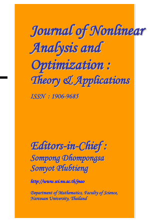
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 6, No. 1, (2015), 27-39
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



SENSITIVITY ANALYSIS FOR AN OPTIMAL CONTROL PROBLEM OF PRODUCTION SYSTEM BASED ON NONLINEAR CONSERVATION LAW

TANMAY SARKAR*

Department of Mathematics, Indian Institute of Technology Madras,
Chennai-600036, India

ABSTRACT. Dynamics of a production system having a large number of items can be represented through hyperbolic conservation law. Due to nonlinear dependence on the work in progress, the resulting partial differential equation becomes nonlinear. Further, occurrence of yield loss during the process makes it nonhomogeneous. In this paper, an optimal control problem has been studied incorporating hyperbolic conservation law as a constraint. One of the few ways to control the output of production system is by adjusting the influx in the system. Moreover, yield loss can also be controlled in a time dependent manner. It is well known that the solutions of nonlinear conservation laws may develop discontinuities known as shock waves that forbid the use of classical variational techniques. This paper studies sensitivity analysis with the presence of shocks. Adjoint technique has been implemented to evaluate gradients of cost functionals.

KEYWORDS : Production System; Hyperbolic Conservation Laws; Optimal Control; Shocks; Sensitivity.
AMS Subject Classification: 49K40, 35L67, 49J20

1. INTRODUCTION

State quantities such as density, velocity and energy, often give rise to nonlinear conservation laws in various fields of science and technology. One such area is production system. Armbruster et al. [1] have introduced a continuum model to study the dynamics of a production system. It is shown that the part density of the materials in a production system can be approximated by hyperbolic conservation law. Further, it has been studied by several authors [2, **?**, 10, 17, 18]. Taking into account customer satisfaction, many important aspects such as velocity form, yield loss are incorporated at macroscopic level.

The main objective behind any model of production network is to control the system in such a way that it should satisfy the demand as closely as possible. Since the demand is so stochastic over a given period of time, a manufacturing system

* *Corresponding author.*
Email address : tanmaysemilo@gmail.com (T. Sarkar).
Article history : Received April 8, 2014. Accepted September 29, 2014.

needs to generate demand forecast quite frequently and functions accordingly. This motivates us to study the optimal control problem arising in a production system. However, it is well known that optimal control for hyperbolic conservation law is a difficult topic due to its considerable analytical effort as well as computational expense in practice. Only few attempts have been made to control hyperbolic equations as they can not be treated straightway with the development of elliptic and parabolic type equations.

One way is to control the production system by adjusting the inflow rate of the materials. In this context, existence of optimal control has been shown by Shang et al. [20] taking influx rate as a control variable. Further, it has been studied numerically by La Marca et al. [14]. Considering a separate equation for queue, optimal control problem with coupled system has been studied by Kirchner et al. [12]. Moreover, for this problem existence of an optimal control has been demonstrated by D'Apice et al. [8].

In order to perform numerical computations, we try to replace continuous optimization problem by discrete approximation. Subsequently, we need to develop an efficient algorithm to achieve discrete minimizer. In practice most efficient methods to approximate minimizers are gradient-based methods. The approach can be summarized as follows: we first linearize the differential equation to obtain a descent direction for the cost functional J. Then, we take the descent direction with the discrete values obtained from numerical scheme. For the linearization step, we consider a small variation of the conserved quantity with respect to the control variables. One may refer [3, 11] for general linearization technique. However, the procedure is justified only when the solution is smooth enough. For discontinuous solution, it is not justified due to the occurrence of singular terms on the linearization over the shock location. To take into account shock location, sensitivity analysis is necessary. The linearized system for the variation of the solution must be complemented with some new equations for the sensitivity of the shock position. Moreover, it is also necessary to perform the sensitivity analysis of the optimal control problem for the numerical perspective. This will be evident from the discussion in remaining sections.

In the present article, we study the sensitivity analysis of an optimal control problem for production system incorporating yield loss. Sensitivity analysis has been studied for many real life problems [16]. For Burgers equation, sensitivity analysis has been performed by Castro et al. [5, 4]. For general conservation law sensitivity analysis has been accomplished by Kowalewski et al. [13] and Ulbrich [22]. Song [21] and Godlewski et al. [9] have extended the analysis for systems. Recently, for scalar conservation law sensitivity analysis has been performed through vanishing viscosity method [15].

In this paper, we perform the sensitivity analysis for a production system by taking into account certain form of the yield loss. Further, we consider more general form of the yield loss wherein another control variable is also considered. We have carried out the analysis without the presence of shocks. In case of shocks, variation of shock position will be given by ODE's to complement the linearized equation. We evaluate the gradient through adjoint calculus. We demonstrate via numerical illustrations that the customer demand can be matched by controlling the influx in the system. Apart from the theoretical investigations, numerical illustrations for yield loss case of production system are new up to our knowledge.

The remaining part of the paper is organized as follows: In Section 2, we provide some preliminaries to describe the model of production system along with the optimal control problem. Sensitivity analysis without the presence of shocks is presented in Section 3. In Section 4, we take into account shocks in the solution and perform sensitivity analysis with the state variable. Sensitivity analysis will be carried out for cost functional in Section 5. We will end up with the numerical results of the presented optimal control problem in Section 6. Concluding remarks and further scopes of development have been pointed out in Section 7.

2. PRELIMINARIES

The nonlinear conservation law model of manufacturing system can be represented as follows:

$$\partial_t \rho(x,t) + \partial_x f(x,t,\rho) + y_l(x,t,\rho) = 0,\ x \in (0,L),\ t > 0, \tag{2.1}$$

where the flux function $f(x,t,\rho)$ is given by

$$f(x,t,\rho) = \min\{\mu(x,t), v(x,t)\rho(x,t)\}.$$

Completion of the product within the supplier is represented by continuous variable x. In (2.1), $\rho(x,t)$ represents the density of goods at stage x and time t. Raw materials entered into the suppliers are described by the parts at $x = 0$. The finished products are going out of the suppliers at $x = L$. The term $\mu(x,t)$ represents maximal capacity of the suppliers. Yield loss phenomena is expressed as $y_l(x,t,\rho)$. Influx and initial situation in the system are prescribed below.

$$\text{Initial condition: } \rho(x,0) = \rho_0(x),\ x \in [0,L]. \tag{2.2}$$

$$\text{Influx condition: } f(0,t,\rho(0,t)) = \lambda(t),\ t > 0. \tag{2.3}$$

Form of velocity function is given in [19] as follows: $v(x,t) = v(W(t))$, where $W(t)$ represents work in progress in supplier at time t. Mathematically, $W(t) = \int_0^L \rho(s,t)ds$. Similar nonlocal velocity form is also considered in [6]. Now we introduce an optimal control problem related to production system.

The profit of a manufacturing system can be affected significantly by two different aspects. One is overproduction. Producing too much of items lead to high inventory cost in the system. The other one is underproduction. Producing not sufficient number of items lead to lost sales which result into backlog cost. It is quite evident that to maximize the profit, a manufacturing system must be able to match the demand of the customers as closely as possible. We consider cost functional J as

$$J(\rho,\lambda) := \frac{1}{2}\int_0^T [y_d(t) - y(t)]^2 dt + \frac{1}{2}\int_0^T |\lambda(t)|^2 dt. \tag{2.4}$$

Here $y_d(t)$ represents the demand rate and $y(t) = v(W(t))\rho(L,t)$ measures the output of the system. The term $\lambda(t)$ provides the influx rate in the system over time T. The objective behind the choice of cost functional is to minimize the amount of influx and mismatch between the outflux and demand of the customers. We assume that the maximal capacity does not exceed the flux in the suppliers, and maximum speed of the materials denoted by V_M. Optimization problem will be studied in the subsequent sections can be formulated as follows:

$$\min J(\rho,\lambda) \text{ subject to the constraints}$$

$$\begin{cases} \partial_t \rho(x,t) + \partial_x (v(W(t))\rho(x,t)) + y_l(x,t,\rho) = 0, \\ \rho(x,0) = \rho_0(x),\ x \in [0,L], \\ v(W(t))\rho(0,t) = \lambda(t),\ t \in (0,T], \\ v(W(t)) = \frac{V_M}{1+\int_0^L \rho(s,t)ds}. \end{cases} \tag{2.5}$$

3. Sensitivity analysis without shocks

In this section, we derive an expression for the sensitivity of the functional J with respect to influx rate by considering certain form of yield loss. After that the analysis will be carried out for general yield loss case involving another control variable.

3.1. Control on Influx.

We consider the yield loss form as $y_l(x,t,\rho) = -g(x,t)\rho(x,t)$, where $g(x,t)$ is a continuous function. The following theorem ensures the existence of at least one minimizer for J given in (2.4).

Theorem 3.1. *Let $U_{ad} = \{f \in L^2(0,T) : \ f$ is non-negative almost everywhere$\}$. Assume that $y_d \in L^2(0,T)$. Then the infimum of the functional J is achieved, i.e., there exists $\lambda_{\min} \in U_{ad}$ such that $J(\lambda_{\min}) = \inf_{\lambda \in U_{ad}} J(\rho(\lambda),\lambda)$.*

Proof of Theorem 3.1 can be carried out in the same manner as in Shang et al. [20] by considering a minimizing sequence of J in U_{ad}. Let us assume that there exist a classical solution $\rho(x,t)$ of (2.5) in $(x,t) \in [0,L] \times [0,T]$. Let $\delta\lambda$ be any possible variation of the influx rate λ. Then for $\epsilon > 0$ sufficiently small, the solution $\rho^\epsilon(x,t)$ corresponding to the influx

$$\lambda^\epsilon(t) = \lambda(t) + \epsilon\delta\lambda(t)$$

is also a solution for $(x,t) \in (0,L) \times (0,T)$ and $\rho^\epsilon(x,t)$ can be written as

$$\rho^\epsilon = \rho + \epsilon(\delta\rho) + o(\epsilon),$$

where $\delta\rho$ is the solution of the linearized equation

$$\begin{cases} \partial_t\delta\rho + \partial_x\Big(v(W)\delta\rho - \frac{v(W)^2}{V_M}\rho(x,t)\int_0^L \delta\rho(s,t)ds\Big) = g(x,t)\delta\rho(x,t), \\ \delta\rho(x,0) = 0, \\ v(W)\delta\rho(0,t) \approx \delta\lambda(t). \end{cases} \tag{3.1}$$

In order to find the adjoint system for the problem (2.5), we seek for first order necessary condition of (2.5). Thus, we introduce the Lagrangian function $L(\rho,p,\lambda)$ by considering $p(x,t)$ as a multiplier:

$$\begin{aligned} L(\rho,p,\lambda) := &\frac{1}{2}\int_0^T [y_d(t) - y(t)]^2 dt + \frac{1}{2}\int_0^T |\lambda(t)|^2 dt \\ &+ \int_0^L\int_0^T [\partial_t\rho(x,t) + \partial_x(v(W(t))\rho) - g(x,t)\rho(x,t)]p(x,t)dtdx. \end{aligned}$$

Taking into account the variation of Lagrangian L with respect to ρ and λ, we obtain the following adjoint system:

$$\begin{cases} \partial_t p + \partial_x(v(W)p) + g(x,t)p = \frac{v(W)^2}{V_M}\big[\rho(L,t)y_d(t) \quad -v(W)\rho(L,t)^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad - \int_0^L p(s,t)\rho_x(s,t)ds\big], \\ p(x,T) = 0, \\ p(L,t) = y_d(t) - v(W)\rho(L,t). \end{cases} \tag{3.2}$$

Multiplying the linearized equation (3.1) by $p(x,t)$ and integrating by parts, we get the following:

$$\begin{aligned}
&\int_0^L \int_0^T \Big(\partial_t p(x,t) + \partial_x(v(W)p(x,t)) + g(x,t)p(x,t)\Big)\delta\rho\; dtdx + \int_0^L p(x,0)\delta\rho(x,0)dx \\
&- \int_0^L \int_0^T \frac{v(W)^2}{V_M}\rho(x,t)\int_0^L \delta\rho(s,t)ds\partial_x p(x,t)dxdt - \int_0^L p(x,T)\delta\rho(x,T)dx \\
&- \int_0^T \frac{v(W)^2}{V_M}\Big[p(0,t)\rho(0,t) - p(L,t)\rho(L,t)\Big]\int_0^L \delta\rho(s,t)dsdt \\
&- \int_0^T v(W)p(L,t)\delta\rho(L,t)dt + \int_0^T p(0,t)v(W)\delta\rho(0,t)dt = 0.
\end{aligned} \tag{3.3}$$

Making use of simple calculus it is not difficult to obtain the following:

$$\begin{aligned}
-\int_0^L \int_0^T \frac{v(W)^2}{V_M}\rho(x,t)&\int_0^L \delta\rho(s,t)ds\partial_x p(x,t)dtdx \\
&= \int_0^L \int_0^T \frac{v(W)^2}{V_M}\Big[\int_0^L \partial_x\rho(s,t)p(s,t)ds\Big]\delta\rho(x,t)dtdx \\
&+ \int_0^T \frac{v(W)^2}{V_M}[p(0,t)\rho(0,t) - p(L,t)\rho(L,t)]\int_0^L \delta\rho(s,t)dsdt.
\end{aligned}$$

Let δJ be the Gateaux derivative of functional J at λ in the direction $\delta\lambda$.

$$\begin{aligned}
\delta J = &\int_0^T \frac{v(W)^2}{V_M}\big(\rho(L,t)y_d(t) - v(W)\rho(L,t)^2\big)\int_0^L \delta\rho(s,t)dsdt \\
&- \int_0^T \big(v(W)^2\rho(L,t) - v(W)y_d(t)\big)\delta\rho(L,t)dt + \int_0^T \lambda(t)\delta\lambda(t)dt.
\end{aligned}$$

Taking into account adjoint system (3.2) in (3.3) we can rewrite the variation of J in the following way.

$$\delta J = \int_0^T \lambda(t)\delta\lambda dt - \int_0^T p(0,t)\delta\lambda dt = \int_0^T (\lambda(t) - p(0,t))\delta\lambda dt.$$

In order to evaluate δJ, we need to get the information from adjoint system. Adjoint state $p(x,t)$ can be computed from the prescribed influx $\lambda(t)$, $\delta\lambda(t)$, boundary data $p(L,t)$ and the terminal data $p(x,T)$. Therefore, the descent direction for the functional J can be chosen as $\delta\lambda = -d(t)$, where $d(t) = (\lambda(t) - p(0,t))$.

3.2. **Control on yield loss.** In order to achieve maximum profit, controlling the yield loss is a crucial objective for any production system. Motivated by this, we include time-dependent control variable $u(t)$ in the yield loss term of continuum model. Form of the yield loss will be considered as $y_l(x,t) = h(x,u(t),\rho(x,t))$, which is assumed to be continuous. In similar way as above, we consider small perturbations $\delta\lambda$ and δu for the influx $\lambda(t)$ and control variable $u(t)$ respectively. This results in small variation on solution $\rho(x,t)$. Let us denote the small variation by $\delta\rho(x,t)$. In this subsection, we carry out the sensitivity analysis considering $\rho(x,t)$ as a classical solution. For discontinuous solution, sensitivity analysis will be performed in the next section. We choose the cost functional as

$$J(\rho,\lambda,u) := \frac{1}{2}\int_0^T [y_d(t) - y(t)]^2 dt + \frac{1}{2}\int_0^T |u(t)|^2 dt.$$

Again $y_d(t)$ and $y(t)$ represent demand and outflux of the system respectively. The first term in the objective functional measures the difference between demand and outflux and the second term can be considered as regularization term.

The small variation $\delta\rho$ satisfies the following linearized problem

$$\begin{cases} \partial_t \delta\rho(x,t) + \partial_x\Big(v(W)\delta\rho - \frac{v(W)^2}{V_M}\rho(x,t)\int_0^L \delta\rho(s,t)ds\Big) \\ +\partial_\rho h(x,u(t),\rho(x,t))\delta\rho + \partial_u h(x,u(t),\rho(x,t))\delta u(t) = 0, \\ \delta\rho(x,0) = 0, \\ v(W)\delta\rho(0,t) \approx \delta\lambda(t). \end{cases} \tag{3.4}$$

By considering usual notion of Lagrangian formulation, we derive the adjoint system. Adjoint variable $p(x,t)$ satisfies the following system:

$$\begin{cases} -\partial_t p(x,t) - \partial_x(v(W)p(x,t)) + \partial_\rho h(x,u(t),\rho(x,t))p(x,t) \\ -\frac{v(W)^2}{V_M}\int_0^L p(s,t)\rho_x(s,t)ds + \frac{v(W)^2}{V_M}(\rho(L,t)y_d(t) - v(W)\rho(L,t)^2) = 0, \\ p(x,T) = 0, \\ p(L,t) = y_d(t) - v(W)\rho(L,t). \end{cases} \tag{3.5}$$

The Gateaux derivative [15] of functional J, denoted by δJ at (u,λ) in the direction $(\delta u, \delta\lambda)$ can be derived as

$$\begin{aligned} \delta J = &\int_0^T \frac{v(W)^2}{V_M}\big(\rho(L,t)y_d(t) - v(W)\rho(L,t)^2\big)\int_0^L \delta\rho(s,t)dsdt \\ &+ \int_0^T \big(v(W)^2\rho(L,t) - v(W)y_d(t)\big)\delta\rho(L,t)dt + \int_0^T u(t)\delta u(t)dt. \end{aligned}$$

From the linearized equation (3.4) multiplying by adjoint variable $p(x,t)$, we obtain

$$\begin{aligned} &\int_0^L\int_0^T \Big(-\partial_t p(x,t) - \partial_x\big(v(W)p(x,t)\big) + \partial_\rho h(x,u,\rho)p(x,t)\Big)\delta\rho(x,t)\, dtdx \\ &- \int_0^L p(x,0)\delta\rho(x,0)dx - \int_0^L\int_0^T \frac{v(W)^2}{V_M}\Big[\int_0^L \partial_x\rho(s,t)p(s,t)ds\Big]\delta\rho(x,t)dtdx \\ &+ \int_0^L p(x,T)\delta\rho(x,T)dx + \int_0^L\int_0^T \big(\partial_u h(x,u,\rho)\delta u\big)p(x,t)dtdx \\ &+ \int_0^T v(W)p(L,t)\delta\rho(L,t)dt - \int_0^T p(0,t)v(W)\delta\rho(0,t)dt = 0. \end{aligned} \tag{3.6}$$

With the help of (3.5) and (3.6), variation δJ can be reduced to

$$\begin{aligned} \delta J = &\int_0^T -p(0,t)\delta\lambda(t)dt + \int_0^T u(t)\delta u(t)dt \\ &+ \int_0^L\int_0^T (\partial_u h(x,u,\rho)\delta u(t))p(x,t)dtdx. \end{aligned}$$

Above expression for δJ provides a descent directions for functional J. Descent direction for control variable u can be chosen as $\delta u(t) = -d_1(t)$, where $d_1(t)$ is having the following expression $d_1(t) = -u(t) - \int_0^L \partial_\rho h(x,u,\rho)p(x,t)dx$. Similarly, we can choose the descent direction for influx as $\delta\lambda(t) = -d_2(t)$, where $d_2(t) = p(0,t)$. Information about adjoint variable $p(x,t)$ can be obtained by solving the system (3.5). Once we have computed adjoint state, we immediately get the descent directions.

4. Sensitivity of the state in presence shocks

The hyperbolic conservation laws may develop singularities in finite time even for the smooth input data. Therefore in practical applications, we need to consider optimal control problems in which the solutions have discontinuities. We shall study the optimal control problem of production system, described in the previous subsection in the presence of shocks. We focus on the analysis of conservation laws with a finite number of noninteracting shocks. In order to develop efficient numerical methods for the optimal control problems in the presence of shocks, we need to investigate the sensitivity of the states in production system with respect to the input data and control variable along with the infinitesimal translation of shock positions.

The conservation laws model is to be studied as in (2.5) with the yield loss and cost functional described in Section 3.2. We assume that $\rho(x,t)$ is a weak solution of conservation law model with discontinuities along Γ_j for $j = 1, 2, ..., S$, where $\Gamma_j = \{(\phi_j(t), t) : t \in [t_j^0, T]\}$. The solution $\rho(x,t)$ is defined in strong sense outside $\cup_j \Gamma_j$. The Rankine-Hugoniot condition on Γ_j can be given as following:

$$\phi_j^{'}(t)[\rho(.,t)]_{x=\phi_j(t)} = [v(W(t))\rho(.,t)]_{x=\phi_j(t)}.$$

The notation $[f]_{x_d} = f(x_d^+) - f(x_d^-)$ denotes the jump at x_d of any piecewise continuous function f with a discontinuity at $x = x_d$. We need to analyze the sensitivity of (ρ, ϕ_j, u) with respect to the variation of $\delta\lambda, \delta\phi_j$ and δu.

We recall that the linearized equation (3.4) must be interpreted in a weak sense. It is reasonable to choose the solution of the linearized equation (3.4) of the following form:

$$\delta\rho = \delta\rho_r + \sum_{j=1}^{S} q_j \chi_{\Gamma_j},$$

where $\delta\rho_r$ is the regular part and the other one is singular part at the shock locations. We observe that the regular part $\delta\rho_r$ satisfies the following linearized system in an analytical sense outside $\cup_j \Gamma_j$

$$\begin{cases} \frac{\partial}{\partial t}\delta\rho_r + \frac{\partial}{\partial x}\Big(v(W)\delta\rho_r - \frac{v(W)^2}{V_M}\rho(x,t)\int_0^L \delta\rho_r(s,t)ds\Big) \\ +\frac{\partial}{\partial\rho}h(x,u(t),\rho(x,t))\delta\rho_r + \frac{\partial}{\partial u}h(x,u(t),\rho(x,t))\delta u = 0, \\ \delta\rho_r(x,0) = 0, \\ v(W)\delta\rho_r(0,t) \approx \delta\lambda(t). \end{cases} \tag{4.1}$$

In order to analyze the singular part, we again get back to the linearized system (3.4). Weak formulation of the linearized system (3.4) can be expressed in the following way by considering $\psi(x,t)$ as a test function having compact support

$$\begin{aligned} &\int_0^L\int_0^T \delta\rho(\partial_t\psi + v(W)\partial_x\psi)dtdx - \int_0^L\int_0^T \psi(x,t)\Big(\partial_\rho h(x,u(t),\rho)\delta\rho \\ &+ \partial_u h(x,u(t),\rho)\delta u\Big)dtdx + \int_0^L\int_0^T \frac{v(W)^2}{V_M}\Big(\int_0^L \psi(s,t)\rho_x(s,t)ds\Big)\delta\rho dtdx \\ &+ \int_0^L \psi(x,0)\delta\rho(x,0)dx - \int_0^L \psi(x,T)\delta\rho(x,T)dx \\ &+ \int_0^T [\psi(0,t)v(W)\delta\rho(0,t) - \psi(L,t)v(W)\delta\rho(L,t)]dt = 0. \end{aligned} \tag{4.2}$$

Let D_c denotes the region $D \setminus \cup_j \Gamma_j$, where D represents the complete domain $[0, L] \times [0, T]$. Using Green's theorem and integration by parts in (4.2), we obtain

$$\begin{aligned}
&\int_{D_c} \Big(-\partial_t \delta\rho - \partial_x \big(v(W)\delta\rho - \frac{v(W)^2}{V_M}\rho(x,t) \int_0^L \delta\rho(s,t)ds \big) \Big) \psi(x,t) dtdx \\
&\quad - \int_0^L \int_0^T \psi(x,t) \Big(\partial_\rho h(x,u(t),\rho)\delta\rho + \partial_u h(x,u(t),\rho)\delta u \Big) dtdx \\
&\quad + \sum_{j=1}^{S} \int_{t_j^0}^{T} \Big[\dot{\phi}_j \delta\rho - v(W)\delta\rho - \frac{v(W)^2}{V_M}\rho \int_0^L \delta\rho(s,t)ds \Big]_{x=\phi_j(t)} \psi|_{x=\phi_j(t)} = 0.
\end{aligned} \tag{4.3}$$

We would like to consider the form of $\delta\rho$ in the weak form of linearized equation (3.4). It is not difficult to derive the following expression

$$\begin{aligned}
&\int_D \sum_{j=1}^{S} q_j \chi_{\Gamma_j} \Big(\partial_t \psi + \partial_x \big(v(W)\delta\rho - \frac{v(W)^2}{V_M}\rho(x,t) \int_0^L \delta\rho(s,t)ds \big) \\
&\quad - \partial_\rho h(x,u(t),\rho)\psi \Big) dxdt = q_j(t)|_{t=t_j^0} \psi(x,t)|_{x=\phi_j(t)} \\
&\quad + \sum_{j=1}^{S} \int_{t_j^0}^{T} \Big(-\frac{dq_j}{dt} - \frac{\partial}{\partial\rho} h(x,u(t),\rho(x,t))|_{x=\phi_j(t)} q_j \Big) \psi|_{x=\phi_j(t)}.
\end{aligned}$$

We observe that for $j = 1, 2, ..., S$, $q_j(t)$ are considered as the solutions of the following ODE's

$$\begin{cases}
\frac{dq_j}{dt} = & -\frac{\partial}{\partial\rho} h(x,u(t),\rho(x,t))|_{x=\phi_j(t)} q_j + \sum_{j=1}^{S} \int_{t_j^0}^{T} \Big[\dot{\phi}_j \delta\rho_r - v(W)\delta\rho_r \\
& -\frac{v(W)^2}{V_M}\rho(x,t) \int_0^L \delta\rho_r(s,t)ds \Big]_{x=\phi_j(t)}, \\
q_j(t_j^0) & = 0.
\end{cases} \tag{4.4}$$

Remark 4.1. In practice the solution of PDE (4.1) should be computed first, for instance with the method of characteristics in D_c as it is interpreted in strong sense. Then we solve the ordinary differential equations to obtain q_j's. It requires the values of $\delta\rho_r$ which will be available from the previous step.

Remark 4.2. If the discontinuities occur after certain time T_0 then the linearization can be done separately for $t \in [0, T_0)$ and $t \in [T_0, T]$. For $t \in [0, T_0)$, the linearization can be carried out as described in Section 3 since the solution is regular. After that the linearization can be done as presented above. The intermediate condition can be obtained from weak formulation of the linearized PDE by choosing appropriate test function.

5. Sensitivity of J in presence shocks

In this section, we study sensitivity of the functional J with respect to the variations of influx $\lambda(t)$ and control $u(t)$. It helps us to evaluate the gradient of cost functional and identify descent directions of the control variables. Furthermore, we describe the solution procedure of the presented optimal control problem in a concise way.

We again make use of adjoint calculus to remove the dependent variables from the variation of cost functional. The variation of cost functional J, denoted by δJ, with respect to the perturbations of (u, λ) can be derived as following

$$\delta J = \int_0^T \frac{v(W)^2}{V_M}\big(\rho(L,t)y_d(t) - v(W)\rho(L,t)^2\big)\int_0^L \delta\rho(s,t)dsdt \\ + \int_0^T \big(v(W)^2\rho(L,t) - v(W)y_d(t)\big)\delta\rho(L,t)dt + \int_0^T u(t)\delta u dt, \tag{5.1}$$

where the pair $(\delta\rho, \delta u)$ solves the linearized equation (3.4).

Incorporating the results of Section 4, the complete system of first variation (3.4) can be rewritten as

$$\begin{cases} \frac{\partial}{\partial t}\delta\rho_r + \frac{\partial}{\partial x}\Big(v(W)\delta\rho_r - \frac{v(W)^2}{V_M}\rho(x,t)\int_0^L \delta\rho_r(s,t)ds\Big) \\ +\frac{\partial}{\partial \rho}h(x,u(t),\rho(x,t))\delta\rho_r + \frac{\partial}{\partial u}h(x,u(t),\rho(x,t))\delta u = 0, \\ \frac{dq_j}{dt} = -\frac{\partial}{\partial\rho}h(x,u(t),\rho(x,t))|_{x=\phi_j(t)}q_j + \sum_{j=1}^S \int_{t_j^0}^T \Big[\dot{\phi}_j\delta\rho_r - v(W)\delta\rho_r \\ -\frac{v(W)^2}{V_M}\rho(x,t)\int_0^L \delta\rho_r(s,t)ds\Big]\big|_{x=\phi_j(t)}, \\ \delta\rho_r(x,0) = 0, \\ v(W)\delta\rho_r(0,t) \approx \delta\lambda(t), \\ q_j(t_j^0) = 0. \end{cases} \tag{5.2}$$

We consider adjoint state variables $p(x,t)$ and $\theta(t) = (\theta_1(t), \theta_2(t), ..., \theta_S(t))$ for $\delta\rho_r$ and $q(t) = (q_1(t), q_2(t), ..., q_S(t))$ respectively. Multiplying equations of $\delta\rho_r$ and q with adjoint variables p and θ respectively and performing integration by parts we have

$$0 = \int_0^T\int_0^L p(x,t)\Big(\partial_t\delta\rho_r + \partial_x\big(v(W)\delta\rho_r - \frac{v(W)^2}{V_M}\rho(x,t)\int_0^L \delta\rho_r(s,t)ds\big) \\ + \partial_\rho h(x,u(t),\rho(x,t))\delta\rho_r + \frac{\partial}{\partial u}h(x,u(t),\rho(x,t))\delta u\Big)dxdt \\ + \sum_{j=1}^S \int_{t_j^0}^T \theta_j\Big(\frac{dq_j}{dt} + \partial_\rho h(x,u,\rho)q_j - \Big[\dot{\phi}_j\delta\rho_r - v(W)\delta\rho_r \\ - \frac{v(W)^2}{V_M}\rho(x,t)\int_0^L \delta\rho_r(s,t)ds\Big]\big|_{x=\phi_j(t)}\Big)$$

$$= \int_0^T\int_0^L \partial_u h(x,u(t),\rho)p(x,t)\delta u dxdt + \int_0^L [p(x,T)\rho(x,T) - p(x,0)\rho(x,0)]dx \\ + \sum_{j=1}^S \int_{t_j^0}^T q_j\Big(-\frac{d\theta_j}{dt} + \partial_\rho h(x,u(t),\rho(x,t))|_{x=\phi_j(t)}\theta_j\Big)dt + \theta_j q_j|_{t_j^0}^T \\ + \sum_{j=1}^S \int_{t_j^0}^T -\theta_j\Big[\dot{\phi}_j\delta\rho_r - v(W)\delta\rho_r - \frac{v(W)^2}{V_M}\rho\int_0^L \delta\rho_r(s,t)ds\Big]\big|_{x=\phi_j(t)} \\ + \int_0^T [p(L,t)v(W)\delta\rho_r(L,t) - p(0,t)v(W)\delta\rho_r(0,t)]dt \\ + \int_0^T\int_0^L \delta\rho_r\Big(-\partial_t p - \partial_x(v(W)p) + \partial_\rho h(x,u,\rho)p(x,t)\Big)dxdt$$

$$- \int_0^T \int_0^L \frac{v(W)^2}{V_M} \big[\int_0^L p(s,t)\rho_x(s,t)ds \big] \delta\rho_r(x,t)dxdt + \sum_{j=1}^{S} \int_{t_j^0}^{T} \big[\dot{\phi}_j(t)p(x,t)$$

$$\delta\rho_r - p(x,t)v(W)\delta\rho_r - p(x,t)\frac{v(W)^2}{V_M}\rho(x,t)\int_0^L \delta\rho_r(s,t)dx\big]|_{x=\phi_j(t)}.$$

In the process of removing dependent variables, we obtain the following system of adjoint variables

$$\begin{cases} \frac{d\theta_j}{dt} = \frac{\partial}{\partial\rho}h(x,u(t),\rho(x,t))|_{x=\phi_j(t)}\theta_j(t), \\ \theta_j(T) = 0, \\ -\partial_t p - \partial_x(v(W)p) + \partial_\rho h(x,u,\rho)p(x,t) + \frac{v(W)^2}{V_M}\big[\rho(L,t)y_d(t) \\ -\rho(L,t)^2 v(W) - \int_0^L p(s,t)\rho_x(s,t)ds\big] = 0, \\ p(x,T) = 0, \\ p(L,t) = y_d(t) - v(W)\rho(L,t), \\ p^-(\phi_j(t),t) = p^+(\phi_j(t),t) = \theta_j(t). \end{cases} \tag{5.3}$$

Taking into account (5.3), variation δJ in (5.1) can be written as

$$\delta J = \int_0^T \int_0^L \partial_u h(x,u(t),\rho)p(x,t)\delta u dxdt + \int_0^T -p(0,t)\delta\lambda dt + \int_0^T u(t)\delta u dt.$$

The above expression provides the information about the gradient of cost functional with respect to the decision variables u and λ. We can choose the descent directions as follows:

$$\delta u = -u(t) - \int_0^L \partial_\rho h(x,u,\rho)p(x,t)dx,$$
$$\delta\lambda = p(0,t),$$

where $p(x,t)$ can be obtained from (5.3).

6. Numerical Illustrations

In this section, we describe the numerical approach which is applied for an optimal control problem considering yield loss. Numerical results presented here can be considered as generalization of [14] for yield loss case. To start the process we require input values of control variable influx $\lambda(t)$ from respectable admissible set. The amount of yield loss is considered as 20 percent of density. As described in previous sections, we obtain first variation and subsequently a system with associated adjoint variable as constraints. We discretize the hyperbolic PDE's of density and adjoint variable respectively with the input values of λ. The density will be evaluated forward in time while adjoint variable is computed backward in time. We minimize the mismatch between outflux and demand over a time period $t = 10$. Then, we can evaluate the descent direction for λ as $\delta\lambda$. We update the control variable influx λ as $\lambda^{new} = \lambda^{old} + c\delta\lambda$ so that λ^{new} belong to admissible set, where $c \in (0,1)$. We proceed in this way until the gradient of cost functionals becomes sufficiently small.

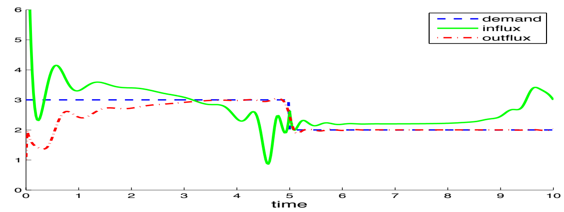


FIGURE 1. Demand, influx, outflux as a function of time with yield loss

We minimize the mismatch between outflux and demand over a time period $t = 10$. Then we can evaluate the descent direction for λ as $\delta\lambda$. We update the control variable influx λ as $\lambda^{new} = \lambda^{old} + c\delta\lambda$ so that λ^{new} belong to admissible set, where $c \in (0, 1)$. We proceed in this way until the gradient of cost functionals becomes sufficiently small.

Motivated by discontinuous nature of demand, for first experiment we have considered a demand function with a steep decrease at time $t = 5$. We start with constant influx $\lambda = 3$, initial density 1 and $v_{max} = 4$. Influx, outflux and demand are presented in Figure 1 as a function of time.

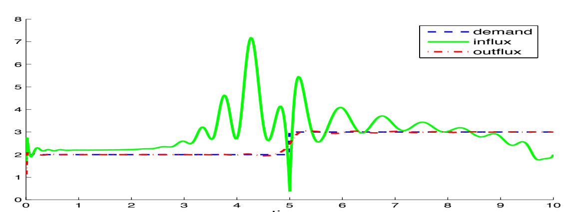


FIGURE 2. Influx, outflux with discontinuous demand

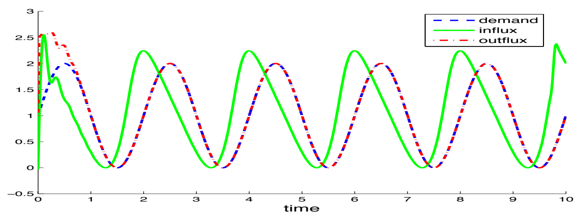


FIGURE 3. Influx, outflux with periodic demand

For second experiment, we have taken a demand function with a jump at time $t = 5$. Starting parameters except influx $\lambda = 2$, are remain same as of first case. Influx and outflux are displayed with discontinuous demand in Figure 2.

We consider a demand function with periodic nature. Demand rate and v_{max} is taken as $sin\pi t + 1$ and 5 respectively. Influx, outflux and demand are presented in Figure 3. The above figures demonstrate that we are able to generate outflux which can match the demand quite closely. It is also observed that initially there are mismatch between outflux and demand but as time progresses it has reduced significantly. Further, it is noticed that discontinuous demand leads to oscillation in influx of the system. Incorporating yield loss the presented results are quite satisfactory. In order to control the yield loss, improved optimization techniques are desirable. So we realize that several theoretical as well as numerical investigations are still to be done in this direction.

7. CONCLUSION

We have studied sensitivity analysis for an optimal control problem of production system. Special attention is given when the solution has discontinuities. By considering singular part at the shock locations, the analysis has been carried out in presence of shocks. Linearized equation is complemented by the equation of shock positions. We have discussed how to identify descent directions to find the minimizer of the optimal control problem. Numerical results are presented for yield loss case considering influx as a control variable. The presented results are new to the author knowledge. This also open several new possibilities in the area of optimal control for partial differential equation based production system models.

REFERENCES

1. D. Armbruster, P. Degond and R. Christian, A model for the dynamics of large queuing networks and supply chains, SIAM Journal on Applied Mathematics, 66 (2006), no.3, 896-920.
2. D. Armbruster, S. Göttlich and M. Herty, A scalar conservation law with discontinuous flux for supply chains with finite buffers, SIAM Journal on Applied Mathematics, 71(2011), no.4, 1070-1087.
3. A. Bressan and A. Marson, A variational calculus for discontinuous solutions of systems of conservation laws, Communications in partial differential equations 20(1995), no.9, 1491-1552.

4. C. Castro, F. Palacios and E. Zuazua, An alternating descent method for the optimal control of the inviscid Burgers equation in the presence of shocks, Mathematical Models and Methods in Applied Sciences 18(2008), no.3, 369-416.
5. C. Castro, F. Palacios and E. Zuazua, Optimal control and vanishing viscosity for the Burgers equation, Integral Methods in Science and Engineering, Volume 2, Springer, 2010, 65-90.
6. R. M. Colombo, M. Herty and M. Mercier, Control of the continuity equation with a non local flow, ESAIM: Control, Optimisation and Calculus of Variations 17(2011), no.2, 353-379.
7. C. D'Apice, R. Manzo and B. Piccoli, Optimal input flows for a PDE–ODE model of supply chains, Communication in Mathematical Sciences 10(2012), no. 4, 1225-1240.
8. C. D'Apice, R. Manzo and B. Piccoli, Numerical schemes for the optimal input flow of a supply chain, SIAM Journal on Numerical Analysis 51(2013), no.5, 2634-2650.
9. E. Godlewski and P. A. Raviart, The linearized stability of solutions of nonlinear hyperbolic systems of conservation laws: A general numerical approach, Mathematics and Computers in Simulation 50(1999), no.1, 77-95.
10. S. Göttlich, M. Herty, C. Ringhofer and U. Ziegler, Production systems with limited repair capacity, Optimization 61(2012), no.8, 915-948.
11. D. Jacquet, M. Krstic and C.C. de Wit, Optimal control of scalar one-dimensional conservation laws, American Control Conference, 2006, IEEE, 5213-5218.
12. C. Kirchner, M. Herty, S. Gottlich, Simone and A. Klar, Optimal control for continuous supply network models, Networks and Heterogeneous Media 1(2006), no.4, 675-688.
13. A. Kowalewski, I. Lasiecka and J. Sokołowski, Sensitivity analysis of hyperbolic optimal control problems, Computational Optimization and Applications 52(2012), no.1, 147-179.
14. M. L. Marca, D. Armbruster, M. Herty and C. Ringhofer, Control of continuum models of production systems, Automatic Control, IEEE Transactions on 55(2010), no.11, 2511-2526.
15. Y. Ou and P. Zhu, The vanishing viscosity method for the sensitivity analysis of an optimal control problem of conservation laws in the presence of shocks, Nonlinear Analysis: Real World Applications 14(2013), no.5, 1947-1974.
16. R. Pinnau and M. Ulbrich, Optimization with PDE constraints, vol.23, Springer, 2009.
17. T. Sarkar and S. Sundar, Conservation law model of serial supply chain network incorporating various velocity forms, International Journal of Applied Mathematics 26(2013), no.3, 363-378.
18. T. Sarkar and S. Sundar, Nonlinear conservation law model for production network considering yield loss, J. Nonlinear Sci. Appl. 7(2014), no.3, 205-217.
19. T. Sarkar and S. Sundar, On existence and stability analysis of a nonlinear conservation law model appearing in production system, Nonlinear Studies 21(2014), no.2, 339-347.
20. P. Shang and Z. Wang, Analysis and control of a scalar conservation law modeling a highly re-entrant manufacturing system, Journal of Differential Equations 250(2011), no.2, 949-982.
21. C. Song, Optimality Condition-Based Sensitivity Analysis of Optimal Control for Hybrid Systems and Its Application, Mathematical Problems in Engineering 22 pages, 2012.
22. S. Ulbrich, A sensitivity and adjoint calculus for discontinuous solutions of hyperbolic conservation laws with source terms, SIAM journal on control and optimization 41(2002), no.3, 740-797