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COMMON FIXED POINT THEOREMS FOR ASYMPTOTIC REGULARITY IN GENERALIZED b-METRIC SPACES

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ABSTRACT. In this paper, we introduce the concept of a b-metric-like space, which is an intriguing extension of the b-metric space, and it encompasses certain sufficient conditions for the existence of a common fixed point. The discoveries presented here build upon and broaden previous research in this area..

KEYWORDS:b-metric-like space, Common fixed point, Asymptotic regularity.
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1. Introduction

In 1920, Banach [1] introduced the Banach contraction principle, which has long been one of the most essential methods for approximating solutions to nonlinear problems. Several authors have expanded and developed it in various disciplines due to its usefulness in various branches.

The Banach contraction principle states

Theorem 1.1. [1] *Let (X, d) be a complete metric space and let f be a contraction on X, there exists $M \in [0, 1)$ such that*

$$d(fx, fy) \leq Md(x, y), \forall x, y \in X.$$

Then f has a unique fixed point.

In recent years, many scholars have proposed a series of new concepts of contraction mapping and new fixed point theorems [2, 3, 4, 5, 6, 7]. In 1993, Bakhtin [2] introduced the concept of b-metric space which is a generalization of metric space. He proved the famous Banach Contraction Principle in the b-metric space, also see [3].

In 2013, the concept of a b-metric-like space was introduced first by Alghamdi

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[5]. For some fixed point results on b-metric-like spaces, see [8] and [9].

In 2019, Bisht and Singh [10] obtain the existence of common fixed point theorems for mappings satisfying Lipschitz–Kannan type condition.

Theorem 1.2. [10] *If (X, d) is a complete metric space and $f, g : X \longrightarrow X$. Suppose that f is asymptotically regular with respect to g and there exist $M \in [0, 1)$ and $K \in [0, \infty)$ satisfying*

$$d(fx, fy) \leq Md(gx, gy) + K\{d(fx, gx) + d(fy, gy)\}$$

for all $x, y \in X$. Further, suppose that f and g are (f, g)-orbitally continuous and compatible. Then $C(f, g) \neq \emptyset$ and f and g have a unique common fixed point.

In 2020, Arunchai, Mungchai and Thala [11] propose the common fixed point theorems for asymptotic regularity on b-metric spaces. The results presented in the paper improve and extend some previous results.

Theorem 1.3. [11] *If (X, b) is a complete b-metric space and $f, g : X \longrightarrow X$. Suppose that f is asymptotically regular with respect to g and there exist $M \in [0, 1)$ and $K \in [0, \infty)$ satisfying*

$$b(fx, fy) \leq Mb(gx, gy) + K\{b(fx, gx) + b(fy, gy)\}$$

for all $x, y \in X$. Further, suppose that f and g are (f, g)-orbitally continuous and compatible. Then $C(f, g) \neq \emptyset$ and f and g have a unique common fixed point.

In this paper, we introduce common fixed point theorems for asymptotic regularity in generalized b-metric spaces, which is a fascinating extension of the b-metric space and contains some sufficient conditions for the presence of a common fixed point. The findings here improve and expand upon prior research.

2. Preliminaries

The following concepts and results are needed for the results.

Definition 2.1. [5] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $b_l : X \times X \longrightarrow \mathbb{R}^+$ is a b-metric-like if, for all $x, y, z \in X$, the following conditions are satisfied:

(b_l1) $b_l(x, y) = 0$ implies $x = y$
(b_l2) $b_l(x, y) = b_l(y, x)$
(b_l3) $b_l(x, z) \leq s[b_l(x, y) + b_l(y, z)]$

A b-metric-like space is a pair (X, b_l) such that X is a nonempty set and b_l is a b-metric-like on X. The number s is called the coefficient of (X, b_l).

Definition 2.2. [5] Let (X, b_l) be a b-metric-like space with coefficient s , $\{x_n\}$ be any sequence in X and $x \in X$. Then:

(i) The sequence $\{x_n\}$ is said to be convergent to x with respect to τ_{b_l} if $\lim_{n \longrightarrow \infty} b_l(x_n, x) = b_l(x, x)$.
(ii) The sequence $\{x_n\}$ is said to be a Cauchy sequence in (X, b_l), if $\lim_{n,m \longrightarrow \infty} b_l(x_n, x_m)$ exists and is finite.
(iii) (X, b_l) is said to be a complete b-metric-like space if for every Cauchy sequence $\{x_n\}$ in X there exists $x \in X$ such that

$$\lim_{n,m \longrightarrow \infty} b_l(x_n, x_m) = \lim_{n \longrightarrow \infty} b_l(x_n, x) = b_l(x, x).$$

Note that in a b-metric-like space the limit of a convergent sequence may not be unique.

Definition 2.3. [12] Let f and g be self-mapings of a set X (*i.e.*, $f, g : X \longrightarrow X$). If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. The set of coincidence points of f and g are denoted by $C(f,g)$ and the set point of coincidences of f and g are denoted by $PC(f,g)$. If $w = x$ then x is a common fixed point of f and g and the set of common fixed points is denoted by $F(f,g)$.

Definition 2.4. [12] Let f and g be self-mapings of a set X (*i.e.*, $f, g : X \longrightarrow X$). Then f and g are called weakly compatible if they commute at every coincidence point,*i.e.*, if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

Definition 2.5. Let f and g be two self-mappings of a b-metric-like space (X, b_l). Then f and g are called compatible if $\lim_{n \longrightarrow \infty} b_l(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \longrightarrow \infty} fx_n = \lim_{n \longrightarrow \infty} gx_n = z$$

for some $t \in X$.

Definition 2.6. Let f and g be two self-mappings of a b-metric-like space (X, b_l). Then f is asymptotically regular with respect to g at $x_0 \in X$, if there exists a sequence $\{x_n\}$ in X such that $gx_{n+1} = fx_n$, for $n = 0, 1, 2, ,$ and

$$\lim_{n \longrightarrow \infty} b_l(gx_{n+1}, gx_{n+2}) = 0.$$

Definition 2.7. Let f and g be two self-mappings of a b-metric-like space (X, b_l) and let $\{x_n\}$ be a sequence in X such that $fx_n = gx_{n+1}$. Then the set $O(x_0, f, g) = \{fx_n : n = 0, 1, 2, ...\}$ is called the (f,g)-orbit at x_0 and g is called (f,g)-orbitally continuous if $\lim_{n \longrightarrow \infty} fx_n = z$ implies $\lim_{n \longrightarrow \infty} gfx_n = gz$ or f is called (f,g)-orbitally continuous if $\lim_{n \longrightarrow \infty} fx_n = z$ implies $\lim_{n \longrightarrow \infty} ffx_n = fz$. We say f and g are orbitally continuous if f is (f,g)-orbitally continuous and g is (f,g)-orbitally continuous.

3. Main Results

In this section, we shall prove the existence of common fixed point in b-metric-like space under some conditions.

Theorem 3.1. *If (X, b_l) is a complete b-metric-like space and $f, g : X \longrightarrow X$. Suppose that f is asymptotically regular with respect to g and there exist $M \in [0,1)$ and $K \in [0, \infty)$ satisfying*

$$b_l(fx, fy) \leq Mb_l(gx, gy) + K\{b_l(fx, gx) + b_l(fy, gy)\} \tag{3.1}$$

for all $x, y \in X$. Further, suppose that f and g are (f,g)-orbitally continuous and compatible. Then $C(f,g) \neq \emptyset$ and f and g have a unique common fixed point.

Proof. Since f is asymptotically regular with respect to g at $x_0 \in x$, there exists a sequence $\{y_n\} \in X$ in X such that $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lim_{n \longrightarrow \infty} b_l(gx_{n+1}, gx_{n+2}) = \lim_{n \longrightarrow \infty} b_l(y_n, y_{n+1}) = 0$. To show that $\{y_n\} \in$ is a Cauchy sequence. Using (3.1), for any n and any $p > 0$,

$$\begin{aligned} b_l(fx_{n+p}, fx_n) &= b_l(y_{n+p}, y_n) \\ &\leq s[b_l(y_{n+p}, y_{n+p+1}) + b_l(y_{n+p+1}, y_n)] \end{aligned}$$

$$
\begin{aligned}
&\leq sb_l(y_{n+p},y_{n+p+1})+sb_l(y_{n+p+1},y_n)\\
&\leq sb_l(y_{n+p},y_{n+p+1})+s^2[b_l(y_{n+p+1},y_{n+1})+b_l(y_{n+1},y_n)]\\
&\leq sb_l(y_{n+p},y_{n+p+1})+s^2b_l(y_{n+p+1},y_{n+1})+s^2b_l(y_{n+1},y_n)\\
&\leq sb_l(y_{n+p},y_{n+p+1})+s^2\Big[Mb_l(y_{n+p},y_n)+K\{b_l(y_{n+p+1},y_{n+p})\\
&\quad+b_l(y_{n+1},y_n)\}\Big]+s^2b_l(y_{n+1},y_n)\\
&\leq sb_l(y_{n+p},y_{n+p+1})+s^2Mb_l(y_{n+p},y_n)+s^2Kb_l(y_{n+p+1},y_{n+p})\\
&\quad+s^2Kb_l(y_{n+1},y_n)+s^2b_l(y_{n+1},y_n).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
b_l(y_{n+p},y_n)-s^2Mb_l(y_{n+p},y_n)\leq{}& sb_l(y_{n+p},y_{n+p+1})+s^2Kb_l(y_{n+p+1},y_{n+p})\\
&+s^2Kb_l(y_{n+1},y_n)+s^2b_l(y_{n+1},y_n)
\end{aligned}
$$

and so,

$$(1-s^2M)b_l(y_{n+p},y_n)\leq(s+s^2K)b_l(y_{n+p},y_{n+p+1})+(s^2+s^2K)b_l(y_{n+1},y_n).$$

Then,

$$b_l(y_{n+p},y_n)\leq\frac{(s+s^2K)}{(1-s^2M)}b_l(y_{n+p},y_{n+p+1})+\frac{(s^2+s^2K)}{(1-s^2M)}b_l(y_{n+1},y_n).$$

By f is asymptotic regularity with respect to g, we get that $\lim_{n\longrightarrow\infty}b_l(y_{n+p},y_n)=0$. Since $\lim_{m,n\longrightarrow\infty}b_l(y_m,y_n)=0$ exists and finite, so $\{y_n\}$ is a Cauchy sequence. Since (X,b_l) is a complete b-metric-like space, we have $\{y_n\}\in X$ converges to $z\in X$ so that

$$\lim_{n\longrightarrow\infty}b_l(y_n,z)=b_l(z,z)=\lim_{m,n\longrightarrow\infty}b_l(y_m,y_n)=0.$$

Hence,

$$\lim_{n\longrightarrow\infty}b_l(y_n,z)=\lim_{n\longrightarrow\infty}b_l(fx_n,z)=\lim_{n\longrightarrow\infty}b_l(gx_{n+1},z)=0.$$

Thus,

$$\lim_{n\longrightarrow\infty}fx_n=\lim_{n\longrightarrow\infty}gx_{n+1}=z.$$

Since f and g are (f,g)-orbitally continuous, we get

$$\lim_{n\longrightarrow\infty}ffx_n=\lim_{n\longrightarrow\infty}fgx_{n+1}=fz$$

and

$$\lim_{n\longrightarrow\infty}gfx_n=\lim_{n\longrightarrow\infty}ggx_{n+1}=gz.$$

Since f and g are compatible, we obtain that $\lim_{n\longrightarrow\infty}b_l(fgx_{n+1},gfx_n)=0$.
Thus,

$$fz=\lim_{n\longrightarrow\infty}fgx_{n+1}=\lim_{n\longrightarrow\infty}gfx_n=gz.$$

Hence $fz=gz$ so $z\in C(f,g)$. That is $C(f,g)\neq\emptyset$.
By compatibility of f and g, we have $gfz=fgz=ffz=ggz$ and (f,g)-orbitally continuous of f and g implies that $b_l(fz,fz)=0$.
Using (3.1), we obtain

$$
\begin{aligned}
b_l(fz,ffz)&\leq Mb_l(gz,gfz)+K\{b_l(fz,gz)+b_l(ffz,gfz)\}\\
&=Mb_l(fz,ffz)+K\{b_l(fz,fz)+b_l(ffz,ffz)\}.
\end{aligned}
$$

So,

$$b_l(fz,ffz)\leq Mb_l(fz,ffz).$$

Hence,

$$b_l(fz, ffz) - Mb_l(fz, ffz) \leq 0$$

and then,

$$(1 - M)b_l(fz, ffz) \leq 0.$$

Therefore,

$$b_l(fz, ffz) = 0.$$

Hence $fz = ffz = gfz$. This implies that fz is a common fixed point of f and g.

Suppose that fz and sz are common fixed point of f and g. This implies that $fz = ffz = gfz$ and $sz = fsz = gsz$ .To show that $fz = sz$.
Using (3.1) , we obtain that

$$\begin{aligned} b_l(fz, sz) &= b_l(ffz, fsz) \\ &\leq Mb_l(gfz, gsz) + K\{b_l(ffz, gfz) + b_l(fsz, gsz)\} \\ &\leq Mb_l(ffz, fsz) + K\{b_l(ffz, ffz) + b_l(fsz, fsz)\}. \end{aligned}$$

So,

$$b_l(ffz, fsz) \leq Mb_l(ffz, fsz).$$

Hence,

$$b_l(ffz, fsz) - Mb_l(ffz, fsz) \leq 0$$

and then,

$$(1 - M)b_l(ffz, fsz) \leq 0.$$

Therefore,

$$b_l(ffz, fsz) = 0.$$

Hence $fz = ffz = fsz = sz$. That is f and g have a unique common fixed point. □

Corollary 3.2. *If (X, b_l) is a complete b-metric-like space and $f, g : X \longrightarrow X$. Suppose that f is asymptotically regular with respect to g and there exist $K \in [0, \infty)$ satisfying*

$$b_l(fx, fy) \leq K\{b_l(fx, gx) + b_l(fy, gy)\} \tag{3.2}$$

for all $x, y \in X$. Further, suppose that f and g are (f, g)-orbitally continuous and compatible. Then $C(f, g) \neq \emptyset$ and f and g have a unique common fixed point.

Proof. Since f is asymptotically regular with respect to g at $x_0 \in x$, there exists a sequence $\{y_n\} \in X$ in X such that $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lim_{n \longrightarrow \infty} b_l(gx_{n+1}, gx_{n+2}) = \lim_{n \longrightarrow \infty} b_l(y_n, y_{n+1}) = 0$. To show that $\{y_n\} \in$ is a Cauchy sequence. By (3.2), for any n and any $p > 0$,

$$\begin{aligned} b_l(fx_{n+p}, fx_n) &= b_l(y_{n+p}, y_n) \\ &\leq s[b_l(y_{n+p}, y_{n+p+1}) + b_l(y_{n+p+1}, y_n)] \\ &\leq sb_l(y_{n+p}, y_{n+p+1}) + sb_l(y_{n+p+1}, y_n) \\ &\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2[b_l(y_{n+p+1}, y_{n+1}) + b_l(y_{n+1}, y_n)] \\ &\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2b_l(y_{n+p+1}, y_{n+1}) + s^2b_l(y_{n+1}, y_n) \\ &\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2[K\{b_l(y_{n+p+1}, y_{n+p}) + b_l(y_{n+1}, y_n)\}] \\ &+ s^2b_l(y_{n+1}, y_n) \\ &\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2Kb_l(y_{n+p+1}, y_{n+p}) + s^2Kb_l(y_{n+1}, y_n) \\ &+ s^2b_l(y_{n+1}, y_n) \\ &\leq (s + s^2K)b_l(y_{n+p}, y_{n+p+1}) + (s^2K + s^2)b_l(y_{n+1}, y_n). \end{aligned}$$

Since f is asymptotic regularity with respect to g, we have $\lim_{n\longrightarrow\infty} b_l(y_{n+p}, y_n) = 0$. By $\lim_{m,n\longrightarrow\infty} b_l(y_m, y_n) = 0$ exists and finite, $\{y_n\}$ is a Cauchy sequence. Since (X, b_l) is a complete b-metric-like space, we obtain $\{y_n\} \in X$ converges to $z \in X$ so that

$$\lim_{n\longrightarrow\infty} b_l(y_n, z) = b_l(z, z) = \lim_{m,n\longrightarrow\infty} b_l(y_m, y_n) = 0.$$

Hence,

$$\lim_{n\longrightarrow\infty} b_l(y_n, z) = \lim_{n\longrightarrow\infty} b_l(fx_n, z) = \lim_{n\longrightarrow\infty} b_l(gx_{n+1}, z) = 0.$$

Therefore,

$$\lim_{n\longrightarrow\infty} fx_n = \lim_{n\longrightarrow\infty} gx_{n+1} = z.$$

By f and g are (f, g)-orbitally continuous, we get

$$\lim_{n\longrightarrow\infty} ffx_n = \lim_{n\longrightarrow\infty} fgx_{n+1} = fz$$

and

$$\lim_{n\longrightarrow\infty} gfx_n = \lim_{n\longrightarrow\infty} ggx_{n+1} = gz.$$

Since f and g are compatible, we have $\lim_{n\longrightarrow\infty} b_l(fgx_{n+1}, gfx_n) = 0$.
Thus,

$$fz = \lim_{n\longrightarrow\infty} fgx_{n+1} = \lim_{n\longrightarrow\infty} gfx_n = gz.$$

Hence $fz = gz$ such that $z \in C(f, g)$, so $C(f, g) \neq \emptyset$.
By compatibility of f and g, we have $gfz = fgz = ffz = ggz$ and (f, g)-orbitally continuous of f and g implies that $b_l(fz, fz) = 0$.
Using (3.2), we obtain that

$$\begin{aligned} b_l(fz, ffz) &\leq K\{b_l(fz, gz) + b_l(ffz, gfz)\} \\ &= K\{b_l(fz, fz) + b_l(ffz, ffz)\}. \end{aligned}$$

So,

$$b_l(fz, ffz) \leq 0.$$

Thus,

$$b_l(fz, ffz) = 0.$$

Hence $fz = ffz = gfz$. This implies that fz is a common fixed point of f and g.

Suppose that fz and sz are common fixed point of f and g implies that $fz = ffz = gfz$ and $sz = fsz = gsz$ .To show that $fz = sz$.
Using (3.2) , we have

$$\begin{aligned} b_l(fz, sz) &= b_l(ffz, fsz) \\ &\leq K\{b_l(ffz, gfz) + b_l(fsz, gsz)\} \\ &\leq K\{b_l(ffz, ffz) + b_l(fsz, fsz)\}. \end{aligned}$$

So,

$$b_l(fz, sz) \leq 0.$$

Thus,

$$b_l(fz, sz) = 0.$$

Hence $fz = sz$. This implies that f and g have a unique common fixed point. □

In the next theorem, we relax the condition of orbital continuity for a pair of mappings considered in Theorem 3.1, while also relaxing compatibility by introducing the minimal non-commuting notion, i.e., non-trivial weak compatibility.

Theorem 3.3. *If (X, b_l) is b-metric-like space and f and g be self-mappings on an arbitrary non-empty set Y with values in a b-metric-like space X. Suppose that f is asymptotically regular with respect to g and gY is a complete subset of X for $M, K \in [0,1)$ satisfying*

$$b_l(fx, fy) \leq Mb_l(gx, gy) + K\{b_l(fx, gx) + b_l(fy, gy)\}$$

for all $x, y \in Y$. Then $C(f,g) \neq \emptyset$ Moreover, if $Y = X$, then f and g have a unique common fixed point provided f and g are non-trivially weakly compatible.

Proof. By f is asymptotically regular with respect to g at $x_0 \in x$, there exists a sequence $\{y_n\} \in X$ in X such that $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lim_{n \longrightarrow \infty} b_l(gx_{n+1}, gx_{n+2}) = \lim_{n \longrightarrow \infty} b_l(y_n, y_{n+1}) = 0$. To show that $\{y_n\} \in$ is a Cauchy sequence. Using (3.1), for any n and any $p > 0$,

$$\begin{aligned} b_l(fx_{n+p}, fx_n) &= b_l(y_{n+p}, y_n) \\ &\leq s[b_l(y_{n+p}, y_{n+p+1}) + b_l(y_{n+p+1}, y_n)] \\ &\leq sb_l(y_{n+p}, y_{n+p+1}) + sb_l(y_{n+p+1}, y_n) \\ &\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2[b_l(y_{n+p+1}, y_{n+1}) + b_l(y_{n+1}, y_n)] \\ &\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2 b_l(y_{n+p+1}, y_{n+1}) + s^2 b_l(y_{n+1}, y_n) \\ &\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2 \Big[Mb_l(y_{n+p}, y_n) + K\{b_l(y_{n+p+1}, y_{n+p}) \\ &\quad + b_l(y_{n+1}, y_n)\}\Big] + s^2 b_l(y_{n+1}, y_n) \\ &\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2 Mb_l(y_{n+p}, y_n) + s^2 Kb_l(y_{n+p+1}, y_{n+p}) \\ &\quad + s^2 Kb_l(y_{n+1}, y_n) + s^2 b_l(y_{n+1}, y_n). \end{aligned}$$

Thus,

$$\begin{aligned} b_l(y_{n+p}, y_n) - s^2 Mb_l(y_{n+p}, y_n) \leq{}& sb_l(y_{n+p}, y_{n+p+1}) + s^2 Kb_l(y_{n+p+1}, y_{n+p}) \\ &+ s^2 Kb_l(y_{n+1}, y_n) + s^2 b_l(y_{n+1}, y_n) \end{aligned}$$

and then,

$$(1 - s^2 M)b_l(y_{n+p}, y_n) \leq (s + s^2 K)b_l(y_{n+p}, y_{n+p+1}) + (s^2 + s^2 K)b_l(y_{n+1}, y_n).$$

Hence,

$$b_l(y_{n+p}, y_n) \leq \frac{(s + s^2 K)}{(1 - s^2 M)} b_l(y_{n+p}, y_{n+p+1}) + \frac{(s^2 + s^2 K)}{(1 - s^2 M)} b_l(y_{n+1}, y_n).$$

By f is asymptotic regularity with respect to g, we have $\lim_{n \longrightarrow \infty} b_l(y_{n+p}, y_n) = 0$. Since $\lim_{m,n \longrightarrow \infty} b_l(y_m, y_n) = 0$ exists and finite, so $\{y_n\}$ is a Cauchy sequence in gY. By gY is a complete subset of X and $y_n = fx_n = gx_{n+1}$is a Cauchy sequence in gY, there exists some $z \in X$ such that

$$\lim_{n \longrightarrow \infty} b_l(gx_{n+1}, gz) = b_l(gz, gz) = \lim_{m,n \longrightarrow \infty} b_l(gx_{n+1+m}, gx_{n+1}) = 0.$$

Hence,

$$\lim_{n \longrightarrow \infty} b_l(gx_{n+1}, gz) = \lim_{n \longrightarrow \infty} b_l(fx_n, gz) = 0.$$

Therefore,

$$\lim_{n \longrightarrow \infty} fx_n = \lim_{n \longrightarrow \infty} gx_{n+1} = gz.$$

Using (3.1), we obtain

$$b_l(fx_n, fz) \leq Mb_l(gx_n, gz) + K\{b_l(fx_n, gx_n) + b_l(fz, gz)\} = Kb_l(fz, gz).$$

Thus,

$$b_l(gz, fz) = b_l(fx_n, fz) \leq Kb_l(fz, gz)$$

and so,

$$b_l(gz, fz) - Kb_l(fz, gz) \leq 0.$$

Hence,

$$(1-K)b_l(fz, gz) \leq 0.$$

Therefore,

$$b_l(fz, gz) = 0.$$

Hence $fz = gz$ such that $z \in C(f,g)$. That is $C(f,g) \neq \emptyset$.
Since $Y = X$ and f and g are non-trivially weakly compatible, we have $gfz = fgz$.
Moreover, implies that $gfz = fgz = ffz = ggz$.
Using (3.1), we obtain

$$\begin{aligned} b_l(fz, ffz) &\leq Mb_l(gz, gfz) + K\{b_l(fz, gz) + b_l(ffz, gfz)\} \\ &= Mb_l(fz, ffz) + K\{b_l(fz, fz) + b_l(ffz, ffz)\}. \end{aligned}$$

Thus,

$$b_l(fz, ffz) \leq Mb_l(fz, ffz).$$

Hence,

$$b_l(fz, ffz) - Mb_l(fz, ffz) \leq 0$$

and so,

$$(1-M)b_l(fz, ffz) \leq 0.$$

Therefore,

$$b_l(fz, ffz) = 0.$$

Hence $fz = ffz = gfz$. This implies that fz is a common fixed point of f and g.

Suppose that fz and sz are common fixed point of f and g, we get that $fz = ffz = gfz$ and $sz = fsz = gsz$. To show that $fz = sz$.
Using (3.1) , we have

$$\begin{aligned} b_l(fz, sz) &= b_l(ffz, fsz) \\ &\leq Mb_l(gfz, gsz) + K\{b_l(ffz, gfz) + b_l(fsz, gsz)\} \\ &\leq Mb_l(ffz, fsz) + K\{b_l(ffz, ffz) + b_l(fsz, fsz)\}. \end{aligned}$$

Thus,

$$b_l(ffz, fsz) \leq Mb_l(ffz, fsz).$$

Hence,

$$b_l(ffz, fsz) - Mb_l(ffz, fsz) \leq 0$$

and so,

$$(1-M)b_l(ffz, fsz) \leq 0.$$

Therefore,

$$b_l(ffz, fsz) = 0.$$

Hence $fz = ffz = fsz = sz$. This implies that f and g have a unique common fixed point. □

Corollary 3.4. *If (X, b_l) is b-metric-like space and f and g be self-mappings on an arbitrary non-empty set Y with values in a b-metric-like space X. Suppose that f is asymptotically regular with respect to g and gY is a complete subset of X for $M, K \in [0,1)$ satisfying*

$$b_l(fx, fy) \leq Mmax\left\{b_l(gx, gy), b_l(fx, gx), b_l(fy, gy), \frac{b_l(fx, gy) + b_l(fy, gx)}{2}\right\}$$

for all $x, y \in Y$. Then $C(f, g) \neq \emptyset$ Moreover, if $Y = X$, then f and g have a unique common fixed point provided f and g are non-trivially weakly compatible.

Remark 3.5. Let $K = 0$ in Theorem 3.3, we obtain

$$\begin{aligned} b_l(fx, fy) &\leq Mb_l(gx, gy) + K\{b_l(fx, gx) + b_l(fy, gy) \\ &= Mb_l(gx, gy) \\ &\leq Mmax\left\{b_l(gx, gy), b_l(fx, gx), b(fy, gy), \frac{b_l(fx, gy) + b_l(fy, gx)}{2}\right\} \end{aligned}$$

Hence satisfy the condition in Corollary 3.4.

Corollary 3.6. [11] *If (X, b) is b-metric space and f and g be self-mappings on an arbitrary non-empty set Y ith values in a b-metric space X. Suppose that f is asymptotically regular with respect to g and gY is a complete subset of X for $M, K \in [0,1)$ satisfying*

$$b(fx, fy) \leq Mmax\left\{b(gx, gy), b(fx, gx), b(fy, gy), \frac{b(fx, gy) + b(fy, gx)}{2}\right\}$$

for all $x, y \in Y$. Then $C(f, g) \neq \emptyset$ Moreover, if $Y = X$, then f and g have a unique common fixed point provided f and g are non-trivially weakly compatible.

4. Conclusion

We have introduced a new extension of the concept of a b-metric space, termed a b-metric-like space. Furthermore, we have established Common Fixed Point Theorems for Asymptotic Regularity in b-Metric-Like Spaces.

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