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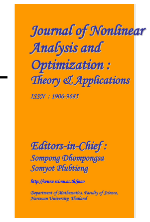

SOME GENERALIZED TRIPLE SEQUENCE SPACES OF REAL NUMBERS

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ABSTRACT. The idea of difference sequence spaces was first introduced by Kizmaz in 1981 and the idea of triple sequences was first introduced by Sahiner et.al. 2007. In this article we introduce the notion of triple sequence spaces $c_0^3(\Delta)$, $c^3(\Delta)$, and $l_\infty^3(\Delta)$ using the difference operator Δ. We study some of their algebraic and topological properties and prove some inclusion results.

KEYWORDS : Difference operator, triple sequence space, solidity.

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1. Introduction and Preliminaries

A triple sequence (real or complex) can be defined as a function $x : N \times N \times N \longrightarrow R(C)$, where N, R and C denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequences was introduced and investigated at the initial stage by Sahiner, et. al. [1, 2] and Dutta, et. al. [3] and others..

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [6] as follows:

$Z(\Delta) = \{(x_n) \in w : (\Delta x_n) \in Z\}$, for $Z = c, c_0, l_\infty$, the spaces of convergent, null and bounded sequences, respectively, where $\Delta x_n = x_n - x_{n+1}$ for all $n \in N$. Later on it was further investigated by Tripathy [4] and many others. Tripathy and Sarma[5] introduced difference double sequence spaces as follows:

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$Z(\Delta) = \{(x_{mn}) \in w : (\Delta x_{mn}) \in Z\}$, for $Z = c^2, c_0^2, l_\infty^2$, the spaces of convergent, null and bounded double sequences respectively, where $\Delta x_{mn} = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbf{N}$.

Definition 1.1 [1]: A triple sequence (x_{lmn}) is said to be convergent to L in Pringsheim′s sense if for every $\epsilon > 0$, there exists $\mathbf{N}(\epsilon) \in N$ such that

$|x_{lmn} - L| < \epsilon$ whenever $l \geq \mathbf{N}$, $m \geq \mathbf{N}$, $n \geq \mathbf{N}$ and we write $lim_{l,m,n \longrightarrow \infty} x_{lmn} = L$.

Note: A triple sequence is convergent in Pringsheim′s sense may not be bounded [2].

Definition 1.2 [1]: A triple sequence (x_{lmn}) is said to be Cauchy sequence if for every $\epsilon > 0$, there exists $\mathbf{N}(\epsilon) \in N$ such that

$|x_{lmn} - x_{pqr}| < \epsilon$ whenever $l \geq p \geq \mathbf{N}$, $m \geq q \geq \mathbf{N}$, $n \geq r \geq \mathbf{N}$.

Definition 1.3 [1]: A triple sequence (x_{lmn}) is said to be bounded if there exists $M > 0$, such that $|x_{lmn}| < M$ for all $l, m, n \in N$.

Definition 1.4 [3]: A triple sequence (x_{lmn}) is said to be converge regularly if it is convergent in Pringsheim′s sense and in addition the following limits holds:

$lim_{n \longrightarrow \infty} x_{lmn} = L_{lm} \ (l, m \in N)$

$lim_{m \longrightarrow \infty} x_{lmn} = L_{ln} \ (l, n \in N)$

$lim_{l \longrightarrow \infty} x_{lmn} = L_{mn} \ (m, n \in N)$

Let w^3 denote the set of all triple sequence of real numbers. We can define the class of triple sequences as follows:

$c_0^3 = \{x = (x_{lmn}) \in w^3 : (x_{lmn})$ is convergent to zero in Pringsheim′s sense $\}$

$c^3 = \{x = (x_{lmn}) \in w^3 : (x_{lmn})$ is convergent in Pringsheim ′s sense $\}$

$l_\infty^3 = \{x = (x_{lmn}) \in w^3 : (x_{lmn})$ is bounded in Pringsheim′s sense $\}$

$c^{3R} = \{x = (x_{lmn}) \in w^3 : (x_{lmn})$ is regularly convergent $\}$

$c^{3B} = \{x = (x_{lmn}) \in w^3 : (x_{lmn})$ is convergent in Pringsheim ′s sense and bounded $\}$

These classes are all linear spaces.

It is obvious that $c_0^3 \subset c^3$; $c^{3R} \subset c^{3B} \subset l_\infty^3$ and the inclusion is strict.

Theorem 1.1: The spaces c_0^3, c^3, l_∞^3, c^{3R} and c^{3B} are complete normed linear spaces with the normed.

$\|x\| = sup_{l,m,n}|x_{lmn}| < \infty$

Proof: simple.

Example 1.1 [1]: Let $x_{lmn} = \begin{cases} mn, & l = 3 \\ nl, & m = 5 \\ lm, & n = 7 \\ 8, & otherwise \end{cases}$

Then $(x_{lmn}) \to 8$ in Pringsheim$'$s sense but not bounded as well as not regularly convergent.

Example 1.2: Let $x_{lmn} = 1$, for all $l, m, n \in N$. Then (x_{lmn}) is convergent in Pringsheim$'$s sense, bounded and regularly convergent.

Definition 1.5[3]: A triple sequence space E is said to be solid if $(\alpha_{lmn} x_{lmn}) \in E$ whenever $(x_{lmn}) \in E$ and for all sequences (α_{lmn}) of scalars with $|\alpha_{lmn}| \leq 1$, for all $l, m, n \in \mathbf{N}$.

Definition 1.6 [3]: A triple sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.

Remark 1.1 [3]: A sequence space is solid implies that it is monotone.

Definition 1.7 [3]: A triple sequence space E is said to be convergence free if $(y_{lmn}) \in E$, whenever $(x_{lmn}) \in E$ and $x_{lmn} = 0$ implies $y_{lmn} = 0$.

Definition 1.8 [3]: A triple sequence space E is said to be symmetric if $(x_{lmn}) \in E$ implies $(x_{\pi(l)\pi(m)\pi(n)}) \in E$, where π is a permutation of $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$.

Now we introduced the difference triple sequence spaces as follows:

$c_0^3(\Delta)$={$(x_{lmn}) \in w^3 : (\Delta x_{lmn})$ is regularly null }

$c^3(\Delta)$={$(x_{lmn}) \in w^3 : (\Delta x_{lmn})$ is convergent in Pringsheim's sense }

$c^{3R}(\Delta)$={$(x_{lmn}) \in w^3 : (\Delta x_{lmn})$ is regularly convergent }

$l_\infty^3(\Delta)$={$(x_{lmn}) \in w^3 : (\Delta x_{lmn})$ is bounded }

$c^{3B}(\Delta)$={$(x_{lmn}) \in w^3 : (\Delta x_{lmn})$ is convergent in Pringsheim's sense and bounded }

Where $\Delta x_{lmn} = x_{lmn} - x_{lmn+1} - x_{lm+1n} + x_{lm+1n+1} - x_{l+1mn} + x_{l+1mn+1} + x_{l+1m+1n} - x_{l+1m+1n+1}$

2. MAIN RESULTS

Theorem 2.1: The classes of sequences $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$, $l_\infty^3(\Delta)$, $c^{3B}(\Delta)$ are linear spaces.

Proof: Obvious.

Theorem 2.2: The classes of sequences $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$, $l_\infty^3(\Delta)$, $c^{3B}(\Delta)$ are complete normed linear spaces with the norm

$$\|x\| = sup_l|x_{l11}| + sup_m|x_{1m1}| + sup_n|x_{11n}| + sup_{l,m,n}|\Delta x_{lmn}| < \infty$$

Proof: Let (x^i) be a Cauchy sequence in $l_\infty^3(\Delta)$, where $x^i = (x^i_{lmn}) \in l_\infty^3(\Delta)$ for each $i \in \mathbf{N}$.
Then we have,
$\|x^i - x^j\| = sup_l|x^i_{l11} - x^j_{l11}| + sup_m|x^i_{1m1} - x^j_{1m1}| + sup_n|x^i_{11n} - x^j_{11n}| + sup_{l,m,n}|\Delta x^i_{lmn} - \Delta x^j_{lmn}| \to 0$

as $i, j \to \infty$

Therefore, $|x^i_{lmn} - x^j_{lmn}| \to 0$, for $i, j \to \infty$ and each $l, m, n \in \mathbf{N}$

Hence (x^i_{lmn})=$(x^1_{lmn}, x^2_{lmn}, x^3_{lmn}, \dots \dots \dots \dots \dots \dots)$ is a Cauchy sequence in R (Real numbers).

Whence by the completeness of R, it converges to x_{lmn} say, i.e., there exists

$lim\ x^i_{lmn} = x_{lmn}$ for each $l, m, n \in \mathbf{N}$

Further for each $\epsilon > 0$, there exists $\mathbf{N} = \mathbf{N}(\epsilon)$, such that for all $i, j \geq \mathbf{N}$, and for all $l, m, n \in \mathbf{N}$

$|x^i_{l11} - x^j_{l11}| < \epsilon$, $|x^i_{1m1} - x^j_{1m1}| < \epsilon$, $|x^i_{11n} - x^j_{11n}| < \epsilon$

$|\Delta x^i_{lmn} - \Delta x^j_{lmn}| = |(x^i_{l+1m+1n+1} - x^j_{l+1m+1n+1}) - (x^i_{l+1m+1n} - x^j_{l+1m+1n}) - (x^i_{l+1mn+1} - x^j_{l+1mn+1}) + (x^i_{l+1mn} - x^j_{l+1mn}) - (x^i_{lm+1n+1} - x^j_{lm+1n+1}) + (x^i_{lm+1n} - x^j_{lm+1n}) + (x^i_{lmn+1} - x^j_{lmn+1}) - (x^i_{lmn} - x^j_{lmn})| < \epsilon$

and

$lim_j\ |x^i_{l11} - x^j_{l11}| = |x^i_{l11} - x_{l11}| \leq \epsilon$,

$lim_j\ |x^i_{1m1} - x^j_{1m1}| = |x^i_{1m1} - x_{1m1}| \leq \epsilon$,

$lim_j\ |x^i_{11n} - x^j_{11n}| = |x^i_{11n} - x_{11n}| \leq \epsilon$,

Now

$lim_j|\Delta x^i_{lmn} - \Delta x^j_{lmn}|$

$= |(x^i_{l+1m+1n+1} - x^j_{l+1m+1n+1}) - (x^i_{l+1m+1n} - x^j_{l+1m+1n}) - (x^i_{l+1mn+1} - x^j_{l+1mn+1}) + (x^i_{l+1mn} - x^j_{l+1mn}) - (x^i_{lm+1n+1} - x^j_{lm+1n+1}) + (x^i_{lm+1n} - x^j_{lm+1n}) + (x^i_{lmn+1} - x^j_{lmn+1}) - (x^i_{lmn} - x^j_{lmn})|$

$= |(x^i_{l+1m+1n+1} - x_{l+1m+1n+1}) - (x^i_{l+1m+1n} - x_{l+1m+1n}) - (x^i_{l+1mn+1} - x_{l+1mn+1}) + (x^i_{l+1mn} - x_{l+1mn}) - (x^i_{lm+1n+1} - x_{lm+1n+1}) + (x^i_{lm+1n} - x_{lm+1n}) + (x^i_{lmn+1} - x_{lmn+1}) - (x^i_{lmn} - x_{lmn})| \leq \epsilon$
for all $i \geq \mathbf{N}$

Since ϵ is not dependent on l, m, n

$sup_{l,m,n}$ $|(x^i_{l+1m+1n+1} - x_{l+1m+1n+1}) - (x^i_{l+1m+1n} - x_{l+1m+1n}) - (x^i_{l+1mn+1} - x_{l+1mn+1}) + (x^i_{l+1mn} - x_{l+1mn}) - (x^i_{lm+1n+1} - x_{lm+1n+1}) + (x^i_{lm+1n} - x_{lm+1n}) + (x^i_{lmn+1} - x_{lmn+1}) - (x^i_{lmn} - x_{lmn})| \leq \epsilon,$

Consequently we have, $\|x^i_{lmn} - x_{lmn}\| \leq 4\epsilon$, for all $i \geq \mathbf{N}$

Hence we obtain $x^i_{lmn} \to x_{lmn}$ as $i \to \infty$ in $l^3_\infty(\Delta)$

Now we have to show that $(x_{lmn}) \in l^3_\infty(\Delta)$

$|x_{lmn} - x_{l+1m+1n+1}| = |x_{lmn} - x^N_{lmn} + x^N_{lmn} - x^N_{l+1m+1n+1} + x^N_{l+1m+1n+1} - x_{l+1m+1n+1}|$

$\leq |x^N_{lmn} - x_{l+1m+1n+1}| + \|x^N_{lmn} - x_{lmn}\| = O(1)$

This implies x = $(x_{lmn}) \in l^3_\infty(\Delta)$, (Since $l^3_\infty(\Delta)$ is a linear space.)

Hence $l^3_\infty(\Delta)$ is complete.

Similarly the others.

Theorem 2.3:

(i) $c^3_0(\Delta) \subset c^3(\Delta)$ and the inclusion is strict. .

(ii) $c^{3R}(\Delta) \subset c^3(\Delta)$ and the inclusion is strict.

Proof: The inclusion being strict follows from the following example:

Example 2.1: For theorem (i)we consider the sequence (x_{lmn}) defined by

(x_{lmn})= $-lmn$, for all $l, m, n \in \mathbf{N}$

Then $(\Delta x_{lmn}) \in c^3$, but $(\Delta x_{lmn}) \notin c^3_0$

Hence the inclusion is strict.

Example 2.2: For theorem (ii) we consider the sequence defined by

$$x_{lmn} = \begin{cases} (-1)^n lmn, & \text{for } l = 1, m = 1, 2, 3 \text{ for all } n \in \mathbf{N} \\ 1, & \text{otherwise} \end{cases}$$

Clearly $(\Delta x_{lmn}) \in c^3$, but the sequence $(\Delta x_{lmn}) \notin c^{3R}$

Hence the inclusion $c^{3R}(\Delta) \subset c^3(\Delta)$, is strict.

Theorem 2.4: The classes of sequences $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$, $l_\infty^3(\Delta)$ and $c^{3B}(\Delta)$ are not solid in general.

Proof: This is clear from the following examples:

Example 2.3: We consider the sequence (x_{lmn}) defined by

(x_{lmn})= 2, for all $l, m, n \in \mathbf{N}$

Clearly the difference triple sequence $(\Delta x_{lmn}) \in c_0^3$, c^3, c^{3R} and c^{3B}

Consider the sequence of scalars defined by

α_{lmn} = $(-1)^{l+m+n}$, for all $l, m, n \in \mathbf{N}$

Then the sequence $(\alpha_{lmn} x_{lmn})$ takes the following form

$\alpha_{lmn} x_{lmn}$ = $2.(-1)^{l+m+n}$, for all $l, m, n \in \mathbf{N}$

Clearly $(\Delta \alpha_{lmn} x_{lmn}) \notin c_0^3$, c^3, c^{3R} and c^{3B}

Hence $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$ and $c^{3B}(\Delta)$ are not solid.

Example 2.4: We consider the sequence (x_{lmn}) defined by

(x_{lmn})= lmn, for all $l, m, n \in \mathbf{N}$

Clearly the sequence $(\Delta x_{lmn}) \in l_\infty^3$

Consider the sequence of scalars defined by

α_{lmn} = $(-1)^{m+n}$, for all $l, m, n \in \mathbf{N}$

Then the sequence $(\alpha_{lmn} x_{lmn})$ takes the following form

$\alpha_{lmn} x_{lmn}$ = $(-1)^{m+n} lmn$, for all $l, m, n \in \mathbf{N}$

Clearly, $(\Delta \alpha_{lmn} x_{lmn}) \notin l_\infty^3$,

Hence $l_\infty^3(\Delta)$ are not solid.

Theorem 2.5: The spaces $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$, $l_\infty^3(\Delta)$ and $c^{3B}(\Delta)$ are not symmetric in general.

Proof: The proof is clear from the following examples:

Example 2.5: Consider the sequence (x_{lmn}) defined by

(x_{lmn})= m, for all $l, m, n \in \mathbf{N}$

Clear the sequence $(\Delta x_{lmn}) \in c_0^3$, c^3, c^{3R} and c^{3B}

Now Consider a rearrange sequence (y_{lmn}) of (x_{lmn}) defined by

$$y_{lmn} = \begin{cases} m+1 \text{ , for } m = l, n \text{ is even} \\ m-1 \text{ , for } m = l+1, n \text{ is even} \\ m \text{ , otherwise} \end{cases}$$

Clearly $(\Delta y_{lmn}) \notin c_0^3$, c^3, c^{3R} and c^{3B}

Hence $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$ and $c^{3B}(\Delta)$ are not symmetric.

Example 2.6: Consider the sequence (x_{lmn}) defined by

(x_{lmn})= lmn, for all $l, m, n \in \mathbf{N}$

Clear the sequence $(\Delta x_{lmn}) \in l_\infty^3$

Now Consider a rearrange sequence (y_{lmn}) of (x_{lmn}) defined by

$$y_{lmn} = \begin{cases} m+1 \text{ , for } m = l, n \text{ is even} \\ m-1 \text{ , for } m = l+1, n \text{ is even} \\ m \text{ , otherwise} \end{cases}$$

Then the sequence $(\Delta y_{lmn}) \notin l_\infty^3$

Hence $l_\infty^3(\Delta)$ are not symmetric.

Theorem 2.6: The classes of sequences $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$, $l_\infty^3(\Delta)$ and $c^{3B}(\Delta)$ are not convergence free in general.

Proof: We provide an example to prove the result.

Example 2.7: Consider the sequence defined by

$$x_{lmn} = \begin{cases} 0, & \text{if } n = 1, \text{ for all } l, m \in \mathbf{N} \\ -2, & \text{otherwise} \end{cases}$$

Clearly the triple sequence $(\Delta x_{lmn}) \in c_0^3$, c^3, c^{3R}, l_∞^3 and c^{3B}

Let the sequence (y_{lmn}) be defined by

$$y_{lmn} = \begin{cases} 0, & \text{if } n \text{ is odd, for all } l, m \in \mathbf{N} \\ lmn, & \text{otherwise} \end{cases}$$

Clearly $(\Delta y_{lmn}) \notin c_0^3$, c^3, c^{3R}, l_∞^3 and c^{3B}, Hence $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$, $l_\infty^3(\Delta)$ and $c^{3B}(\Delta)$, are not convergence free.

Theorem 2.7: The classes of sequences $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$, $l_\infty^3(\Delta)$ and $c^{3B}(\Delta)$ are all sequence algebra.

Proof: It is obvious.

Conclusion: We have introduced the notions of null, convergent and bounded triple sequence spaces based on the difference operator Δ and have investigated its different properties, which are the generalizations of null, convergent and bounded triple sequence spaces. Further generalizations may be possible based on the difference operator Δ^m.

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