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MODIFIED GERAGHTY TYPE VIA SIMULATION FUNCTIONS

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ABSTRACT. We explore the Geraghty contraction through a simulation function, elucidating certain conditions for the existence and uniqueness of coincidence points for multi-class mappings involving the Geraghty function in metric spaces. The results presented in this work are consistent with those found in existing literature.

KEYWORDS: Geraghty type contraction mapping, simulation function, point of coincidence, common fixed point.
AMS Subject Classification: 47H09: 47H10.

1. Introduction

The field of fixed point theory emerged in the last quarter of the nineteenth century, and it has since been utilized extensively to establish the existence and uniqueness of solutions, particularly for functional equations. A significant contribution to this area is the Banach contraction principle, attributed to Banach [1], which has found widespread application in various contemporary research endeavors [2, 3, 4, 5, 6]. Fixed point theory finds applications across diverse fields such as engineering, economics, and computer science.

Geraghty [22] introduced the Cauchy criteria for convergence of contractive iterations in complete metric spaces, which led to the development of the Geraghty contraction. Subsequently, Khojasteh et al. [21] introduced the concept of $\mathcal{Z}$-contractions, which has been further investigated and summarized by numerous researchers [7, 8, 9, 10, 11, 12, 13, 14, 15]. Fixed point theory offers a rich platform for conducting interesting research.

Let Ω and Ψ be two self-maps defined on a non-empty set Π. If $\eta = \Omega\mu = \Psi\mu$ for some $\mu \in \Pi$, then μ is termed a coincidence point of Ω and Ψ. Consequently, η is referred to as a point of coincidence of Ω and Ψ. Furthermore, η is deemed

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a common fixed point of $\mathit{\Omega}$ and $\mathit{\Psi}$ if $\mu = \eta$. A pair $(\mathit{\Omega}, \mathit{\Psi})$ of self-maps is termed weakly compatible if they commute at their coincidence points.

In this article, we modify the Geraghty contraction using a simulation function and investigate the requisite conditions for its existence. We also focus on non-commuting type mappings, which are crucial for establishing the existence of common fixed points and the uniqueness of coincidence points, as well as common fixed points for classes of mappings in complete metric spaces. Finally, we provide an illustrative example to corroborate our theorem.

2. Preliminaries

Definition 2.1. [17] Two self-mappings $\mathit{\Omega}$ and $\mathit{\Psi}$ of a metric space $(\mathit{\Pi}, \mathit{\Lambda})$ are compatible if

$$\lim_{n\longrightarrow\infty} \mathit{\Lambda}(\mathit{\Psi}\mathit{\Omega}(\mu_n), \mathit{\Omega}\mathit{\Psi}(\mu_n)) = 0$$

whenever $\{\mu_n\}$ is a sequence in $\mathit{\Pi}$ such that

$$\lim_{n\longrightarrow\infty} \mathit{\Omega}(\mu_n) = \lim_{n\longrightarrow\infty} \mathit{\Psi}(\mu_n) = t$$

for some $t \in \mathit{\Pi}$.

Theorem 2.1. [18] *Let $\mathit{\Omega}$ and $\mathit{\Psi}$ be weakly compatible self-maps defined on a non-empty set $\mathit{\Pi}$. If $\mathit{\Omega}$ and $\mathit{\Psi}$ have a unique point of coincidence $\eta = \mathit{\Omega}\mu = \mathit{\Psi}\eta$, then η is the unique common fixed point of $\mathit{\Omega}$ and $\mathit{\Psi}$.*

Definition 2.2. [19] Let $(\mathit{\Pi}, \mathit{\Lambda})$ is a metric space and $\mathit{\Omega}, \mathit{\Psi} : \mathit{\Pi} \longrightarrow$ be two mappings. The mappings $\mathit{\Omega}$ and $\mathit{\Psi}$ are said to satisfy the common limit in the range of $\mathit{\Psi}$ (shortly, $(CLR_{\mathit{\Psi}})$ property) if there exists a sequence $\{\mu_n\}$ in $\mathit{\Pi}$ such that

$$\lim_{n\longrightarrow\infty} \mathit{\Omega}(\mu_n) = \lim_{n\longrightarrow\infty} \mathit{\Psi}(\mu_n) = \mathit{\Psi}(\mu)$$

for some $\mu \in \mathit{\Pi}$. The importance of $(CLR_{\mathit{\Psi}})$-property ensures that one does not require the closeness of range subspaces.

Lemma 2.3. [20] *Let $(\mathit{\Pi}, \mathit{\Lambda})$ be a metric space and let $\{\mu_n\}$ be a sequence in $\mathit{\Pi}$ such that $\mathit{\Lambda}(\mu_n, \mu_{n+1}) \longrightarrow 0$ as $n \longrightarrow \infty$. If $\{\mu_n\}$ is not a Cauchy sequence in $\mathit{\Pi}$, then there exist $\varepsilon > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that $n_k > m_k > k$ and the following sequences tend to ε when $k \longrightarrow \infty$:*

$$\{\mathit{\Lambda}(\mu_{m_k}, \mu_{n_k})\}, \{\mathit{\Lambda}(\mu_{m_k}, \mu_{n_k+1})\}, \{\mathit{\Lambda}(\mu_{m_k-1}, \mu_{n_k})\},$$
$$\{\mathit{\Lambda}(\mu_{m_k-1}, \mu_{n_k+1})\}, \{\mathit{\Lambda}(\mu_{m_k+1}, \mu_{n_k+1})\}.$$

Definition 2.4. [21] A mapping $\zeta : [0, \infty)^2 \to \mathbb{R}$ is called a simulation function if it satisfies the following conditions:

- (ζ_1) $\zeta(0, 0) = 0$;
- (ζ_2) $\zeta(t, s) < s - t$ for all $t, s > 0$;
- (ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\lim_{n\to\infty} \zeta(t_n, s_n) < 0$.

Denoted by $\mathcal{Z}$ is the set of all simulation functions.

Example 2.5. [21] The following are some examples of simulation functions.

- (i) $\zeta(t, s) = \alpha s - t$ for all $t, s \in [0, \infty)$, where $\alpha \in [0, 1)$;
- (ii) $\zeta(t, s) = \frac{s}{1+s} - t$ for all $t, s \in [0, \infty)$;
- (iii) $\zeta(t, s) = sf(s) - t$ for all $t, s \in [0, \infty)$, where $f : [0, \infty) \to [0, 1)$ such that $\lim_{t\to c} f(t) < 1$ for all $c > 0$.

Definition 2.6. [21] Let (Π, Λ) be a metric space and $\zeta \in \mathcal{Z}$. A mapping $\Omega : \Pi \longrightarrow \Pi$ is called a $\mathcal{Z}$-contraction with respect to ζ if

$$\zeta(\Lambda(\Omega\mu, \Omega\upsilon), \Lambda(\mu, \upsilon)) \geq 0$$

holds for all $\mu, \upsilon \in \Pi$.

We denote by $\mathcal{F}$ the class of all functions $\beta : [0, \infty) \longrightarrow [0, 1)$satisfying $\beta(t_n) \longrightarrow 1$, implies $t_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Definition 2.7. [22] Let (Π, Λ) be a metric space. A map $\Omega : \Pi \longrightarrow \Pi$ is called Geraghty contraction if there exists $\beta \in \mathcal{F}$ such that for all $\mu, \upsilon \in \Pi$,

$$\Lambda(\Omega\mu, \Omega\upsilon) \leq \beta(\Lambda(\mu, \upsilon))\Lambda(\mu, \upsilon)$$

Theorem 2.2. [22] *Let (Π, Λ) be a complete metric space. Mapping $\Omega : \Pi \longrightarrow \Pi$ is Geraghty contraction. Then Ω has a fixed point $\mu \in \Pi$, and $\{\Omega^n \mu_1\}$ converges to μ.*

3. Main Results

Theorem 3.1. *Let (Π, Λ) be a complete metric space and $\Omega, \Psi : \Pi \longrightarrow \Pi$ be two self-mappings. Suppose that there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta\left(\Lambda(\Omega\mu, \Omega\upsilon), \beta(\Upsilon_{\Psi\Omega}(\mu, \upsilon))\Upsilon_{\Psi\Omega}(\mu, \upsilon)\right) \geq 0, \tag{3.1}$$

for all $\mu, \upsilon \in \Pi$ with $\Psi\mu \neq \Psi\upsilon$, where $\beta : [0, \infty) \longrightarrow (0, 1)$ and

$$\Upsilon_{\Psi\Omega}(\mu, \upsilon) = \max\left\{\Lambda(\Psi\mu, \Psi\upsilon), \frac{[1 + \Lambda(\Psi\mu, \Omega\mu)]\Lambda(\Psi\upsilon, \Omega\upsilon)}{1 + \Lambda(\Psi\mu, \Psi\upsilon)}\right\}.$$

Suppose that there exists a Picard-Jungck sequence $\{j_n\}$ of (Ω, Ψ). Also assume that, at least, one of the following conditions holds:

(i) *$(\Omega\Pi, \Lambda)$ or $(\Psi\Pi, \Lambda)$ is complete*;
(ii) *(Π, Λ) is complete, Ψ is continuous, Ω and Ψ are compatible.*

Then Ω and Ψ have a unique point of coincidence.

Proof. Firstly, we will show that the point of coincidence of Ω and Ψ is unique. Suppose that η_1 and η_2 are distinct points of coincidence of Ω and Ψ. It follows that there exist two points θ_1 and $\theta_2\,(\theta_1 \neq \theta_2)$ such that $\Omega\theta_1 = \Psi\theta_1 = \eta_1$ and $\Omega\theta_2 = \Psi\theta_2 = \eta_1$. Then $d(\Omega\theta_1, \Omega\theta_2) > 0$ and using (ζ_2), we obtain

$$0 \leq \zeta\left(\Lambda(\Omega\theta_1, \Omega\theta_2), \beta(\Upsilon_{\Psi\Omega}(\theta_1, \theta_2))\Upsilon_{\Psi\Omega}(\theta_1, \theta_2)\right), \tag{3.2}$$

where

$$\begin{aligned}
\Upsilon_{\Psi\Omega}(\theta_1, \theta_2) &= \max\left\{\Lambda(\Psi\theta_1, \Psi\theta_2), \frac{[1 + \Lambda(\Psi\theta_1, \Omega\theta_1)]\Lambda(\Psi\theta_2, \Omega\theta_2)}{1 + \Lambda(\Psi\theta_1, \Psi\theta_2)}\right\} \\
&= \max\left\{\Lambda(\eta_1, \eta_2), \frac{[1 + \Lambda(\eta_1, \eta_1)]\Lambda(\eta_2, \eta_2)}{1 + \Lambda(\eta_1, \eta_2)}\right\} \\
&= \max\{\Lambda(\eta_1, \eta_2), 0\} \\
&= \Lambda(\eta_1, \eta_2).
\end{aligned}$$

This together with (3.2) show that

$$\begin{aligned}0 &\le \zeta\left(\Lambda(\Omega\theta_1,\Omega\theta_2),\beta(\Upsilon_{\Psi\Omega}(\theta_1,\theta_2))\Upsilon_{\Psi\Omega}(\theta_1,\theta_2)\right)\\ &= \zeta\left(\Lambda(\eta_1,\eta_2),\beta(\Lambda(\eta_1,\eta_2))\Lambda(\eta_1,\eta_2)\right)\\ &< \beta(\Lambda(\eta_1,\eta_2))\Lambda(\eta_1,\eta_2)-\Lambda(\eta_1,\eta_2)\\ &< \Lambda(\eta_1,\eta_2)-\Lambda(\eta_1,\eta_2)\\ &= 0\end{aligned}$$

which is a contradiction. Suppose that there is a Picard-Jungck sequence $\{j_n\}$ such that $j_n = \Omega\mu_n = \Psi\mu_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$. If $j_m = j_{m+1}$ for some $m \in \mathbb{N}\cup\{0\}$, then $\Psi\mu_{m+1} = j_m = j_{m+1} = \Omega\mu_{m+1}$. Hence Ψ and Ω have a coiucidence point μ_{m+1}. Therefore, we assume that $j_n \neq j_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$. Also, $\Lambda(j_{n+1}, j_{n+2}) > 0$ and taking $\mu = \mu_{n+1}$, $\upsilon = \mu_{n+2}$ in (3.1), we get that

$$\zeta\left(\Lambda(\Omega\mu_{n+1},\Omega\mu_{n+2}),\beta(\Upsilon_{\Psi\Omega}(\mu_{n+1},\mu_{n+2}))\Upsilon_{\Psi\Omega}(\mu_{n+1},\mu_{n+2})\right) \ge 0, \tag{3.3}$$

where

$$\begin{aligned}&\Upsilon_{\Psi\Omega}(\mu_{n+1},\mu_{n+2})\\ &= \max\left\{\Lambda(\Psi\mu_{n+1},\Psi\mu_{n+2}),\frac{[1+\Lambda(\Psi\mu_{n+1},\Omega\mu_{n+1})]\Lambda(\Psi\mu_{n+2},\Omega\mu_{n+2})}{1+\Lambda(\Psi\mu_{n+1},\Psi\mu_{n+2})}\right\}\\ &= \max\left\{\Lambda(j_n,j_{n+1}),\frac{[1+\Lambda(j_n,j_{n+1})]\Lambda(j_{n+1},j_{n+2})}{1+\Lambda(j_n,j_{n+1})}\right\}.\end{aligned}$$

This together with (3.3) show that

$$\begin{aligned}0 &\le \zeta\left(\Lambda(\Omega\mu_{n+1},\Omega\mu_{n+2}),\beta(\Upsilon_{\Psi\Omega}(\mu_{n+1},\mu_{n+2}))\Upsilon_{\Psi\Omega}(\mu_{n+1},\mu_{n+2})\right)\\ &= \zeta\left(\Lambda(j_{n+1},j_{n+2}),\beta(\Upsilon_{\Psi\Omega}(\mu_{n+1},\mu_{n+2}))\Upsilon_{\Psi\Omega}(\mu_{n+1},\mu_{n+2})\right)\\ &< \beta(\Upsilon_{\Psi\Omega}(\mu_{n+1},\mu_{n+2}))\Upsilon_{\Psi\Omega}(\mu_{n+1},\mu_{n+2})-\Lambda(j_{n+1},j_{n+2})\\ &< \Upsilon_{\Psi\Omega}(\mu_{n+1},\mu_{n+2})-\Lambda(j_{n+1},j_{n+2}).\end{aligned} \tag{3.4}$$

If $\Upsilon_{\Psi\Omega}(\mu_{n+1},\mu_{n+2}) = \Lambda(j_{n+1},j_{n+2})$, inequality (3.4) gives

$$\Lambda(j_{n+1},j_{n+2}) < \Lambda(j_{n+1},j_{n+2})$$

which is a contradiction. Hence, $\Upsilon_{\Psi\Omega}(\mu_{n+1},\mu_{n+2}) = \Lambda(j_n,j_{n+1})$. This implies that

$$\Lambda(j_{n+1},j_{n+2}) < \Lambda(j_n,j_{n+1})$$

for all $n \in \mathbb{N}\cup\{0\}$. Thus, there exists $\rho > 0$ such that $\lim\limits_{n\longrightarrow\infty}\Lambda(j_n,j_{n+1}) = \rho$. Assme that $\rho > 0$. In this case we get that

$$\frac{\Lambda(j_{n+1},j_{n+2})}{\Lambda(j_n,j_{n+1})} \le \beta\left(\Lambda(j_n,j_{n+1})\right) < 1,$$

taking $n \longrightarrow \infty$, we get $\lim\limits_{n\longrightarrow\infty}\beta\left(\Lambda(j_n,j_{n+1})\right) = 1$ which is a contradiction to the fact that $\lim\limits_{n\longrightarrow\infty}\Lambda(j_n,j_{n+1}) = \rho > 0$. Hence, $\lim\limits_{n\longrightarrow\infty}\Lambda(j_n,j_{n+1}) = 0$. Next , we will show that $j_n \neq j_m$, whenever $n \neq m$. Assume that $j_n = j_m$ for some $n > m$. Then we can claim that $\mu_{n+1} = \mu_{m+1}$. If $\mu_{n+1} \neq \mu_{m+1}$, then

$$\Omega\mu_n \neq \Omega\mu_m \Rightarrow j_n \neq j_m$$

which is obviously impossible . Hence

$$\begin{aligned}\mu_{n+1} = \mu_{m+1} &\Rightarrow \Omega\mu_{n+1} = \Omega\mu_{m+1}\\ &\Rightarrow j_{n+1} = j_{m+1}.\end{aligned}$$

Then following above, we obtain

$$\begin{aligned}\Lambda(j_{m+1}, j_m) &< \Lambda(j_m, j_{m-1})\\ &\vdots\\ &< \Lambda(j_{n+1}, j_n)\\ &= \Lambda(j_{m+1}, j_m)\end{aligned}$$

which is a contradiction. Now, we will show that $\{j_n\}$ is a Cauchy sequence. Assume that $\{j_n\}$ is not a Cauchy sequence. Taking $\mu = \mu_{m_k+1}$, $\upsilon = \mu_{n_k+1}$ in (3.1), we get that

$$\zeta\left(\Lambda(\Omega\mu_{m_k+1}, \Omega\mu_{n_k+1}, \beta(\Upsilon_{\Psi\Omega}(\mu_{m_k+1}, \mu_{n_k+1}))\Upsilon_{\Psi\Omega}(\mu_{m_k+1}, \mu_{n_k+1})\right) \geq 0, \tag{3.5}$$

where

$$\begin{aligned}&\Upsilon_{\Psi\Omega}(\mu_{m_k+1}, \mu_{n_k+1})\\ &= \max\left\{\Lambda(\Psi\mu_{m_k+1}, \Psi\mu_{n_k+1}), \frac{[1+\Lambda(\Psi\mu_{m_k+1}, \Omega\mu_{m_k+1})]\Lambda(\Psi\mu_{n_k+1}, \Omega\mu_{n_k+1})}{1+\Lambda(\Psi\mu_{m_k+1}, \Psi\mu_{n_k+1})}\right\}\\ &= \max\left\{\Lambda(j_{m_k}, j_{n_k}), \frac{[1+\Lambda(j_{m_k}, j_{m_k+1})]\Lambda(j_{n_k}, j_{n_k+1})}{1+\Lambda(j_{m_k}, j_{n_k})}\right\}\\ &= \Lambda(j_{m_k}, j_{n_k}).\end{aligned}$$

This together with (3.5) show that

$$\begin{aligned}0 &\leq \zeta\left(\Lambda(\Omega\mu_{m_k+1}, \Omega\mu_{n_k+1}, \beta(\Upsilon_{\Psi\Omega}(\mu_{m_k+1}, \mu_{n_k+1}))\Upsilon_{\Psi\Omega}(\mu_{m_k+1}, \mu_{n_k+1})\right)\\ &= \zeta(\Lambda(j_{m_k+1}, j_{n_k+1}), \beta(\Lambda(j_{m_k}, j_{n_k}))\Lambda(j_{m_k}, j_{n_k}))\\ &\leq \zeta(\phi_k, \varphi_k)\end{aligned} \tag{3.6}$$

where $0 < \phi_k = \Lambda(j_{m_k+1}, j_{n_k+1})$ and $0 < \varphi_k = \beta(\Lambda(j_{m_k}, j_{n_k}))\Lambda(j_{m_k}, j_{n_k})$. Since the sequence $\{j_n\}$ is not a Cauchy sequence and usiug Lemma 2.3, we have $\{\Lambda(j_{m_k}, j_{n_k})\}$ and $\{\Lambda(j_{m_k+1}, j_{n_k+1})\}$ both the sequence tend to $\varepsilon > 0$ as $k \longrightarrow \infty$. So,

$$\begin{aligned}\phi_k &= \Lambda(j_{m_k+1}, j_{n_k+1})\\ &\leq \beta(\Lambda(j_{m_k}, j_{n_k}))\Lambda(j_{m_k}, j_{n_k})\\ &= \varphi_k\\ &< \Lambda(j_{m_k}, j_{n_k})\end{aligned} \tag{3.7}$$

and using the sandwich theorem, $\{\varphi_k\}$, where $\varphi_k = \beta(\Lambda(j_{m_k}, j_{n_k}))\Lambda(j_{m_k}, j_{n_k})) \longrightarrow \varepsilon$ as $k \longrightarrow \infty$. Hence , we have $0 < \phi_k, \varphi_k \longrightarrow \varepsilon$.
Thus,

$$0 \leq \overline{\lim_{k\longrightarrow\infty}}\zeta(\phi_k, \varphi_k) = \overline{\lim_{k\longrightarrow\infty}}(\varphi_k - \phi_k) = \varepsilon - \varepsilon = 0$$

which is a contradiction. Hence, the Picard-Jungck sequence $\{j_n\}$ is a Cauchy sequence. from condition(i), $(\Psi\Pi, \Lambda)$ is complete, then there exists $\omega \in \Pi$ such that $j_n = \Psi\mu_{n+1} \longrightarrow \Psi\omega$ as $n \longrightarrow \infty$ which implies

$$\lim_{n\longrightarrow\infty} \Lambda(\Psi\mu_{n+1}, \Psi\omega) = 0. \tag{3.8}$$

We will show that $\Omega\omega = \Psi\omega$. Let $\Omega\omega \neq \Psi\omega$ and $\Lambda(\Omega\omega, \Psi\omega) > \sigma$. From (3.8), there exists $n_0 \in \mathbb{N}$ such that

$$\Lambda(\Omega\mu_n, \Psi\omega) < \sigma = \Lambda(\Omega\omega, \Psi\omega)$$

for all $n \geq n_0$. So,

$$\Omega\mu_n \neq \Omega\omega \Rightarrow \Lambda(\Omega\mu_n, \Omega\omega) > 0 \tag{3.9}$$

for all $n \geq n_0$. Now, there dose not exist some $n \geq n_3$

$$\Psi\mu_{n+1} = \Psi\omega.$$

Hence, there exists a partial subsequence $\{\Psi\mu_{t_k}\}$ of $\{\Psi\mu_{n+1}\}$ such that

$$\Psi\mu_{t_k} \neq \omega \tag{3.10}$$

for all $k \in \mathbb{N}$. Let $n_2 \in \mathbb{N}$ be such that $t_{n_2} \geq n_0$. Using (3.9) and (3.10), we have $\Lambda(\Omega\mu_{t_n}, \Omega\omega) > 0$ and $\Lambda(\Psi\mu_{n+1}, \omega) > 0$ for all $n > n_2$. Using (ζ_2), we get

$$\begin{aligned}
0 &\leq \zeta\left(\Lambda(\Omega\omega, \Omega\mu_{t_n}, \beta(\Upsilon_{\Psi\Omega}(\omega, \mu_{n+1}))\Upsilon_{\Psi\Omega}(\omega, \mu_{n+1})\right) \\
&= \zeta\left(\Lambda(\Omega\omega, \Omega\mu_{t_n}, \beta(\Lambda(\Psi\omega, \Psi\mu_{n+1}))\Lambda(\Psi\omega, \Psi\mu_{n+1})\right) \\
&< \beta(\Lambda(\Psi\omega, \Psi\mu_{n+1}))\Lambda(\Psi\omega, \Psi\mu_{n+1}) - \Lambda(\Omega\omega, \Omega\mu_{t_n}) \\
&< \Lambda(\Psi\omega, \Psi\mu_{n+1}) - \Lambda(\Omega\omega, \Omega\mu_{t_n}).
\end{aligned}$$

Taking $n \longrightarrow \infty$, we obtain

$$\begin{aligned}
0 &< \Lambda(\Psi\omega, \Psi\omega) - \Lambda(\Omega\omega, \Psi\omega) \\
&= 0 - \Lambda(\Omega\omega, \Psi\omega).
\end{aligned}$$

This implies that $\eta = \Psi\omega = \Omega\omega$ and η is the unique point coincidence of Ω and Ψ. In the same way, we can show that $\varrho = \Omega\omega = \Psi\omega$ is a unique point of coincidence of Ω and Ψ when $(\Omega\Pi, \Lambda)$ is complete.
From condition(ii), (Π, Λ) is complete, there exists $\omega \in \Pi$ such that $j_n = \Omega\mu_n = \Psi\mu_{n+1} \longrightarrow \omega$ as $n \longrightarrow \infty$. Since Ψ is continuous, we get

$$\lim_{n\longrightarrow\infty} \Psi(\Omega\mu_n) = \Psi\omega \Rightarrow \lim_{n\longrightarrow\infty} \Lambda(\Psi(\Omega\mu_n), \Psi\omega) = 0 \tag{3.11}$$

and

$$\lim_{n\longrightarrow\infty} \Psi(\Psi\mu_{n+1}) = \Psi\omega \Rightarrow \lim_{n\longrightarrow\infty} \Lambda(\Psi(\Psi\mu_{n+1}), \Psi\omega) = 0. \tag{3.12}$$

We claim that $\lim_{n\longrightarrow\infty} \Omega(\Psi\mu_n) = \Omega\omega$. If not, there exists a subsequence $\{\Omega(\Psi\mu_{t_k})\}$ of $\{\Omega(\Psi\mu_n)\}$ such that

$$\Lambda(\Omega(\Psi\mu_{t_k}), \Omega\omega) > 0 \tag{3.13}$$

for all $k \in \mathbb{N}$. Then there does not exist some $k_1 \in \mathbb{N}$ for all $n > k_1$

$$\Psi(\Psi\mu_{n+1}) = \Psi\omega.$$

Thus, there exists a partial subsequence $\{\Psi(\Psi\mu_{t_r})\}$ of $\{\Psi(\Psi\mu_{n+1})\}$ such that

$$\Psi(\Psi\mu_{t_r}) \neq \Psi\omega \tag{3.14}$$

for all $r \in \mathbb{N}$. Hence, using (3.13) and (3.14), we have $\Lambda(\Omega(\Psi\mu_{t_k}), \Omega\omega) > 0$ and $\Lambda(\Psi(\Psi\mu_{t_r}), \Psi\omega) > 0$ for all $k, r \in \mathbb{N}$. Using (ζ_2), we obtain

$$\begin{aligned}
0 &\leq \zeta(\Lambda(\Omega(\Psi\mu_{t_k}), \Omega\omega), \beta(\Upsilon_{\Psi\Omega}(\Psi\mu_{t_r}, \omega))\Upsilon_{\Psi\Omega}(\Psi\mu_{t_r}, \omega) \\
&= \zeta\left(\Lambda(\Omega(\Psi\mu_{t_k}), \Omega\omega), \beta(\Lambda(\Psi(\Psi\mu_{t_r}), \Psi\omega))\Lambda(\Psi(\Psi\mu_{t_r}), \Psi\omega)\right) \\
&< \beta(\Lambda(\Psi(\Psi\mu_{t_r}), \Psi\omega))\Lambda(\Psi(\Psi\mu_{t_r}), \Psi\omega)) - \Lambda(\Omega(\Psi\mu_{t_k}), \Omega\omega) \\
&< \Lambda(\Psi(\Psi\mu_{t_r}), \Psi\omega) - \Lambda(\Omega(\Psi\mu_{t_k}), \Omega\omega).
\end{aligned}$$

Hence, we have $\Lambda(\Omega(\Psi\mu_{t_k}), \Omega\omega) < \Lambda(\Psi(\Psi\mu_{t_r}), \Psi\omega) \longrightarrow 0$ as $k \longrightarrow \infty$ which is a contradiction. This implies that

$$\lim_{n\longrightarrow\infty} \Lambda(\Omega(\Psi\mu_n), \Omega\omega) = 0. \tag{3.15}$$

Further, since Ω and Ψ are compatible, we have

$$\lim_{n\longrightarrow\infty} \Lambda(\Omega(\Psi\mu_n), \Psi(\Omega\mu_n)) = 0. \tag{3.16}$$

Finally, using (3.11), (3.15) aud (3.16), we have

$$\begin{aligned}\Lambda(\Omega\omega,\Psi\omega) &= \Lambda(\Omega\omega,\Omega(\Psi\mu_n)) + \Lambda(\Omega(\Psi\mu_n),\Psi(\Omega\mu_n)) + \Lambda(\Psi(\Omega\mu_n),\Psi\omega)\\ &\Rightarrow \Lambda(\Omega\omega,\Psi\omega) \leq 0\\ &\Rightarrow \Lambda(\Omega\omega,\Psi\omega) = 0.\end{aligned}$$

This implies that $\varrho = \Psi\omega = \Omega\omega$ and ϱ is the unique point of coincidence of Ω and Ψ. Thus, the mappings Ω and Ψ have a unique point of coincidence. □

Theorem 3.2. *Let $\Omega, \Psi : \Pi \longrightarrow \Pi$ be two self-maps defined on a complete metric space (Π, Λ). Assume there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta\left(\Lambda(\Omega\mu,\Omega\upsilon), \beta(\Upsilon_{\Psi\Omega}(\mu,\upsilon))\Upsilon_{\Psi\Omega}(\mu,\upsilon)\right) \geq 0, \tag{3.17}$$

for all $\mu, \upsilon \in \Pi$ with $\Psi\mu \neq \Psi\upsilon$, where $\beta : [0,\infty) \longrightarrow (0,1)$ and

$$\Upsilon_{\Psi\Omega}(\mu,\upsilon) = \max\left\{\Lambda(\Psi\mu,\Psi\upsilon), \frac{[1+\Lambda(\Psi\mu,\Omega\mu)]\Lambda(\Psi\upsilon,\Omega\upsilon)}{1+\Lambda(\Psi\mu,\Psi\upsilon)}\right\}.$$

Suppose that, there exists a Picard-Jungck sequence $\{\mu_n\}$ of (Ω,Ψ). Also assume that, $(\Omega\Pi,\Lambda)$ or $(\Psi\Pi,\Lambda)$ is complete and Ω and Ψ are weakly compatible. Then Ω and Ψ have a unique common fixed point in Π.

Proof. It follows Theorem 3.1, Ω and Ψ have a unique point of coincidence. Further, since Ω and Ψ are weakly compatible, then according to Theorem 2.1, they have a unique common fixed point in Π. □

Theorem 3.3. *Let $\Omega, \Psi : \Pi \longrightarrow \Pi$ be two self-maps defined on a complete metric space (Π, Λ). Assume there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta\left(\Lambda(\Omega\mu,\Omega\upsilon), \beta(\Upsilon_{\Psi\Omega}(\mu,\upsilon))\Upsilon_{\Psi\Omega}(\mu,\upsilon)\right) \geq 0, \tag{3.18}$$

for all $\mu, \upsilon \in \Pi$ with $\Psi\mu \neq \Psi\upsilon$, where $\beta : [0,\infty) \longrightarrow (0,1)$ and

$$\Upsilon_{\Psi\Omega}(\mu,\upsilon) = \max\left\{\Lambda(\Psi\mu,\Psi\upsilon), \frac{[1+\Lambda(\Psi\mu,\Omega\mu)]\Lambda(\Psi\upsilon,\Omega\upsilon)}{1+\Lambda(\Psi\mu,\Psi\upsilon)}\right\}.$$

Suppose that, there exists a Picard-Jungck sequence $\{\mu_n\}$ of (Ω,Ψ). Also assume that, $(\Omega\Pi,\Lambda)$ or $(\Psi\Pi,\Lambda)$ is complete, Ω and Ψ are satisfy (CLR_g)-property. Then Ω and Ψ have a unique common fixed point in Π.

Proof. Using Ω and Ψ are satisfy (CLR_g)-property in Definition 2.2 and Theorem 3.1. □

Example 3.1. Let $\Pi = \{0,4,5\}$ and $\Lambda : \Pi \times \Pi \longrightarrow [0,\infty)$ be defined by $\Lambda(\mu,\upsilon) = |\mu - \upsilon|$. Define $\Omega, \Psi : \Pi \longrightarrow \Pi$ as

$$\Omega\mu = \begin{pmatrix} 0 & 4 & 5\\ 4 & 4 & 4\end{pmatrix} \quad \text{and} \quad \Psi\mu = \begin{pmatrix} 0 & 4 & 5\\ 5 & 4 & 0\end{pmatrix}.$$

Suppose $\zeta(t,s) = \dfrac{s}{s+1} - t$, $\beta(t) = \dfrac{1}{1+\dfrac{t}{9}}$ for $t > 0$ and $\beta(t) = \dfrac{1}{2}$ for $t = 0$.

Case (i): For $\mu = 0$, $\upsilon = 4$. From (3.1), we obtain

$$\begin{aligned}\zeta\left(\Lambda(\Omega 0,\Omega 4), \beta(\Upsilon_{\Psi\Omega}(0,4))\Upsilon_{\Psi\Omega}(0,4)\right) &= \zeta\left(\Lambda(4,4), \beta(\Upsilon_{\Psi\Omega}(0,4))\Upsilon_{\Psi\Omega}(0,4)\right)\\ &= \zeta\left(0, \beta(\Upsilon_{\Psi\Omega}(0,4))\Upsilon_{\Psi\Omega}(0,4)\right),\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}\Upsilon_{\Psi\Omega}(0,4) &= \max\left\{\Lambda(\Psi 0,\Psi 4), \frac{[1+\Lambda(\Psi 0,\Omega 0)]\Lambda(\Psi 4,\Omega 4)}{1+\Lambda(\Psi 0,\Psi 4)}\right\}\\ &= \max\left\{\Lambda(5,4), \frac{[1+\Lambda(5,4)]\Lambda(4,4)}{1+\Lambda(5,4)}\right\}\\ &= \max\{1,0\}\\ &= 1.\end{aligned}$$

This together with (3.19) show that

$$\begin{aligned}\zeta\left(0,\beta(\Upsilon_{\Psi\Omega}(0,4))\Upsilon_{\Psi\Omega}(0,4)\right) &= \zeta\left(0,\beta(1)\cdot 1\right)\\ &= \frac{\beta(1)}{\beta(1)+1}\\ &\geq 0.\end{aligned}$$

Case (ii): For $\mu = 0$, $\upsilon = 5$. From (3.1), we obtain

$$\begin{aligned}\zeta\left(\Lambda(\Omega 0,\Omega 5),\beta(\Upsilon_{\Psi\Omega}(0,5))\Upsilon_{\Psi\Omega}(0,5)\right) &= \zeta\left(\Lambda(4,4),\beta(\Upsilon_{\Psi\Omega}(0,5))\Upsilon_{\Psi\Omega}(0,5)\right)\\ &= \zeta\left(0,\beta(\Upsilon_{\Psi\Omega}(0,5))\Upsilon_{\Psi\Omega}(0,5)\right),\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}\Upsilon_{\Psi\Omega}(0,5) &= \max\left\{\Lambda(\Psi 0,\Psi 5), \frac{[1+\Lambda(\Psi 0,\Omega 0)]\Lambda(\Psi 5,\Omega 5)}{1+\Lambda(\Psi 0,\Psi 5)}\right\}\\ &= \max\left\{\Lambda(5,0), \frac{[1+\Lambda(5,4)]\Lambda(0,4)}{1+\Lambda(5,0)}\right\}\\ &= \max\left\{5,\frac{4}{3}\right\}\\ &= 5.\end{aligned}$$

This together with (3.20) show that

$$\begin{aligned}\zeta\left(0,\beta(\Upsilon_{\Psi\Omega}(0,5))\Upsilon_{\Psi\Omega}(0,5)\right) &= \zeta\left(0,\beta(5)\cdot 5\right)\\ &= \frac{5\beta(5)}{5\beta(5)+1}\\ &\geq 0.\end{aligned}$$

Case (iii): For $\mu = 4$, $\upsilon = 5$. From (3.1), we obtain

$$\begin{aligned}\zeta\left(\Lambda(\Omega 4,\Omega 5),\beta(\Upsilon_{\Psi\Omega}(4,5))\Upsilon_{\Psi\Omega}(4,5)\right) &= \zeta\left(\Lambda(4,4),\beta(\Upsilon_{\Psi\Omega}(4,5))\Upsilon_{\Psi\Omega}(4,5)\right)\\ &= \zeta\left(0,\beta(\Upsilon_{\Psi\Omega}(4,5))\Upsilon_{\Psi\Omega}(4,5)\right),\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}\Upsilon_{\Psi\Omega}(4,5) &= \max\left\{\Lambda(\Psi 4,\Psi 5), \frac{[1+\Lambda(\Psi 4,\Omega 4)]\Lambda(\Psi 5,\Omega 5)}{1+\Lambda(\Psi 4,\Psi 5)}\right\}\\ &= \max\left\{\Lambda(4,0), \frac{[1+\Lambda(4,4)]\Lambda(0,4)}{1+\Lambda(4,0)}\right\}\\ &= \max\left\{4,\frac{4}{5}\right\}\\ &= 4.\end{aligned}$$

This together with (3.21) show that

$$\begin{aligned}\zeta\left(0, \beta(\Upsilon_{\Psi\Omega}(4,5))\Upsilon_{\Psi\Omega}(4,5)\right) &= \zeta\left(0, \beta(4)\cdot 4\right)\\ &= \frac{4\beta(4)}{4\beta(4)+1}\\ &\geq 0.\end{aligned}$$

Therefore, all the assumptions of Theorem 3.1 are satisfied, and as per its conclusion, Ω and Ψ have a unique point of coincidence $\mu = 4$, making it their unique common fixed point.

4. Conclusion

This paper focuses on investigating the existence and uniqueness of coincidence points and Geraghty-type common fixed points under contractive conditions using simulation functions within the context of complete metric spaces. The obtained results are illustrated with examples to demonstrate their applicability.

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