Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 6, No. 1, (2015), 81-88
ISSN : 1906-9605
http://www.math.sci.nu.ac.th

ON THE SYMMETRIC VECTOR QUASI-EQUILIBRIUM PROBLEM VIA NONLINEAR SCALARIZATION METHOD

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ABSTRACT. The aim of this paper, among other things, is, using a nonlinear scalarization function and its properties, to study an existence theorem for a solution of SVQEP in the setting of real topological vector space. One can consider this note as a new version of the reference [5] by replacing a nonlinear scalarization function by a linear functional.

KEYWORDS : Symmetric vector quasi-equilibrium problem; Properly quasi-convex; Acyclic map; Admissible set

1. INTRODUCTION

Let X and Y be real Hausdorff topological vector spaces (for short, t.v.s.), C and D be nonempty subsets of X and Y, respectively. Let Z be a real Hausdorff t.v.s. with its topological dual space Z^*. The pairing between Z and Z^* is denoted by $\langle ., . \rangle$. Let $P \subsetneqq Z$ be a convex cone with $\operatorname{int} P \neq \emptyset$, where $\operatorname{int} P$ denotes the interior of P. Let $S : C \times D \longrightarrow 2^C$ and $T : C \times D \longrightarrow 2^D$ be set-valued mappings and let f, $g : C \times D \longrightarrow Z$ be two vector-valued functions.

In 2003, Fu [8] introduced the symmetric vector quasi-equilibrium problem (for short, SVQEP) that consists in finding $(\bar{x}, \bar{y}) \in C \times D$ such that $\bar{x} \in S(\bar{x}, \bar{y})$, $\bar{y} \in T(\bar{x}, \bar{y})$ and

$$f(x, \bar{y}) - f(\bar{x}, \bar{y}) \notin -\text{int}P, \quad \forall x \in S(\bar{x}, \bar{y}),$$
$$g(\bar{x}, y) - g(\bar{x}, \bar{y}) \notin -\text{int}P, \quad \forall y \in T(\bar{x}, \bar{y}).$$

The SVQEP is a generalization of the (scalar) symmetric quasi-equilibrium problem (for short, SQEP) posed by Noor and Oettli [10] which this problem is a generalization of the equilibrium problem that, at the first, proposed by Blum and Oettli [3]. The equilibrium problem contains as special cases, for instance, optimization problems, problems of Nash equilibria, variational inequalities, and complementarity problems (see [3]).

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Article history : Received May 18, 2014. Accepted May 13, 2015.

The aim of this paper, among other things, is, using a nonlinear scalarization function and its properties, to study an existence theorem for a solution of SVQEP in the setting real of t.v.s. This method for obtaining a solution of SVQEP is different from that which is used by Fu in [8]. Fu?s method is based on the notion of weak minimal points and well-known Kakutani-Fan- Glicksberg Fixed point theorem in locally convex Hausdorff space. Also our method enables us extends some results in [4, 8, 10, 11].

2. DEFINITIONS AND PRELIMINARIES

In the rest of this section we recall some definitions and preliminaries results which we need in the sequel.

In this paper, all topological spaces are assumed to be Hausdorff. As mentioned before, let $P \subsetneqq Z$ be a convex cone with $\operatorname{int} P \neq \emptyset$. We can define a vector ordering in Z by setting

$$x \preceq y \Leftrightarrow y - x \in P,$$

and a weak ordering by setting

$$x \prec y \Leftrightarrow y - x \in \operatorname{int} P.$$

We will denote usual ordering on real numbers by $\leq$.

It is clear that $P \cap \text{-int}\, P = \emptyset$, since $P + \operatorname{int} P \subseteq \operatorname{int} P$ and $P \neq Z$ (this fact will be used in Lemma 3.1).

Let E be a t.v.s. and $C : E \longrightarrow 2^E$ a multi-valued map and for all $x \in E$, $C(x)$ is a solid cone (that is, $intC(x)$ is non empty). Let $e : E \longrightarrow E$ be a map with $e(x) \in C(x)$ for $x \in E$. The non linear scalarization function $\xi : E \times E \longrightarrow R$ is defined as follows:

$$\xi(x, y) = \inf\{r \in R : y \in re(x) - C(x)\}.$$

Definition 2.1 [8]. Let B be a nonempty subset of Z. Element $b \in B$ is called a weak minimal point of B if $B \cap (b - \operatorname{int} P) = \emptyset$. The set of all weak minimal points of B will be denoted by $\min_{w} B$.

Lemma 2.1 [7]. Let B be a nonempty compact subset of Z. Then

(i) $\min_w B \neq \emptyset$,

(ii) $B \subset \min_w B + (\operatorname{int} P \cup \{0\})$.

In the following definition (i)-(iv) is due to Ferro [7] and (v) to Tanaka [12].

Definition 2.2. Let (Z, P) be an ordered topological vector space, and let C be a nonempty convex subset of a vector space X. Let a vector mapping $f : C \longrightarrow Z$ be given.

(i) f is called convex if for every x, $y \in C$ and $t \in [0, 1]$, one has $f(tx + (1-t)y) \preceq tf(x) + (1-t)f(y)$.

(ii) f is called properly quasi-convex if for every $x, y \in C$ and $t \in [0, 1]$, one has either $f(tx + (1-t)y) \preceq f(x)$ or $f(tx + (1-t)y) \preceq f(y)$.

(iii) f is called P-l.s.c. if, for all $z \in Z$, the set $L(z) = \{x \in C : z \not\prec f(x)\}$ is closed in C.

(iv) f is called P-u.s.c. if, for all $z \in Z$, the set $U(z) = \{x \in C : f(x) \nless z\}$ is closed in C.

(v) f is called natural quasi-convex if for every $x, y \in C$ and $t \in [0, 1]$, there exists $\mu \in [0, 1]$ such that $f(tx + (1-t)y) \preceq \mu f(x) + (1-\mu) f(y)$

Also, the function f is said to be natural quasi-concave(respectively, concave, properly quasi-concave) if $-f$ is natural quasi-convex(respectively, convex, properly quasi-convex).

Remark 2.1. Every convex or properly quasi-convex function is natural quasi-convex function (see Lemma 2.1 [14]). A vector mapping may be convex and not properly quasi-convex, and conversely (see [7]). Consequently, the class of natural quasi-convex functions is strictly larger than both the class of convex functions and the class of properly quasi-convex functions. It is easily seen that properly quasi-convexity and quasi-convexity are equivalent to each other in the scalar case, i.e., $Z = \mathbb{R}$ and $P = [0, \infty)$.

Definition 2.3. Let X and Y be two topological spaces. A set-valued mapping $T : X \longrightarrow 2^Y$ is called:

(i) **upper semi-continuous** (u.s.c.) at $x \in X$ if for each open set V containing $T(x)$, there is an open set U containing x such that for each $t \in U$, $T(t) \subseteq V$; T is said to be u.s.c. on X if it is u.s.c. at all $x \in X$.

(ii) **lower semi-continuous** (l.s.c.) at $x \in X$ if for each open set V with $T(x) \cap V \neq \emptyset$, there is an open set U containing x such that for each $t \in U$, $T(t) \cap V \neq \emptyset$; T is said to be l.s.c. on X if it is l.s.c. at all $x \in X$.

(iii) **continuous** on X if it is at the same time u.s.c. and l.s.c. on X.

(iv) **closed** if the graph $G_r(T)$ of T, i.e., $\{(x, y) : x \in X, y \in T(x)\}$, is a closed set in $X \times Y$.

(v) **compact** if the closure of range T, i.e., $\overline{T(X)}$, is compact, where $T(X) = \cup_{x \in X} T(x)$.

Remark 2.2 [13]. T is l.s.c. at $x \in X$ if and only if for any $y \in T(x)$, and any net $\{x_\alpha\}$, $x_\alpha \longrightarrow x$, there is a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ and $y_\alpha \longrightarrow y$.

Definition 2.4. Let X be a topological space, Y be a t.v.s. A function $f : X \longrightarrow Y$ is said to be demicontinuous if

$$f^{-1}(M) = \{x \in X : f(x) \in M\}$$

is closed in X for each closed half space $M \subset Y$.

Lemma 2.2 [14]. Let X be a topological space, Z a t.v.s. and $f : X \longrightarrow Z$ be a demicontinuous function, then for any $x^* \in Z^*$, the composite function $x^* \circ f$ is continuous, where Z^* is the topological dual space of Z.

Definition 2.5 [11]. A nonempty topological space is acyclic if all of its reduced Cech homology groups over rationals vanish. Note that any convex or star-shape subset of a topological vector space is contractible, and that any contractible space is acyclic. A map $T : X \longrightarrow 2^Y$ is said to be acyclic if it is u.s.c. with compact acyclic values .

Definition 2.6 [11]. A nonempty subset X of a t.v.s. E is said to be admissible provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \longrightarrow X$ such that $x - h(x) \in V$, for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E. Note that every nonempty convex subset of a locally convex t.v.s. is admissible (see [9]). Other examples of admissible t.v.s. are l^p and $L^p(0,1)$ for $0 < p < 1$, the space $S(0,1)$ of equivalent class of measurable functions on $[0,1]$, the Hardy spaces H^p for $0 < p < 1$ and certain Orlicz spaces. Ultrabarrelled t.v.s. are also admissible.

We need the following theorem in the sequel.

Theorem 2.1 [11]. Let C and D be admissible convex subsets of t.v.s. X and Y, respectively. Let $S : C \times D \longrightarrow 2^C$ and $T : C \times D \longrightarrow 2^D$ be compact acyclic maps, and $f, g : C \times D \longrightarrow \mathbb{R}$ l.s.c. functions such that (i) The functions

$$F(x,y) = \min\{f(\xi,y) : \xi \in S(x,y)\},$$

$$G(x,y) = \min\{g(x,\eta) : \eta \in T(x,y)\}$$

are u.s.c. on $C \times D$, and

(ii) For each $(x,y) \in C \times D$, the sets

$$A(x,y) = \{\xi \in S(x,y) : f(\xi,y) = F(x,y)\},$$

$$B(x,y) = \{\eta \in T(x,y) : g(x,\eta) = G(x,y)\}$$

are acyclic.

Then there exists an $(\overline{x},\overline{y}) \in C \times D$ such that

$$\overline{x} \in S(\overline{x},\overline{y}), f(x,\overline{y}) \geq f(\overline{x},\overline{y}), \quad \text{for all } x \in S(\overline{x},\overline{y}),$$

$$\overline{y} \in T(\overline{x},\overline{y}), g(\overline{x},y) \geq g(\overline{x},\overline{y}), \quad \text{for all } y \in T(\overline{x},\overline{y}).$$

3. MAIN RESULTS

Throughout this section, let X, Y be real Hausdorff t.v.s., C and D be non empty, admissible convex subsets of X and Y, respectively. Let Z be a real Hausdorff t.v.s. with topological dual space Z^* and $P \subsetneqq Z$ a convex cone with $\operatorname{int} P \neq \emptyset$.

The following Lemma is essential tool for our main results. In the following we establish some important properties of the non linear scalarization function which generalize Propositions 2.3 and 2.4 in [4] from locally convex spaces to topological vector spaces which its proof left to the reader.

Lemma 3.1. Let Z be a t.v.s. and P be a convex cone. Let $e \in intP$ Then the following assertions, for each $r \in R$ and $y \in z$ are satisfied.

(i) $\xi_e(y) = \inf\{r \in R : y \in re - P = \min\{r \in R : y \in re - P\};$
(ii) $\xi_e(y) \leq r \Leftrightarrow y \in re - P$
(iii) $\xi_e(y) < r \Leftrightarrow y \in re - intP$
(iv) If $y_1 \preceq y_2$, then $\xi_e(y_1) \leq \xi_e(y_2);$

(v) The function $y \longrightarrow \xi_e(y)$ is continuous, positively homogeneous and sub additive on Z;

(vi) The function $y \longrightarrow \xi_e(y)$ is bounded on some neighborhood of zero.

Now, we are ready to prove existence theorems that extends the main result in [6], Theorems 1,2 and 3 in [8], and also is a generalization of the Theorem 1.1.

Theorem 3.1. Assume that

(i) $S : C \times D \longrightarrow 2^C$ and $T : C \times D \longrightarrow 2^D$ are continuous and compact; and for each $(x, y) \in C \times D,\ S(x, y),\ T(x, y)$ are nonempty, closed convex subsets;

(ii) $f, g : C \times D \to Z$ are demicontinuous;

(iii) For any fixed $y \in D,\ f(x, y)$ is natural quasi-convex in x; for any fixed $x \in C,\ g(x, y)$ is natural quasi-convex in y.

Then SVQEP has a solution.

Proof. By (ii) and Lemma 3.1 through theorem 2.2 in [6], the composite functions $\xi_e of$ and $\xi_e og$ are l.s.c. We claim that the real-valued continuous functions $\xi_e of$ and $\xi_e og$ satisfy in conditions (i) and (ii) of Theorem 2.1. Indeed, condition (i) follows from Theorem 1 in [1, p. 122].

Now for condition (ii), we must show that for any fixed $(x, y) \in C \times D$ the set A(x, y) is convex, where

$$\begin{aligned} A(x, y) &= \{u \in S(x, y) : \xi_e \circ f(u, y) = F(x, y)\} \\ F(x, y) &= \min\{\xi_e \circ f(u, y) : u \in S(x, y)\}. \end{aligned}$$

To this end, let $t \in]0, 1[$ and $u_1, u_2 \in A(x, y)$. By the definition of $A(x, y)$, $u_1, u_2 \in A(x, y)$ and convexity of the set $S(x, y)$, we get $(1 - t)u_1 + tu_2 \in S(x, y)$ and $F(x, y) = \xi_e \circ f(u_1, y) = \xi_e \circ f(u_2, y)$. Hence by (iii) there exists $\mu \in]0, 1[$ such that

$$\begin{aligned} F(x, y) &\leq \xi_e \circ f((1 - t)u_1 + tu_2, y) \\ &\leq (1 - \mu)\xi_e \circ f(u_1, y) + \mu\xi_e \circ f(u_2, y) \\ &= (1 - \mu)F(x, y) + \mu F(x, y) \\ &= F(x, y). \end{aligned}$$

In the above, the first inequality holds by the definition of $F(x, y)$ and $(1 - t)u_1 + tu_2 \in S(x, y)$, but the second inequality holds by natural quasi-convexity of the function f in the first argument (assumption (iii)) and to preserve ordering on Z by ξ_e (see, Lemma 3.1 (iv,v)). Then, $(1 - t)u_1 + tu_2 \in A(x, y)$. Similarly $\xi_e \circ g$ satisfies in conditions (i) and (ii) of Theorem 2.1. Now, by virtue of Theorem 2.1, there exists $(\overline{x}, \overline{y}) \in C \times D$ such that

$$\begin{aligned} &\overline{x} \in S(\overline{x}, \overline{y}), \xi_e \circ f(x, \overline{y}) \geq \xi_e \circ f(\overline{x}, \overline{y}),\ \forall x \in S(\overline{x}, \overline{y}), \\ &\overline{y} \in T(\overline{x}, \overline{y}), \xi_e \circ g(\overline{x}, y) \geq \xi_e \circ g(\overline{x}, \overline{y}),\ \forall y \in T(\overline{x}, \overline{y}). \end{aligned}$$

Then by Lemma 3.1 (v),

$$\begin{aligned} &\overline{x} \in S(\overline{x}, \overline{y}), \xi_e(f(x, \overline{y}) - f(\overline{x}, \overline{y})) \geq \xi_e(f(x, \overline{y})) - \xi_e(f(\overline{x}, \overline{y})) \geq 0, \forall x \in S(\overline{x}, \overline{y}), \\ &\overline{y} \in T(\overline{x}, \overline{y}), \xi_e(g(\overline{x}, y) - g(\overline{x}, \overline{y})) \geq \xi_e(g(\overline{x}, y)) - \xi_e(g(\overline{x}, \overline{y})) \geq 0, \forall y \in T(\overline{x}, \overline{y}). \end{aligned}$$

Consequently, it follows from Lemma 3.1 (iii) and the relations (1) and (2) that

$$\overline{x} \in S(\overline{x}, \overline{y}), f(x, \overline{y}) - f(\overline{x}, \overline{y}) \notin -intP,\ \forall x \in S(\overline{x}, \overline{y})$$

and

$$\overline{y} \in T(\overline{x}, \overline{y}), g(\overline{x}, y) - g(\overline{x}, \overline{y}) \notin -intP, \ \forall y \in T(\overline{x}, \overline{y}),$$

and so $(\overline{x}, \overline{y})$ is a solution of the SVQEP. This completes the proof. □

The following corollary is one of the applications Theorem 3.1. which extends the existence Theorem 3.1 in [14] from locally convex topological vector spaces to topological vector space.

Corollary 3.1. Let C and D be nonempty compact admissible convex sets, and let the vector-valued function $f : C \times D \longrightarrow Z$ satisfy the following conditions

(i) The function f is demicontinuous;

(ii) For any fixed $y \in D$, $f(x, y)$ is natural quasi-convex in x; for any fixed $x \in C$, $f(x, y)$ is natural quasi-concave in y.

Then the vector-valued function f has at least one P-weak saddle point, that is, there exists $(\bar{x}, \bar{y}) \in C \times D$ such that

$$f(\bar{x}, \bar{y}) - f(x, \bar{y}) \notin \text{int } P \ \forall x \in C$$
$$f(\bar{x}, y) - f(\bar{x}, \bar{y}) \notin \text{int } P \ \forall x \in D.$$

Proof. It is enough in Theorem 3.1, we define the set-valued mappings $S : C \times D \longrightarrow 2^C$ and $T : C \times D \longrightarrow 2^D$ as $S(x, y) = C$, $T(x, y) = D$, and also the vector-valued function g on $C \times D$ as $g(x, y) = -f(x, y)$. □

By using Theorem 2.1 and Lemma 3.1 we can state the following theorem which is another version of Theorem 3.1 without continuity condition of the maps.

Theorem 3.2. Let $S : C \times D \longrightarrow 2^C$ and $T : C \times D \longrightarrow 2^D$ be compact acyclic maps. Suppose that $f, g : C \times D \longrightarrow Z$ and $\xi_e \in S_{-\text{int } P, P}$, be such that

(i) The composite functions $\xi_e \circ f$, $\xi_e \circ g$ are l.s.c.,

(ii) The functions

$$\begin{aligned} F(x, y) &= \min\{\xi_e(f(\xi, y)) \ : \ \xi \in S(x, y)\}, \\ G(x, y) &= \min\{\xi_e(g(x, \eta)) \ : \ \eta \in T(x, y)\} \end{aligned}$$

are u.s.c. on $C \times D$,

(iii) For each $(x, y) \in C \times D$, the sets

$$A(x, y) = \{u \in S(x, y) : \ \xi_e(f(u, y)) = F(x, y)\},$$
$$B(x, y) = \{\eta \in T(x, y) : \ \xi_e(g(x, \eta)) = G(x, y)\}$$

are acyclic.

Then SVQEP has a solution.

Remark 3.1. Let us briefly discuss assumptions (i),(iii) and convexity $C \times D$. The lower semicontinuity of $\xi_e \circ f$ and $\xi_e \circ g : C \times D \longrightarrow Z$ is ensured whenever f and g are P-l.s.c. This follows from Lemma 2.4 in [2]. We can omit compactness condition of the sets C and D in Theorem 1 in [10], by using Himmelberg's Fixed point theorem [11] instead of Berge's maximum Theorem in its proof. Then by using this form of the Theorem 1 in [10] and the property of $\xi_e \in S_{-\text{int } P, P}$, we can omit condition (iii) in Theorem 3.2, if C and D be nonempty convex subsets of real locally convex Hausdorff spaces X and Y, respectively, and S, T be u.s.c. and compact maps with nonempty closed convex values. At last convexity of $C \times D$ is not essential. In fact, $C \times D$ can be any subset of $X \times Y$ which is homomorphic to an admissible convex subset in t.v.s. $X_1 \times Y_1$ (see discussion after Theorem 1 in [11]).

The following examples show that Theorem 3.2 is sharper than Theorem 3.1.

Example 3.1. Let $C = [-1, 1],\ D = [0, 1]$. Define $T : C \times D \longrightarrow 2^D$ by $T(x, y) = [0, 1],\ S : C \times D \longrightarrow 2^C$ by

$$S(x,y) = \begin{cases} \{0\} & \text{if } x \neq 0 \\ [0,1] & \text{if } x = 0, \end{cases}$$

and $f, g : C \times D \longrightarrow \mathbb{R}$ by

$$g(x,y) = x + y, \quad f(x,y) = \begin{cases} 0 & \text{if } x \in \{\frac{-1}{n} : n \in N\} \cup \{0\} \\ 1 & \text{otherwise .} \end{cases}$$

The maps S and T are acyclic. The function f is not quasi-convex but l.s.c. and the function g is convex and continuous such that

$$F(x,y) = \min\{f(\xi, y) : \xi \in S(x,y)\} = 0, \quad \text{for all } (x,y) \in C \times D,$$

$$G(x,y) = \min\{g(x,\eta) = x + \eta : \eta \in T(x,y)\} = x, \text{ for all } (x,y) \in C \times D$$

are continuous and convex. It is clear that,

$$A(x,y) = \{\xi \in S(x,y) : f(\xi,y) = F(x,y)\} = \{0\}, \text{ for all } (x,y) \in C \times D$$

$$B(x,y) = \{\eta \in T(x,y) : g(x,\eta) = G(x,y)\} = \{0\}, \text{ for all } (x,y) \in C \times D$$

are acyclic (sets) for every $(x, y) \in C \times D$. Therefore, SVQEP has a solution by Theorem 3.2. But the example does not satisfy in the conditions of Theorem 3.1.

Acknowledgment. The author would like to thank of Islamic Azad University, Kermanshah Branch, Kermanshah, Iran, for partial supporting of this research.

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