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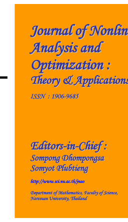

THE SPLIT EQUALITY FIXED POINT PROBLEM FOR DEMI-CONTRACTIVE MAPPINGS

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ABSTRACT. Motivated by the recent work of Moudafi (*Inverse Problems, 26 (2010), 587-600*) and inspired by Xu (*Inverse Problems, 22 (2006), 2021-2034*), Censor and Segal (*J. Convex Anal. 16 (2009), 587-600*) , and Yang (*Inverse Problems, 20 (2004), 1261-1266*), we investigate a Krasnoselskii-type iterative algorithm for solving the split equality fixed point problem recently introduced by Moudafi and Al-Shemas (*Transactions on Mathematical Programming and Applications, Vol. 1, No. 2 (2013), 1-11*). Weak and strong convergence theorems are proved for the class of demi-contractive mappings in Hilbert spaces. Our theorems extend and complement some recent results of Moudafi and a host of other recent important results.

KEYWORDS :Split equality fixed point problem, Uniform Continuity, Demicontractive mappings, iterative scheme, Fixed point.
AMS Subject Classification: 47J25, 47H06, 49J53, 90C25.

1. INTRODUCTION

The split feasibility problem arises in many areas of application such as phase retrieval, medical image reconstruction, image restoration, computer temography and radiation therapy treatment planning (see e.g., Byrne [1], Censor *et al.*[2], Censor *et al.* [3], and Censor and Elfving [4]). It takes the following form: Let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (SFP) is formulated as follows:

$$Find \;\; x^* \in C \;\; such \; that \;\; Ax^* \in Q. \tag{1.1}$$

The SFP was first introduced in 1994 by Censor and Elfving [4] in finite-dimensional Hilbert spaces for modelling inverse problems arising from phase retrieval and medical image reconstruction.

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Assuming that the SFP (1.1) has a solution, one can easily show that $x^* \in C$ solves SFP if and only if it solves the fixed point equation

$$x^* = P_C(I - \gamma A^*(I - P_Q)A)x^*,$$

where P_C and P_Q are the metric projections from H_1 onto C and from H_2 onto Q, respectively, where γ is a positive constant and A^* denotes the adjoint of A.

A popular algorithm used in approximating the solution of the SFP (1.1) is the CQ-algorithm of Byrne [1]:

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n,$$

for each $n \geq 1$, where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A.
Based on the work of Censor and Segal [5], Moudafi [10] proposed the following scheme which does not involve the metric projections P_C and P_Q:

$$x_{n+1} = (1-\alpha_n)\Big(x_n + \gamma A^*(T-I)Ax_n\Big) + \alpha_n U\Big(x_n + \gamma A^*(T-I)Ax_n\Big), \quad n \in \mathbb{N},$$

for approximating a solution of the split feasibility fixed point problem (1.1) and obtained a weak convergence results when U and T are *demi-contractive*.

Very recently, Moudafi and Al-Shemas [9] introduced the following *split equality fixed point problem* as a generalization of the split feasibility problem (1.1):

$$Find \ \ x \in C := F(U) \ \ and \ \ y \in Q := F(T) \ such \ that \ \ Ax = By, \tag{1.2}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators, $U : H_1 \to H_1, \ \ T : H_2 \to H_2$, $F(U)$ and $F(T)$ denote the fixed point sets of U and T, respectively. Note that problem (1.2) reduces to problem (1.1) if $H_2 = H_3$ and $B = I$ (where I is the identity map on H_2) in (1.2).

In order to approximate a solution of problem (1.2), Moudafi and Al-Shemas [9] introduced the following iterative scheme:

$$\begin{cases} x_{n+1} = U(x_n - \gamma_n A^*(Ax_n - By_n)); \\ \\ y_{n+1} = T(y_n + \gamma_n B^*(Ax_n - By_n)), \quad \forall n \geq 1, \end{cases} \tag{1.3}$$

where $U : H_1 \to H_1, \ \ T : H_2 \to H_2$ are two *firmly quasi-nonexpansive mappings*, $A : H_1 \to H_3, \ B : H_2 \to H_3$ are two bounded linear operators, A^* and B^* are the adjoints of A and B, respectively, $\{\gamma_n\} \subset (\epsilon, \frac{2}{\lambda_{A^*A}+\lambda_{B^*B}} - \epsilon)$, λ_{A^*A} and λ_{B^*B} denote the spectral radii of A^*A and B^*B, respectively. Using the iterative scheme (1.3), Moudafi obtained a *weak convergence* result for problem (1.2).

Yuan-Fang $et\ al.$ [15] introduced the following algorithm for solving problem (1.2):

$$\begin{cases} \forall x_1 \in H_1, \quad \forall y_1 \in H_2; \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n U(x_n - \gamma_n A^*(Ax_n - By_n)); \\ \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n T(y_n + \gamma_n B^*(Ax_n - By_n)), \quad \forall n \geq 1, \end{cases} \tag{1.4}$$

where $U : H_1 \to H_1, \ \ T : H_2 \to H_2$ are two *firmly quasi-nonexpansive mappings*, $A : H_1 \to H_3, \ B : H_2 \to H_3$ are two bounded linear operators, A^* and B^* are the adjoints of A and B, respectively, $\{\gamma_n\} \subset (\epsilon, \frac{2}{\lambda_{A^*A}+\lambda_{B^*B}} - \epsilon)$ (for ϵ small enough), where λ_{A^*A} and λ_{B^*B} denote the spectral radii of A^*A and B^*B, respectively and $\{\alpha_n\} \subset [\alpha, 1]$ (for some $\alpha > 0$). Under some conditions, the authors obtained strong and weak convergence results.

Motivated by the work of Moudafi [8], Moudafi and Al-Shemas [9], Moudafi [10] and Yuan-Fang $et\ al.$ [15], we define the following iterative algorithm to solve the split equality fixed point problem (1.2) in the case that U and T are *demi-contractive*.

$$\begin{cases} \forall x_1 \in H_1, \quad \forall y_1 \in H_2; \\ x_{n+1} = (1-\alpha)\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) + \alpha U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big); \\ y_{n+1} = (1-\alpha)\Big(y_n + \gamma B^*(Ax_n - By_n)\Big) + \alpha T\Big(y_n + \gamma B^*(Ax_n - By_n)\Big), \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where $U : H_1 \to H_1, \quad T : H_2 \to H_2$ are two *demi-contractive mappings*. The important class of demi-contractive mappings properly includes the class of firmly quasi-nonexpansive mappings studied by Moudafi and Al-Shemas [9]. Under suitable conditions, we prove weak and strong convergence theorems of the iterative scheme (1.5) to a solution of the split equality problem in real Hilbert spaces. Our theorems extend and complement the results of Censor and Segal [5], Maruster $et\ al.$ [7], Moudafi and Al-Shemas [9], Moudafi [10], [11], Xu [13], Yang [14], Yuan-Fang $et\ al.$ [15], and a host of other results.

2. PRELIMINARIES AND NOTATIONS

We recall some definitions and lemmas which will be needed in the proof of our main theorems.
In the sequel, we denote strong and weak convergence by " $\longrightarrow$" and " $\rightharpoonup$", respectively, the fixed point set of a mapping T by $F(T)$ and the solution set of problem (1.2) by Ω, namely,

$$\Omega := \{(x^*, y^*) \in F(U) \times F(T) \ : \ Ax^* = By^*\}.$$

Definition 2.1. Let H be a real Hilbert space.

(1) Let $T : H \to H$ be a mapping. Then, $(I - T)$ is said to be *demi-closed* at zero if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$, and $x_n - Tx_n \longrightarrow 0$, we have $x^* = Tx^*$.

(2) A mapping $T : H \to H$ is said to be *semi-compact* if for any bounded sequence $\{x_n\} \subset H$ with $x_n - Tx_n \longrightarrow 0$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some $x^* \in H$.

Definition 2.2. Let H be a real Hilbert space.

(1) A mapping $T : H \to H$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \ \forall \ (x, y) \in H \times H. \tag{2.1}$$

(2) A mapping $T : H \to H$ is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - x^*\| \leq \|x - x^*\| \ \forall x^* \in F(T), \ \ x \in H. \tag{2.2}$$

(3) A mapping $T : H \to H$ is said to be *firmly quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 - \|x - Tx\|^2 \;\forall x^* \in F(T), \;\; x \in H. \tag{2.3}$$

(4) Let D be a nonempty subset of H. A map $T : D \to D$ is said to be k-*strictly pseudo-contractive* if there exists a constant $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \;\; \forall x, y \in D.$$

(5) $T : D \to D$ is said to be *demi-contractive* if $F(T) \neq \emptyset$ and there exists a constant $k \in (0, 1)$ such that

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|x - Tx\|^2 \;\; \forall x \in D, \; x^* \in F(T).$$

Remark 2.3. The following inclusions are obvious.

Firmly quasi-nonexpansive $\subset$ *Quasi-nonexpansive* $\subset$ *Demi-contractive.*

We give examples to show that the above inclusions are proper.

Example 2.4. Let $H = l_2$; $D := \{x \in l_2 : \|x\|_2 \leq 1\}$ and $T : D \to D$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots)$. Then, T has a unique fixed point, zero. Clearly, T is a quasi-nonexpansive mapping which is not firmly quasi-nonexpansive.
In fact, we have:

$$\|Tx - 0\| = \|x - 0\|, \tag{$*$}$$

so T is quasi-nonexpansive, and for every $x \neq 0$, suppose

$$\|Tx - 0\|^2 \leq \|x - 0\|^2 - \|x - Tx\|^2.$$

Then, using $(*)$, we obtain that $x = 0$, which is a contradiction. Therefore, T is not firmly quasi-nonexpansive.

Example 2.5. Let $H = l_2$ and $T : l_2 \to l_2$ be defined by $T(x_1, x_2, x_3, \cdots) = -\frac{5}{2}(x_1, x_2, x_3, \cdots)$, for arbitrary $(x_1, x_2, x_3, \cdots) \in l_2$. Then, $F(T) = \{0\}$, and T is a demi-contractive mapping which is not quasi-nonexpansive.
Indeed, for each $x \in l_2$, we have

$$\|Tx - 0\|^2 = \frac{25}{4}\Big\|x - 0\Big\|^2,$$

which implies that T is not quasi-nonexpansive. We also have that

$$\|x - Tx\|^2 = \Big\|x - \Big(-\frac{5}{2}x\Big)\Big\|^2 = \frac{49}{4}\Big\|x - 0\Big\|^2,$$

so that

$$\|x - 0\|^2 = \frac{4}{49}\Big\|x - Tx\Big\|^2. \tag{$**$}$$

Thus, using $(**)$, we have:

$$\|Tx - 0\|^2 = \|x - 0\|^2 + \frac{21}{4}\Big\|x - 0\Big\|^2 = \|x - 0\|^2 + \frac{3}{7}\Big\|x - Tx\Big\|^2.$$

Hence, T is a demi-contractive mapping with constant $k = \frac{3}{7} \in (0, 1)$.

Lemma 2.6. *(Opial's Lemma [12]) Let H be a real Hilbert space and $\{\mu_n\}$ be a sequence in H such that there exists a nonempty set $W \subset H$ satisfying the following conditions:*

(i) *For every* $\mu \in W$, $\lim_{n\to\infty} \|\mu_n - \mu\|$ *exists;*
(ii) *Any weak-cluster point of the sequence* $\{\mu_n\}$ *belongs to* W.

Then, there exists $w^* \in W$ *such that* $\{\mu_n\}$ *converges weakly to* w^*.

Lemma 2.7. *(see e.g., Chidume, [6]) Let* H *be a real Hilbert space and* $\lambda \in [0, 1]$. *Then, for any* x, y, $z \in H$,

$$\|\lambda x + (1-\lambda)y - z\|^2 = \lambda\|x - z\|^2 + (1-\lambda)\|y - z\|^2 - \lambda(1-\lambda)\|x - y\|^2.$$

3. MAIN RESULTS

To approximate a solution of the split equality fixed point problem (1.2), we make the following assumptions:

($\mathbf{A_1}$) H_1, H_2 and H_3 are real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators.
($\mathbf{A_2}$) $U : H_1 \to H_1$, $T : H_2 \to H_2$ are demi-contractive mappings with constants k_1 and k_2, respectively.
($\mathbf{A_3}$) $I - U$ and $I - T$ are demi-closed at zero, and U and T are uniformly continuous.

For arbitrary $x_1 \in H_1$ and $y_1 \in H_2$ define an iterative algorithm by

$$\begin{cases} x_{n+1} = (1-\alpha)\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) + \alpha U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big); \\ \\ y_{n+1} = (1-\alpha)\Big(y_n + \gamma B^*(Ax_n - By_n)\Big) + \alpha T\Big(y_n + \gamma B^*(Ax_n - By_n)\Big), \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\alpha \in (0, 1-k)$ and $\gamma \in \left(0, \frac{2}{(\lambda_{A^*A} + \lambda_{B^*B})}\right)$, where λ_{A^*A} and λ_{B^*B} denote the spectral radii of A^*A and B^*B, respectively and $k = \max\{k_1, k_2\}$.

We now prove the following theorem.

Theorem 3.1. *Suppose assumptions* $(\mathbf{A_1}) - (\mathbf{A_3})$ *hold.*
If $\Omega := \{(x^*, y^*) \in F(U) \times F(T) \ : \ Ax^* = By^*\} \neq \emptyset$, *then the sequence* $\{(x_n, y_n)\}$ *generated by (3.1) converges weakly to a solution of problem (1.2).*

Proof. Let $(x^*, y^*) \in \Omega$. Using lemma 2.7 and assumption $\mathbf{A_2}$, we have

$$\begin{aligned} \Big\|x_{n+1} - x^*\Big\|^2 &= \Big\|(1-\alpha)\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) + \alpha U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) - x^*\Big\|^2 \\ &= (1-\alpha)\Big\|x_n - \gamma A^*(Ax_n - By_n) - x^*\Big\|^2 + \alpha\Big\|U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) - x^*\Big\|^2 \\ &- \alpha(1-\alpha)\Big\|x_n - \gamma A^*(Ax_n - By_n) - U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big)\Big\|^2 \\ &\leq (1-\alpha)\Big\|x_n - \gamma A^*(Ax_n - By_n) - x^*\Big\|^2 + \alpha\Big\|x_n - \gamma A^*(Ax_n - By_n) - x^*\Big\|^2 \\ &+ \alpha k_1\Big\|x_n - \gamma A^*(Ax_n - By_n) - U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big)\Big\|^2 \\ &- \alpha(1-\alpha)\Big\|x_n - \gamma A^*(Ax_n - By_n) - U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big)\Big\|^2 \\ &= \Big\|x_n - \gamma A^*(Ax_n - By_n) - x^*\Big\|^2 \end{aligned}$$

$$
\begin{aligned}
&- \alpha(1-k_1-\alpha)\Big\|x_n-\gamma A^*(Ax_n-By_n)-U\Big(x_n-\gamma A^*(Ax_n-By_n)\Big)\Big\|^2\\
&\leq \Big\|x_n-x^*\Big\|^2-2\gamma\langle Ax_n-By_n, Ax_n-Ax^*\rangle+\gamma^2\lambda_{A^*A}\Big\|Ax_n-By_n\Big\|^2\\
&- \alpha(1-k_1-\alpha)\Big\|x_n-\gamma A^*(Ax_n-By_n)-U\Big(x_n-\gamma A^*(Ax_n-By_n)\Big)\Big\|^2.
\end{aligned}
$$

Similary, we have that

$$
\begin{aligned}
\Big\|y_{n+1}-y^*\Big\|^2 &\leq \Big\|y_n-y^*\Big\|^2+2\gamma\langle Ax_n-By_n, By_n-By^*\rangle+\gamma^2\lambda_{B^*B}\Big\|Ax_n-By_n\Big\|^2\\
&- \alpha(1-k_2-\alpha)\Big\|y_n+\gamma B^*(Ax_n-By_n)-T\Big(y_n+\gamma B^*(Ax_n-By_n)\Big)\Big\|^2.
\end{aligned}
$$

Adding the above two inequalities and using $k=\max\{k_1,k_2\}$ and the fact that $Ax^*=By^*$, we have that

$$
\begin{aligned}
\Big\|x_{n+1}-x^*\Big\|^2+\Big\|y_{n+1}-y^*\Big\|^2 &\leq \Big\|x_n-x^*\Big\|^2+\Big\|y_n-y^*\Big\|^2+\gamma^2(\lambda_{A^*A}+\lambda_{B^*B})\Big\|Ax_n-By_n\Big\|^2\\
&- 2\gamma\Big\|Ax_n-By_n\Big\|^2\\
&- \alpha(1-k-\alpha)\Big\|x_n-\gamma A^*(Ax_n-By_n)-U\Big(x_n-\gamma A^*(Ax_n-By_n)\Big)\Big\|^2\\
&- \alpha(1-k-\alpha)\Big\|y_n+\gamma B^*(Ax_n-By_n)-T\Big(y_n+\gamma B^*(Ax_n-By_n)\Big)\Big\|^2.
\end{aligned}
$$

That is,

$$
\begin{aligned}
\Big\|x_{n+1}-x^*\Big\|^2+\Big\|y_{n+1}-y^*\Big\|^2 &\leq \Big\|x_n-x^*\Big\|^2+\Big\|y_n-y^*\Big\|^2-\gamma\Big(2-\gamma(\lambda_{A^*A}+\lambda_{B^*B})\Big)\Big\|Ax_n-By_n\Big\|^2\\
&- \alpha(1-k-\alpha)\Big\|x_n-\gamma A^*(Ax_n-By_n)-U\Big(x_n-\gamma A^*(Ax_n-By_n)\Big)\Big\|^2\\
&- \alpha(1-k-\alpha)\Big\|y_n+\gamma B^*(Ax_n-By_n)-T\Big(y_n+\gamma B^*(Ax_n-By_n)\Big)\Big\|^2.
\end{aligned}
$$

Now set $\Omega_n(x^*,y^*)=\Big\|x_n-x^*\Big\|^2+\Big\|y_n-y^*\Big\|^2$. Then, it follows that

$$
\begin{aligned}
\Omega_{n+1}(x^*,y^*) \leq \Omega_n(x^*,y^*)&-\gamma\Big(2-\gamma(\lambda_{A^*A}+\lambda_{B^*B})\Big)\Big\|Ax_n-By_n\Big\|^2\\
&-\alpha(1-k-\alpha)\Big\|x_n-\gamma A^*(Ax_n-By_n)-U\Big(x_n-\gamma A^*(Ax_n-By_n)\Big)\Big\|^2\\
&-\alpha(1-k-\alpha)\Big\|y_n+\gamma B^*(Ax_n-By_n)-T\Big(y_n+\gamma B^*(Ax_n-By_n)\Big)\Big\|^2.
\end{aligned}
\tag{3.2}
$$

Since $\alpha\in\Big(0,1-k\Big)$ and $\gamma\in\Big(0,\ \frac{2}{(\lambda_{A^*A}+\lambda_{B^*B})}\Big)$,

we have $2-\gamma(\lambda_{A^*A}+\lambda_{B^*B})>0$ and $1-k-\alpha>0$. It follows that

$$\Omega_{n+1}(x^*,y^*)\leq\Omega_n(x^*,y^*).$$

So, the sequence $\{\Omega_n(x^*, y^*)\}$ is non-increasing and bounded below, therefore, it converges. On the other hand, it follows from inequality (3.2) and the convergence of the sequence $\{\Omega_n(x^*, y^*)\}$ that

$$\lim_{n\to\infty}\Big\|Ax_n - By_n\Big\| = 0, \tag{3.3}$$

$$\lim_{n\to\infty}\Big\|x_n - \gamma A^*(Ax_n - By_n) - U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big)\Big\| = 0, \tag{3.4}$$

and

$$\lim_{n\to\infty}\Big\|y_n + \gamma B^*(Ax_n - By_n) - T\Big(y_n + \gamma B^*(Ax_n - By_n)\Big)\Big\| = 0. \tag{3.5}$$

Furthermore, since $\{\Omega_n(x^*, y^*)\}$ converges, we have that $\{x_n\}$ and $\{y_n\}$ are bounded. Let x^{**} and y^{**} be the weak-cluster points of the sequences $\{x_n\}$ and $\{y_n\}$, respectively. Then, there exists a subsequence of $\{(x_n, y_n)\}$ (without loss of generality, still denoted by $\{(x_n, y_n)\}$) such that $x_n \rightharpoonup x^{**}$ and $y_n \rightharpoonup y^{**}$. Next, we show that $Ux^{**} = x^{**}$ and $Ty^{**} = y^{**}$. Since U is uniformly continuous, it follows from (3.3) that

$$\lim_{n\to\infty}\Big\|U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) - Ux_n\Big\| = 0. \tag{3.6}$$

Similarly, we have that

$$\lim_{n\to\infty}\Big\|T\Big(y_n + \gamma B^*(Ax_n - By_n)\Big) - Ty_n\Big\| = 0. \tag{3.7}$$

We now show that $\lim_{n\to\infty}\Big\|Ux_n - x_n\Big\| = 0$. Using (3.4) and (3.6), we have

$$\begin{aligned}
\Big\|Ux_n - x_n\Big\| &\le \Big\|x_n - \gamma A^*(Ax_n - By_n) - U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big)\Big\| \\
&+ \Big\|U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) - Ux_n\Big\| \\
&+ \Big\|x_n - \gamma A^*(Ax_n - By_n) - x_n\Big\| \\
&\le \Big\|x_n - \gamma A^*(Ax_n - By_n) - U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big)\Big\| \\
&+ \Big\|U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) - Ux_n\Big\| \\
&+ \gamma\Big\|A^*\Big\|\Big\|Ax_n - By_n\Big\| \longrightarrow 0 \text{ as n} \to \infty.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty}\Big\|Ux_n - x_n\Big\| = 0. \tag{3.8}$$

Similarly, we have that

$$\lim_{n\to\infty}\Big\|Ty_n - y_n\Big\| = 0. \tag{3.9}$$

Now, since $x_n \rightharpoonup x^{**}$, $I - U$ is demi-closed at zero, and $\lim_{n\to\infty}\Big\|Ux_n - x_n\Big\| = 0$, we have that $Ux^{**} = x^{**}$, which shows that $x^{**} \in F(U)$. Similarly, we have that $y^{**} \in F(T)$. Since A and B are bounded linear operators, and $\{x_n\}$ and $\{y_n\}$ converge weakly to x^{**} and y^{**}, respectively, we have that for arbitrary $f \in H_3^*$,

$$f(Ax_n) = (f \circ A)(x_n) \longrightarrow (f \circ A)(x^{**}) = f(Ax^{**}).$$

Similarly,

$$f(By_n) = (f \circ B)(y_n) \longrightarrow (f \circ B)(y^{**}) = f(By^{**}).$$

These convergences imply that

$$Ax_n - By_n \rightharpoonup Ax^{**} - By^{**},$$

which, in turn, implies that

$$\left\|Ax^{**} - By^{**}\right\| \leq \liminf_{n\to\infty} \left\|Ax_n - By_n\right\| = 0,$$

so that $Ax^{**} = By^{**}$. Hence, we have $(x^{**}, y^{**}) \in \Omega$.
Summing up, we have proved that:

(1) for each $(x^*, y^*) \in \Omega$, $\lim_{n\to\infty} \left(\left\|x_n - x^*\right\|^2 + \left\|y_n - y^*\right\|^2\right)$ exists;
(2) each weak cluster point of the sequence $\{(x_n, y_n)\}$ belongs to Ω.

Taking $H = H_1 \times H_2$ with the norm $\left\|(x, y)\right\| = \left(\left\|x\right\|^2 + \left\|y\right\|^2\right)^{\frac{1}{2}}$, $W = \Omega$, $\mu_n = (x_n, y_n)$, and $\mu = (x^*, y^*)$ in lemma 2.6, we have that there exists $(\bar{x}, \bar{y}) \in \Omega$ such that $x_n \rightharpoonup \bar{x}$ and $y_n \rightharpoonup \bar{y}$. Hence, the sequence $\{(x_n, y_n)\}$ generated by the iterative scheme (3.1) converges weakly to a solution of problem (1.2) in Ω. This completes the proof. □

We now prove the following strong convergence theorem.

Theorem 3.2. *Suppose assumptions* $(\mathbf{A_1}) - (\mathbf{A_3})$ *hold and let* $\{x_n\}$ *and* $\{y_n\}$ *be as in theorem 3.1. If* $\Omega \neq \emptyset$*, and the mappings* U *and* T *are semi-compact, then, the sequence* $\{(x_n, y_n)\}$ *generated by (3.1) converges strongly to a solution of problem (1.2) in* Ω.

Proof. Since U and T are semi-compact, $\{x_n\}$ and $\{y_n\}$ are bounded (by theorem 3.1), and $\lim_{n\to\infty} \left\|(I - U)x_n\right\| = 0$, $\lim_{n\to\infty} \left\|(I - T)y_n\right\| = 0$, there exist (without loss of generality) subsequences $\{x_{n_j}\} \subset \{x_n\}$ and $\{y_{n_j}\} \subset \{y_n\}$ such that $\{x_{n_j}\}$ and $\{y_{n_j}\}$ converge strongly to some points x^* and y^*, respectively. It follows from the demi-closedness of $I - U$ and $I - T$ that $x^* \in F(U)$ and $y^* \in F(T)$.
Thus,

$$\left\|Ax^* - By^*\right\| = \lim_{j\to\infty} \left\|Ax_{n_j} - By_{n_j}\right\| = 0.$$

This implies that $Ax^* = By^*$. Hence, $(x^*, y^*) \in \Omega$. On the other hand, since $\Omega_n(x, y) = \left\|x_n - x\right\|^2 + \left\|y_n - y\right\|^2$ for any $(x, y) \in \Omega$, we know that $\lim_{j\to\infty} \Omega_{n_j}(x^*, y^*) = 0$. From theorem 3.1, we have $\lim_{n\to\infty} \Omega_n(x^*, y^*)$ exists, therefore $\lim_{n\to\infty} \Omega_n(x^*, y^*) = 0$. So, as in the proof of theorem 3.1, the iterative scheme converges strongly to a solution of problem (1.2) in Ω. The proof is complete. □

Corollary 3.1. *Suppose assumptions* $(\mathbf{A_1}) - (\mathbf{A_3})$ *hold and let* $\{x_n\}$ *and* $\{y_n\}$ *be as in theorem 3.1. If* $\Omega \neq \emptyset$*, and the mappings* U *and* T *have convex and compact domain* D*, then, the sequence* $\{(x_n, y_n)\}$ *generated by (3.1) converges strongly to a solution of problem (1.2) in* Ω.

Proof. Since every map $T : D \subset H \to D$, with D compact, is semi-compact, the proof follows from theorem 3.2. □

Corollary 3.2. *Suppose assumptions* $(\mathbf{A_1})$ *and* $(\mathbf{A_3})$ *hold and let* $\{x_n\}$ *and* $\{y_n\}$ *be as in theorem 3.1. If* $\Omega \neq \emptyset$*, and the mappings* U *and* T *are quasi-nonexpansive and semi-compact, then, the sequence* $\{(x_n, y_n)\}$ *generated by (3.1) converges strongly to a solution of problem (1.2) in* Ω.

Corollary 3.3. *Suppose assumptions* $(\mathbf{A_1})$ *and* $(\mathbf{A_3})$ *hold and let* $\{x_n\}$ *and* $\{y_n\}$ *be as in theorem 3.1. If* $\Omega \neq \emptyset$*, and the mappings* U *and* T *are firmly quasi-nonexpansive and semi-compact, then, the sequence* $\{(x_n, y_n)\}$ *generated by (3.1) converges strongly to a solution of problem (1.2) in* Ω*.*

Remark 3.4. Our theorems 3.1 and 3.2 extend and complement the results of Moudafi $et\ al.$ [9], Moudafi [10], and Yuan-Fang $et\ al.$ [15].

Remark 3.5. The recursion formula considered in this paper is of Krasnoselskii-type which, in general, converges as fast as a geometric progression.

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