the mappings T_1, T_2, S_1 and S_2 has semi-compactness, they established a strong convergence theorem.

Motivated by ongoing research in this area and inspired by Cholamjiak's iteration method and Chang's method, we delve into the minimization and fixed point problems of total asymptotically nonexpansive mappings in CAT(0) spaces in this paper. We introduce a novel algorithm and derive some strong convergence theorems and $\triangle$-convergence theorems by amalgamating the proximal point algorithm with Mann's iterative method. Finally, we apply the key findings to solve the equilibrium problem in CAT(0) spaces.

2. Preliminaries

Let (X,d) be a metric space and $p,q \in X$. A geodesic path joining p to q is an isometry $c:[0,d(p,q)] \to X$ such that $c(0)=p$ and $c(d(p,q))=q$. The image of a geodesic path joining p to q is called a geodesic segment between p and q. When it is unique, this geodesic segment is denoted by $[p,q]$. The metric space (X,d) is said to be a geodesic space, if every two points of X are joined by a geodesic. In this paper, we write $(1-t)p \oplus tq$ for the unique point h in $[p,q]$ such that

$$d(h,p)=td(p,q), d(h,q)=(1-t)d(p,q).$$

A geodesic space (X,d) is called a CAT(0) space, if the geodesic segment connecting two points is unique and the following inequality holds [5]:

$$d^2((1-t)p \oplus tq, h) \leq (1-t)d^2(p,h) + td^2(q,h) - (1-t)d^2(p,q)$$

for all $p,q,h \in X$.

A subset K of a CAT(0) space X is said to be convex if $[p,q] \subseteq K$ for all $p,q \in K$. For more fundamental knowledge about CAT(0) spaces, refer to read [5]-[11].

It is well known that any complete and simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space and the Hilbert ball with the hyperbolic metric [12], Pre-Hilbert space [6], Euclidean building [11] and R-tree [13] are also examples of CAT(0) spaces.

Definition 2.1. Let $T: X \to X$ be a mapping. T is said to be

(i) nonexpansive, if $d(Tp,Tq) \leq d(p,q)$, for any $p,q \in X$.

(ii) asymptotically nonexpansive, if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that $d(T^n p, T^n q) \leq k_n d(p,q)$, for any $n \geq 1$ and any $p,q \in X$.

(iii) total asymptotically nonexpansive, if there exists nonnegative sequences $\{\mu_n\}$ and $\{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\xi:[0,1) \to [0,\infty)$ with $\xi(0)=0$ such that

$$d(T^n p, T^n q) \leq d(p,q) + \nu_n \xi(d(p,q)) + \mu_n, \quad \forall_n \geq 1,\ p,q \in X.$$

(iv) uniformly L-Lipschitzian, if there exists a constant $L>0$ such that

$$d(T^n p, T^n q) \leq L d(p,q), \quad \forall_n \geq 1,\ p,q \in X.$$

Let $\{u_n\}$ be a bounded sequence of a complete CAT(0) space X. Then $A(\{u_n\}) = \{u \in X : \limsup_{n\to\infty} d(u,u_n) \leq \limsup_{n\to\infty} d(h,u_n), \forall\, h \in X\}$ is said to be the asymptotic center of $\{u_n\}$. It is known [26] that in a complete CAT(0) space X, the asymptotic center of $\{u_n\}$ consists of exactly one point.

Definition 2.2. [14, 28] A sequence $\{u_n\}$ in a CAT(0) space X is said to be $\triangle$-convergent to $u \in X$ if u is the unique asymptotic center of any subsequence $\{u_{n_k}\} \subset \{u_n\}$. Symbolically, we write it as $\triangle - \lim_{n\to\infty} u_n = u$.

Lemma 2.3. [27] *Let K be a closed and convex subset of CAT(0) space X and $\{u_n\}$ be a bounded sequence in K. Then $\triangle - \lim_{n\to\infty} u_n = u$ implies that $u_n \to u$ (i.e. $\limsup_{n\to\infty} d(u_n,u) = \inf_{y\in K} \limsup_{n\to\infty} d(u_n,y)$).*

Lemma 2.4. [5] *Let X be a CAT(0) space and $p,q,h \in X$. Then*

(i) $d((1-t)p \oplus tq, h)) \leq (1-t)d(p,h) + td(q,h)$, $t \in [0,1]$,

(ii) $d^2((1-t)p \oplus tq, h)) \leq (1-t)d^2(p,h) + td^2(q,h) - t(1-t)d^2(p,q)$, $t \in [0,1]$.

Lemma 2.5. [27] *Let $\{u_n\}$ be a bounded sequence of complete CAT(0) space X. Then*

(i) *$\{u_n\}$ has a $\triangle$-convergent subsequence,*
(ii) *the asymptotic center of $\{u_n\} \subset K \subset X$ is in K, where K is nonempty closed and convex.*

Lemma 2.6. [5] *Let $\{u_n\}$ be a bounded sequence of a complete CAT(0) space and $A(\{u_n\}) = \{u\}$. Let $\{u_{n_k}\}$ be an arbitrary subsequence of $\{u_n\}$ and $A(\{u_{n_k}\}) = \{q\}$ If $\lim_{n\to\infty} d(u_n, q)$ exists, then $u = q$.*

Definition 2.7. A function $g : K \to (-\infty, \infty]$ is said to be convex if the following inequality holds

$$g(\lambda p \oplus (1-\lambda)q) \leq \lambda g(p) + (1-\lambda)g(q), \quad \text{for all} \quad p, q \in K, \quad \lambda \in [0,1].$$

Definition 2.8. [29] Let $g : X \to (-\infty, \infty]$ be a proper convex lower semi-continuous function, for all $\lambda > 0$, the Moreau-Yosida resolvent of f in CAT(0) space X is defined by

$$J_\lambda^g := \arg\min_{q\in X}[g(y) + \frac{1}{2\lambda} d^2(q,p)], \quad \forall\, p \in X.$$

It is known that the fixed points set $Fix(J_\lambda^g(p))$ of the resolvent of g is consistent with the set $\arg\min_{q\in X} g(q)$ of minimizers of g, and J_λ^g is a nonexpansive mapping [30].

Lemma 2.9. [30] *Let (X,d) be a complete CAT(0) space and $g : X \to (-\infty, \infty]$ be a proper convex lower semi-continuous function. Then*

$$J_\lambda p := J_\mu(\frac{\lambda-\mu}{\lambda} J_\lambda p \oplus \frac{\mu}{\lambda} p), \quad \textit{for all } p \in X \textit{ and } \lambda > \mu > 0.$$

Lemma 2.10. [7] *Let (X,d) be a complete CAT(0) space and $g : X \to (-\infty, \infty]$ be a proper convex lower semi-continuous function. Then*

$$\frac{1}{2\lambda} d^2(J_\lambda p, q) - \frac{1}{2\lambda} d^2(p,q) + \frac{1}{2\lambda} d^2(p, J_\lambda p) + g(J_\lambda p) \leq g(q), \ \textit{for all } p, q \in X, \lambda > 0.$$

Lemma 2.11. [8] *Let C be a closed and convex subset of complete CAT (0) space X and let $T : K \to X$ be a uniformly L-Lipschitzian and total asymptotically nonexpansive mapping. If $\{u_n\}$ is a bounded sequence in K such that $\lim_{n\to\infty} d(u_n, Tu_n) = 0$ and $\triangle - \lim_{n\to\infty} u_n = u$, then $Tu = u$.*

Lemma 2.12. [9] *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \ n \geq 1,$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.13. [8] *Let X be a CAT(0) space,$x \in X$ be a given point and $\{a_n\}$ be a sequence in $[b,c]$, and $b, c \in (0,1), 0 < b(1-c) \leq \frac{1}{2}$, let $\{u_n\}$ and $\{p_n\}$ be any sequences in X such that $\limsup_{n\to\infty} d(u_n, u) \leq r$, $\limsup_{n\to\infty} d(p_n, u) \leq r$ and $\lim_{n\to\infty} d((1-a_n)u_n \oplus a_n p_n, u) = r$, for some $r \geq 0$, then $\lim_{n\to\infty} d(u_n, p_n) = 0$.*

3. Main Results

We suppose the following conditions are satisfied:

(1) (X,d) is a complete CAT(0) space.

(2) $K \subset X$ is a nonempty closed convex subset, $T : K \to K$ is a uniformly L-Lipschitzian total asymptotically nonexpansive mapping, $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, and there exists a constant $M > 0$ such that $\xi(r) \leq Mr, r \geq 0$.

(3) $g : X \to (-\infty, \infty]$ is a proper convex lower semi-continuous function, $J^g_{\lambda_n} : X \to X$ is the Moreau-Yosida resolvent of $g, \lambda_n \geq \lambda > 0$.

(4) $\{\alpha_n\}$ is a sequence in $[b,c]$, and $b, c \in (0,1), 0 < b(1-c) \leq \frac{1}{2}$

Theorem 3.1. *Let $(X,d), K, T, g, J^g_{\lambda_n}, \lambda_n, \{\alpha_n\}$ satisfy the above conditions. Let $u_1 \in X$ be chosen arbitrarily and the sequence $\{u_n\}$ be defined as follows:*

$$\begin{cases} p_n = J^g_{\lambda_n}(u_n) = \arg\min_{q\in X}[g(q) + \frac{1}{2\lambda_n} d^2(q,u_n)], \\ u_{n+1} = T^n((1-\alpha_n)u_n \oplus \alpha_n T^n p_n), \; n \geq 1. \end{cases} \tag{3.1}$$

(I) *If $\Omega = F(T) \cap \arg\min_{q\in X} g(q) \neq \phi$, then $\{u_n\}$ $\triangle$-convergent to a point $u \in \Omega$.*

(II) *In addition, if $\Omega = F(T) \cap \arg\min_{q\in X} g(q) \neq \phi$ and T is semi-compact, then $\{u_n\}$ converges strongly to a point $u \in \Omega$.*

Proof. Now we will demonstrate the conclusion (I). The proof is divided into five steps.

Step 1. Firstly we show that $\{u_n\}$ is bounded.

Let $u^* \in \Omega$, since $J^g_{\lambda_n}$ is a nonexpansive mapping, from (3.1), we have

$$d(p_n, u^*) = d(J^g_{\lambda_n}(u_n), u^*) = d(J^g_{\lambda_n}(u_n), J^g_{\lambda_n}(u^*)) \leq d(u_n, u^*), \tag{3.2}$$

and from Lemma 2.4 (i), we can obtain that

$$\begin{aligned} d(x_{n+1}, u^*) &= d(T^n((1-\alpha_n)u_n \oplus \alpha_n T^n p_n), u^*) \\ &\leq d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, u^*) \\ &\quad + \nu_n \xi(d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, u^*)) + \mu_n \\ &\leq (1+\nu_n M) d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, u^*) + \mu_n \\ &\leq (1+\nu_n M)[(1-\alpha_n)d(u_n,u^*) + \alpha_n d(T^n p_n, u^*)] + \mu_n \\ &\leq (1+\nu_n M)[(1-\alpha_n)d(u_n,u^*) + \alpha_n (d(p_n,u^*) + \nu_n \xi(d(p_n,u^*)) + \mu_n] + \mu_n \\ &\leq (1+\nu_n M)[(1+\nu_n M)d(u_n,u^*) + \mu_n] + \mu_n \\ &\leq (1+\nu_n M)^2 d(u_n,u^*) + (2+\nu_n M)\mu_n. \end{aligned} \tag{3.3}$$

Since $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, it follows from Lemma 2.12 that $\lim_{n\to\infty} d(u_n, u^*)$ exists. This implies that $\{u_n\}$ is bounded. Obviously, $\{p_n\}$ is also bounded.

Step 2. We show that $\lim_{n\to\infty} d(u_n, p_n) = 0$.

By Lemma 2.10, we have

$$\frac{1}{2\lambda_n} d^2(p_n, u^*) - \frac{1}{2\lambda_n} d^2(u_n, u^*) + \frac{1}{2\lambda_n} d^2(u_n, p_n) \leq g(u^*) - g(p_n). \tag{3.4}$$

Since $g(u^*) \leq g(p_n)$, from (3.4), we can get

$$d^2(u_n, p_n) \leq d^2(u_n, u^*) - d^2(p_n, u^*). \tag{3.5}$$

Since $\lim_{n\to\infty} d(u_n, u^*)$ exists, without loss of generality, we may assume $\lim_{n\to\infty} d(u_n, u^*) = c \geq 0$ By (3.2), we have

$$\sum_{n\to\infty} d(p_n, u^*) \leq \sum_{n\to\infty} d(u_n, u^*) = c, \tag{3.6}$$

and from (3.3), we can obtain that

$$d(u_n, u^*) \leq \frac{d(u_n, u^*)}{\alpha_n} - \frac{d(u_{n+1}, u^*)}{\alpha_n(1+\nu_n M)} + d(p_n, u^*) + \nu_n \xi(d(p_n, u^*)) + \mu_n + \frac{\mu_n}{\alpha_n(1+\nu_n M)}. \tag{3.7}$$

It follows from $\lim_{n\to\infty} d(u_n, u^*) = c$, $\mu_n \to 0$, and $\nu_n \to 0$ that

$$c = \liminf_{n\to\infty} d(u_n, u^*) \leq \liminf_{n\to\infty} d(p_n, u^*). \tag{3.8}$$

Combining (3.6) and (3.8), we have

$$\lim_{n\to\infty} d(p_n, u^*) = c. \tag{3.9}$$

Thus it follows from (3.5) that

$$\lim_{n\to\infty} d(u_n, p_n) = 0. \tag{3.10}$$

Step 3. We show that

$$\lim_{n\to\infty} d(u_n, T^n p_n) = \lim_{n\to\infty} d(u_n, u_{n+1}) = \lim_{n\to\infty} d(p_n, p_{n+1}) = 0.$$

Since

$$d(T^n p_n, u^*) = d(T^n p_n, T^n u^*) \leq d(p_n, u^*) + \nu_n \xi(d(p_n, u^*)) + \mu_n \tag{3.11}$$

we have

$$\limsup_{n\to\infty} d(T^n p_n, u^*) \leq \limsup_{n\to\infty} d(p_n, u^*) = c. \tag{3.12}$$

Due to (3.3) we have

$$\begin{aligned} c = \lim_{n\to\infty} d(u_{n+1}, u^*) &= \lim_{n\to\infty} d(T^n((1-\alpha_n)u_n \oplus \alpha_n T^n p_n), u^*) \\ &\leq \lim_{n\to\infty} ((1+\nu_n M)^2 d(u_n, u^*) + (2+\nu_n M)\mu_n) \\ &= c. \end{aligned} \tag{3.13}$$

This implies that

$$\lim_{n\to\infty} d(T^n((1-\alpha_n)u_n \oplus \alpha_n T^n p_n), u^*) = c \tag{3.14}$$

and

$$\begin{aligned} &d(T^n((1-\alpha_n)u_n \oplus \alpha_n T^n p_n), u^*) \\ &\leq (1+\nu_n M) d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, u^*) + \mu_n \end{aligned}$$

which

$$\begin{aligned} \limsup_{n\longrightarrow\infty} d(T^n((1-\alpha_n)u_n \oplus \alpha_n T^n p_n), u^*) &\leq \limsup_{n\longrightarrow\infty} d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, u^*) \\ c &\leq \limsup_{n\longrightarrow\infty} d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, u^*). \end{aligned} \tag{3.15}$$

Also, we have

$$\begin{aligned} &d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, u^*) \leq (1-\alpha_n) d(u_n, u^*) + \alpha_n d(T^n p_n, u^*) \\ &\limsup_{n\longrightarrow\infty} d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, u^*) \leq c. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we have

$$\limsup_{n\longrightarrow\infty} d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, u^*) = c.$$

Since $\{\alpha_n\}$ is a sequence in $[b, c]$, and $b, c \in (0, 1), 0 < b(1-c) \leq \frac{1}{2}$, from (3.12), (3.14), (3.15), (3.16) and Lemma 2.13, we have

$$\lim_{n\to\infty} d(u_n, T^n p_n) = 0. \tag{3.17}$$

In addition, we also have

$$\begin{aligned}
\lim_{n\to\infty} d(u_{n+1}, T^n p_n) &= \lim_{n\to\infty} (d(T^n((1-\alpha_n)u_n \oplus \alpha_n T^n p_n), T^n p_n) \\
&\leq \lim_{n\to\infty} d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, p_n) \\
&\quad + \nu_n \xi(d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, p_n)) + \mu_n) \\
&= 0.
\end{aligned} \tag{3.18}$$

So, from (3.17) and (3.18), we know that

$$\lim_{n\to\infty} d(u_n, u_{n+1}) = 0. \tag{3.19}$$

Since

$$d(p_n, p_{n+1}) \leq d(p_n, u_n) + d(u_n, u_{n+1}) + d(u_{n+1}, p_{n+1}),$$

from (3.10) and (3.19), we have

$$\lim_{n\to\infty} d(p_n, p_{n+1}) = 0. \tag{3.20}$$

Step 4. We show that

$$\lim_{n\to\infty} d(p_n, Tp_n) = \lim_{n\to\infty} d(u_n, Tu_n) = \lim_{n\to\infty} d(u_n, J_\lambda^g u_n) = 0.$$

In the view of (3.10), and (3.17), we can obtain that

$$d(p_n, T^n p_n) \leq d(p_n, u_n) + d(u_n, T^n p_n) \to 0 \quad (\text{as } n \to \infty). \tag{3.21}$$

Since T is uniformly L-Lipschitzian, combining (3.20) and (3.21), we may get

$$\begin{aligned}
d(p_n, Tp_n) &\leq d(p_n, p_{n+1}) + d(p_{n+1}, T^{n+1}p_{n+1}) + d(T^{n+1}p_{n+1}, T^{n+1}p_n) \\
&\quad + d(T^{n+1}p_n, Tp_n) \\
&\leq (1+L)d(p_n, p_{n+1}) + d(p_{n+1}, T^{n+1}p_{n+1}) + Ld(T^n p_n, p_n) \\
&\to 0 \ (\text{as } n \to \infty).
\end{aligned} \tag{3.22}$$

In addition, we also have

$$\begin{aligned}
d(p_n, Tp_n) &\leq d(u_n, p_n) + d(p_n, Tp_n) + d(Tp_n, Tu_n) \\
&\leq (1+L)d(u_n, p_n) + d(p_n, Tp_n) \to 0 \quad (\text{as } n \to \infty).
\end{aligned} \tag{3.23}$$

It follows from (3.10) and Lemma 2.9 that

$$\begin{aligned}
d(J_\lambda^g u_n, u_n) &\leq d(J_\lambda^g u_n, J_{\lambda_n}^g(u_n)) + d(p_n, u_n) \\
&\leq d(J_\lambda^g u_n, J_\lambda^g((\frac{\lambda_n - \lambda}{\lambda_n})J_{\lambda_n}^g(u_n) \oplus \frac{\lambda}{\lambda_n}u_n)) + d(p_n, u_n) \\
&\leq d(u_n, (1 - \frac{\lambda}{\lambda_n})(J_{\lambda_n}^g(u_n) \oplus \frac{\lambda}{\lambda_n}u_n)) + d(p_n, u_n) \\
&\leq (1 - \frac{\lambda}{\lambda_n})d(u_n, p_n) + d(p_n, u_n) \to 0 \quad (\text{as } n \to \infty).
\end{aligned} \tag{3.24}$$

Step 5. Finally we prove that $\{u_n\}$ $\triangle$-convergent to a point $u \in \Omega$.

Denote $\omega_w(u_n) = \bigcup_{\{u_{n_i}\}\subset\{u_n\}} A(\{u_{n_i}\})$. Let $z \in \omega_w(u_n)$, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $A(\{u_{n_i}\}) = \{z\}$. By Lemma 2.5, there exists a subsequence $\{v_{n_j}\}$ of $\{u_{n_i}\}$ such that $\triangle$ - $\lim_{n\to\infty} v_{n_j} = u$. Because J_λ^g is a nonexpansive mapping, it follows from (3.24), (3.23), (3.22), (3.10) and Lemma 2.11 that $u \in F(J_\lambda^g) \cap F(T)$. This implies that $u \in \Omega$. Since $\lim_{n\to\infty} d(u_n, u^*)$ exists for any $u^* \in \Omega$. Then $\lim_{n\to\infty} d(u_n, u)$ also exists.

Next we prove that $\omega_w(u_n)$ consists of exactly one point. Let $\{u_{n_i}\}$ be a subsequence of $\{u_n\}$ such that $A(\{u_{n_i}\}) = \{z\}$ and $A(\{u_n\}) = \{u\}$. Because $z \in \omega_w(u_n) \subset \Omega$, we know that $z \in \Omega$. Thus, $\lim_{n\to\infty} d(u_n, z)$ exists. By Lemma 2.6, we know that $z = u$. This means that $\omega_w(u_n)$ consists of exactly one point. It follows from Definition 2.2 that $\{u_n\}$ $\triangle$-convergent to a point $u \in \Omega$.

Next we prove the conclusion (II).

From T is semi-compact and $\lim_{n\to\infty} d(p_n, Tp_n) = 0$, there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $\{p_{n_k}\} \to u_*$. It follows from $\lim_{n\to\infty} d(u_n, p_n) = 0$ that the subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converges strongly to u_*. Because $\triangle$ - $\lim_{n\to\infty} u_n = u$, then $u_* = u$. Due to $\lim_{n\to\infty} d(u_n, u)$ exists and $\lim_{k\to\infty} d(u_{n_k}, u) = 0$, we know that $\{u_n\}$ converges strongly to a point $u \in \Omega$. The proof is completed. □

Every asymptotically nonexpansive mapping is also a total asymptotically nonexpansive mapping, and every nonexpansive mapping is also a total asymptotically nonexpansive mapping. Therefore, when T is an asymptotically nonexpansive mapping, the following result holds in Theorem 3.1.

Corollary 3.1. *Let $(X,d), K, g, J^g_{\lambda_n}, \lambda_n, \{\alpha_n\}$ be the same as them of Theorem 3.1, $T : K \to K$ be an asymptotically nonexpansive mapping with the sequence $\{k_n\} \subset [1, \infty)$ satisfying $\lim_{n\to\infty} k_n = 1$. Let $u_1 \in X$ be chosen arbitrarily and the sequence $\{u_n\}$ be defined as follows:*

$$\begin{cases} p_n = J^g_{\lambda_n}(u_n) = \arg\min_{q\in X}[g(q) + \frac{1}{2\lambda_n} d^2(q, u_n)] \\ u_{n+1} = T^n((1-\alpha_n)u_n \oplus \alpha_n T^n p_n), \quad n \ge 1, \end{cases} \tag{3.25}$$

(I) *If $\Omega = F(T) \cap \arg\min_{q\in X} g(q) \neq \phi$, then $\{u_n\}$ $\triangle$-convergent to a point $u \in \Omega$.*

(II) *In addition, if $\Omega = F(T) \cap \arg\min_{q\in X} g(q) \neq \phi$ and T is semi-compact, then $\{u_n\}$ converges strongly to a point $u \in \Omega$.*

In Theorem 3.1, when T is a nonexpansive mapping, the following result holds.

Corollary 3.2. *Let $(X,d), K, g, J^g_{\lambda_n}, \lambda_n, \{\alpha_n\}$ be the same as them of Theorem 3.1, $T : K \to K$ be a nonexpansive mapping. Let $u_1 \in X$ be chosen arbitrarily and the sequence $\{u_n\}$ be defined as follows:*

$$\begin{cases} p_n = J^g_{\lambda_n}(u_n) = \arg\min_{q\in X}[g(q) + \frac{1}{2\lambda_n} d^2(q, u_n)] \\ u_{n+1} = T^n((1-\alpha_n)u_n \oplus \alpha_n T^n p_n), \quad n \ge 1, \end{cases} \tag{3.26}$$

(I) *If $\Omega = F(T) \cap \arg\min_{q\in X} g(q) \neq \phi$, then $\{u_n\}$ $\triangle$-convergent to a point $u \in \Omega$.*

(II) *In addition, if $\Omega = F(T) \cap \arg\min_{q\in X} g(q) \neq \phi$ and T is semi-compact, then $\{u_n\}$ converges strongly to a point $u \in \Omega$.*

4. Applications

In this section, we apply the main results to solve equilibrium problem in CAT(0) spaces.

4.1. **Equilibrium problem.** Let (X,d) be a complete CAT(0) space and K be a nonempty closed convex subset of it. Suppose that $F : K \times K \to \mathbb{R}$ is a bifunction, the equilibrium problem $(shortly, EP)$ is to find a point $u^* \in K$ such that

$$F(u^*, q) \geq 0, \quad \forall\, q \in K.$$

Denote the solution set of EP by $(shortly, EP(F))$. In order to solve EP, we need the following assumptions on F :

(i) $F(p,p) = 0$ for all $p \in K$;
(ii) $F(p,q) + F(q,p) \leq 0$ for all $p, q \in K$;
(iii) For each $p \in K, q \mapsto F(p,q)$ is convex;
(iv) For each $\overline{p} \in X, r > 0$, there exists a compact subset $D_{\overline{p}} \subseteq K$ containing a point $h \in D_{\overline{p}} \subseteq K$ such that

$$F(p,h) - \frac{1}{r}\langle \overrightarrow{ph}, \overrightarrow{p\overline{p}} \rangle < 0 \quad \forall\, p \in D_{\overline{p}} \subseteq K.$$

Lemma 4.1. *([10]) Let K be a nonempty closed convex subset of a complete CAT(0) space X and $F : K \times K \to \mathbb{R}$ be a bifunction satisfying (i)-(iv). For any $r > 0$ and $p \in X$, define the following resolvent $T_r : X \to K$ of F:*

$$T_r p = \{h \in K : F(h,q) - \frac{1}{r}\langle \overrightarrow{ph}, \overrightarrow{p\overline{p}} \rangle \geq 0,\ \forall\, q \in K\},$$

then, the following conclusions holds

(i) *T_r is a single-valued firmly nonexpansive mapping;*
(ii) *$F(T_r) = EP(F)$;*
(iii) *$EP(F)$ is closed and convex.*

It follows Corollary 3.2 and Lemma 4.1 that the following result holds.

Theorem 4.1. *Let $K \subset X$ be a nonempty closed convex subset of complete CAT(0) space $(X,d), g : X \to (-\infty, \infty]$ be a proper convex lower semi-continuous function, $F : K \times K \to \mathbb{R}$ be a bifunction satisfying (i)-(iv), T_r be the resolvent of F . Let $u_1 \in X$ be chosen arbitrarily and the sequence $\{u_n\}$ be defined as follows:*

$$\begin{cases} p_n = J^g_{\lambda_n}(u_n) = \arg\min\limits_{q\in X}[g(q) + \dfrac{1}{2\lambda_n} d^2(q,u_n)] \\ x_{n+1} = T((1-\alpha_n)u_n \oplus \alpha_n T_r p_n), \quad n \geq 1, \end{cases} \tag{4.1}$$

where $\lambda_n \geq \lambda > 0$, $\{\alpha_n\}$ be a sequence in $[b,c]$, and $b, c \in (0,1), 0 < b(1-c) \leq \frac{1}{2}$.

(i) *If $\Omega = F(T_r) \cap \arg\min\limits_{q\in X} g(q) \neq \emptyset$ then $\{u_n\}$ $\triangle$-convergent to a point $u \in \Omega$.*
(ii) *In addition, if $\Omega = F(T_r) \cap \arg\min\limits_{q\in X} g(q) \neq \emptyset$ and T_r is semi-compact, then $\{u_n\}$ converges strongly to a point $u \in \Omega$.*

5. Conclusion

This paper introduces an iterative algorithm aimed at tackling the minimization and fixed point problems arising from total asymptotically nonexpansive mappings in CAT(0) spaces. We provide strong convergence theorems and $\triangle$-convergence theorems to address these problems effectively. Additionally, we demonstrate the applicability of our results by solving the equilibrium problem in CAT(0) spaces.

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