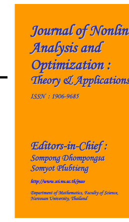
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol.6, No.1, (2015), 11-26
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



ON CHARACTERIZATION OF MULTIWAVELET PACKETS ASSOCIATED WITH A DILATION MATRIX

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ABSTRACT. We provide a complete characterization of multiwavelet packets associated with dilation matrix based on results on affine and quasi-affine frames. Moreover, these characterizations are valid without any decay assumptions on the generators of the system.

KEYWORDS : Multiresolution analysis; Multiwavelet packet; Dilation matrix; Frame; Bessel's sequence; Fourier transform.
AMS Subject Classification: 42C40 42C15 65T60.

1. INTRODUCTION

The fundamental idea of wavelet packet analysis is to construct a library of orthonormal bases for $L^2(\mathbb{R})$, which can be searched in real time for the best expansion with respect to a given application. Wavelet packets, due to their nice characteristics have been widely applied to signal processing, coding theory, image compression, fractal theory and solving integral equations and so on. Coifman *et al.*[8] firstly introduced the notion of univariate wavelet packets. Chui and Li [7] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be applied to the spline wavelets and so on. Shen [18] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets. Other notable generalizations are the p-wavelet packets and p-wavelet frame packets on a half-line $\mathbb{R}^+$ [13, 14, 16], higher dimensional wavelet packets with arbitrary dilation matrix [9], the orthogonal version of vector-valued wavelet packets [6] and the M-band framelet packets [17].

On the other hand, multiwavelets are natural extension and generalization of traditional wavelets. They have received considerable attention from the wavelet research communities both in the theory as well as in applications. They can be seen as vector valued-wavelets that satisfy conditions in which matrices are involved

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Article history : Received December 28, 2013 Accepted May 13, 2015.

rather than scalars as in the wavelet case. Multiwavelets can own symmetry, orthogonality, short support and high order vanishing moments, however traditional wavelets can not possess all these properties at the same time (see [4, 10]). Yang and Cheng [20] generalized the concept of wavelet packets to the case of multiwavelet packets associated with a dilation factor a which were more flexible in applications. Subsequently, Behera [1] extended the results of Yang and Cheng to the multivariate multiwavelet packets associated with a dilation matrix A. He proved lemmas on the so-called splitting trick and several theorems concerning the Fourier transform of the multiwavelet packets and the construction of multiwavelet packets to show that their translates form an orthonormal basis of $L^2(\mathbb{R}^d)$. Recently, Sun and Li [19] have given the construction and properties of generalized orthogonal multiwavelet packets based on the results discussed in [20].

As far as the characterization of multiwavelets is concerned, Calogero studied the characterization of all multiwavelets associated with general expanding maps of $\mathbb{R}^n$ in [5]. The Calogero's work was extended by Bownik [2], taking into consideration the dilation matrices which preserves the standard lattice $\mathbb{Z}^n$ in terms of affine systems. In the same year, another characterization of multiwavelets was given by Rzeszotnik [12] for expanding dilations that preserves the lattice $\mathbb{Z}^n$. However, Bownik [3] has presented a new approach to characterize all orthonormal multiwavelets by means of basic equations in the Fourier domain. This characterization was obtained by using the results about shift invariant systems and quasi-affine systems in [11].

The characterization of multiwavelet packets associated with the general dilation matrix A has been given by Shah and Ahmad in [15] by following dual Gramian approach of Bownik [2]. In the present paper, we study the characterization of multiwavelet packets associated with expansive dilation matrices in terms of the two simple equations in the Fourier domain based on results on affine and quasi-affine frames.

In order to make the paper self-contained, we state some basic preliminaries, notations and definitions including the multiresoltion analysis associated with a dilation matrix A and corresponding multiwavelet packets in Section 2. In Section 3, we establish the characterization of multiwavelet packets associated with a dilation matrix A based on results on affine and quasi-affine frames.

2. NOTATIONS AND PRELIMINARIES

Throughout, this paper, we use the following notations. Let $\mathbb{R}$ and $\mathbb{C}$ be all real and complex numbers, respectively. $\mathbb{Z}$ and $\mathbb{Z}^+$ denote all integers and all nonnegative integers, respectively. $\mathbb{Z}^d$ and $\mathbb{R}^d$ denote the set of all d-tuples integers and d-tuples of reals, respectively. Assume that we have a lattice Γ ($\Gamma = P\mathbb{Z}^d$ for some non-degenerate $d \times d$ matrix P) of $\mathbb{R}^d$. Let A denotes a $d \times d$ dilation matrix, whose determinant is $a (a \in \mathbb{Z}, a \geq 2)$. A $d \times d$ matrix A is said to be a dilation matrix for $\mathbb{R}^d$ if $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$ and all eigenvalues λ of A satisfy $|\lambda| > 1$. Let $a = |\det A|$, $B =$ transpose of A and, if A is expanding, so is B. Let Γ^* be the dual lattice; that is,

$$\Gamma^* = \left\{ \gamma' \in \mathbb{R}^d : \forall\, \gamma \in \Gamma\ \langle \gamma, \gamma' \rangle \in \mathbb{Z} \right\} = (P^t)^{-1}\mathbb{Z}^d.$$

By taking the transpose of $P^{-1}AP$ we observe that $B = A^t$ is a dilation preserving the dual lattice: $B\Gamma^* \subset \Gamma^*$ and let $\mathbb{S} = \Gamma^* \setminus B\Gamma^*$.

We recall the notion of higher dimensional multiresolution analysis associated with multiplicity L and orthogonal multiwavelets of $L^2(\mathbb{R}^d)$.

Definition 2.1. A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2\big(\mathbb{R}^d\big)$ is called a *multiresolution analysis* (MRA) of $L^2\big(\mathbb{R}^d\big)$ of multiplicity L associated with the dilation matrix A if the following conditions are satisfied:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2\big(\mathbb{R}^d\big)$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(iii) $f \in V_j$ if and only if $f(A\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(iv) there exist L-functions $\varphi_\ell \in V_0$, such that the system of functions $\{\varphi_\ell(x-k)\}_{\ell=1,k\in\mathbb{Z}^d}^{L}$, forms an orthonormal basis for subspace V_0.

The L-functions whose existence is asserted in (iv) are called *scaling functions* of the given MRA. Given a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$, we define another sequence $\{W_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2\big(\mathbb{R}^d\big)$ by $W_j = V_{j+1} \ominus V_j,\ j \in \mathbb{Z}$. These subspaces inherit the scaling property of $\{V_j\}$, namely

$$f \in W_j \quad \text{if and only if} \quad f(A\cdot) \in W_{j+1}. \tag{2.1}$$

Further, they are mutually orthogonal, and we have the following orthogonal decompositions:

$$L^2(\mathbb{R}^d) = \bigoplus_{j\in\mathbb{Z}} W_j = V_0 \oplus \Big(\bigoplus_{j\geq 0} W_j\Big). \tag{2.2}$$

A set of functions $\{\psi_\ell^r : 1 \le \ell \le L, 1 \le r \le a-1\}$ in $L^2(\mathbb{R}^d)$ is said to be a set of *basic multiwavelets* associated with the MRA of multiplicity L if the collection

$$\Big\{\psi_\ell^r(.-k) : 1 \le r \le a-1, 1 \le \ell \le L,\ k \in \mathbb{Z}^d\Big\}$$

forms an orthonormal basis for W_0. Now, in view of (2.1) and (2.2), it is clear that if $\{\psi_\ell^r : 1 \le \ell \le L, 1 \le r \le a-1\}$ is a basic set of multiwavelets, then

$$\Big\{|\det A|^{j/2}\psi_\ell^r(A^j.-k) : j \in \mathbb{Z},\ k \in \mathbb{Z}^d, 1 \le \ell \le L, 1 \le r \le a-1\Big\}$$

forms an orthonormal basis for $L^2(\mathbb{R}^d)$ (see [1, 4]).

For any $n \in \mathbb{Z}^+$, we define the *basic multiwavelet packets* $\omega_\ell^n;\ 1 \le \ell \le L$ recursively as follows. We denote $\omega_\ell^0 = \varphi_\ell,\ 1 \le \ell \le L$, the scaling functions and $\omega_\ell^r = \psi_\ell^r, r \in \mathbb{Z}^+, 1 \le \ell \le L$ as the possible candidates for basic multiwavelets. Then, define

$$\omega_\ell^{s+ar}(x) = \sum_{j=1}^{L}\sum_{k\in\mathbb{Z}^d} h_{\ell jk}^s\, a^{1/2}\, \omega_\ell^r(Ax-k),\ 1 \le \ell \le L, 0 \le s \le a-1 \tag{2.3}$$

where $\Big(h_{\ell jk}^s\Big)$ is a unitary matrix (see [1]).

Taking Fourier transform on both sides of (2.3), we obtain

$$\left(\omega_\ell^{s+ar}\right)^\wedge(\xi) = \sum_{j=1}^{L} h_{\ell j}^s(B^{-1}\xi)\left(\omega_\ell^r\right)^\wedge(B^{-1}\xi). \tag{2.4}$$

Note that (2.3) defines ω_ℓ^n for every non-negative integer n and every ℓ such that $1 \le \ell \le L$. The set of functions $\{\omega_\ell^n : n \in \mathbb{Z}^+, 1 \le \ell \le L\}$ as defined above are called the *basic multiwavelet packets* corresponding to the MRA $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^d)$ of multiplicity L associated with matrix dilation A.

Definition 2.2. Let $\{\omega_\ell^n : n \in \mathbb{Z}^+, 1 \le \ell \le L\}$ be the basic multiwavelet packets. The collection

$$\mathcal{P} = \left\{|\det A|^{j/2}\omega_\ell^n(A.-\,k) : 1 \le \ell \le L, j \in \mathbb{Z}, k \in \mathbb{Z}^d\right\}$$

is called the *general multiwavelet packets* associated with MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R}^d)$ of multiplicity L over matrix dilation A.

Corresponding to some orthonormal scaling vector $\Phi = \omega_\ell^0$, the family of multiwavelet packets ω_ℓ^n defines a family of subspaces of $L^2(\mathbb{R}^d)$ as follows:

$$U_j^n = \overline{\text{span}}\left\{|\det A|^{j/2}\omega_\ell^n(A^jx-k) : k \in \mathbb{Z}^d, 1 \le \ell \le L\right\};\ j \in \mathbb{Z},\ n \in \mathbb{Z}^+. \tag{2.5}$$

Observe that

$$U_j^0 = V_j, \quad U_j^1 = W_j = \bigoplus_{r=1}^{a-1} U_j^r, \quad j \in \mathbb{Z}$$

so that the orthogonal decomposition $V_{j+1} = V_j \oplus W_j$, can be written as

$$U_{j+1}^0 = \bigoplus_{r=0}^{a-1} U_j^r. \tag{2.6}$$

A generalization of this result for other values of $n = 1, 2, \ldots$ can be written as

$$U_{j+1}^n = \bigoplus_{r=0}^{a-1} U_j^{an+r}, \quad j \in \mathbb{Z}. \tag{2.7}$$

The following proposition is proved in [1].

Proposition 2.3. If $j \ge 0$, then

$$W_j = \bigoplus_{r=0}^{a-1} U_j^r = \bigoplus_{r=a}^{a^2-1} U_{j-1}^r = \cdots = \bigoplus_{r=a^t}^{a^{t+1}-1} U_{j-t}^r = \bigoplus_{r=a^j}^{a^{j+1}-1} U_0^r$$

where U_j^n is defined in (2.5). Using this decomposition, we get the multiwavelet packets decomposition of subspaces $W_j,\ j \ge 0$.

Let $\{\omega_\ell^n : n \ge 0, 1 \le \ell \le L\}$ be a family of functions in $L^2(\mathbb{R}^d)$. Then, the *affine system* generated by ω_ℓ^n and associated with (A, Γ) is the collection

$$\mathcal{F}(\omega_\ell^n) = \Big\{\omega_{\ell,j,k}^n : j \in \mathbb{Z},\ k \in \Gamma, 1 \le \ell \le L, a^j \le n < a^{j+1}\Big\}, \tag{2.8}$$

where $\omega_{\ell,j,k}^n(x) = D_{A^j} T_k\, \omega_\ell^n(x) = |\det A|^{j/2}\omega_\ell^n(A^j x - k)$. The *quasi-affine system* generated by ω_ℓ^n is

$$\mathcal{F}^q(\omega_\ell^n) = \Big\{\tilde{\omega}_{\ell,j,k}^n : j \in \mathbb{Z}, k \in \Gamma,\ 1 \le \ell \le L, a^j \le n < a^{j+1}\Big\}, \tag{2.9}$$

where

$$\tilde{\omega}_{\ell,j,k}^n(x) = \begin{cases} D_{A^j} T_k\, \omega_\ell^n(x) = |\det A|^{j/2}\omega_\ell^n(A^j x - k), & j \ge 0,\ k \in \Gamma, \\ |\det A|^{j/2} T_k D_{A^j}\omega_\ell^n(x) = |\det A|^j \omega_\ell^n\big(A^j(x-k)\big), & j < 0,\ k \in \Gamma, \end{cases}$$

where $\tau_y f(x) = f(x-y)$ is translation by a vector $y \in \mathbb{R}^d$ and $D_{A^j} f(x) = |\det A|^{j/2} f(Ax)$ is dilation by the matrix A. Since A is a dilation matrix, $A^t = B$ so there exist constants $\lambda > 1$ and $c > 0$ such that

$$\left|B^j \xi\right| > c\lambda^j |\xi|, \quad \left|B^{-j}\xi\right| < 1/c\lambda^{-j}|\xi| \text{ for } j > 0. \tag{2.10}$$

The following two lemma's are proved in [2].

Lemma 2.4. Suppose $b > 0, g \in L^\infty(\mathbb{R}^d)$, $\operatorname{supp} g \subset \left\{\xi \in \mathbb{R}^d : |\xi| > b\right\}$, and $\operatorname{supp} g \subset B^{j_0} I_d + \xi_0$ for some $\xi_0 \in \mathbb{R}^d$ and $j_0 \in \mathbb{Z}$, then

$$\sum_{j\in\mathbb{Z}} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |\det A|^i \Big| g(B^j \xi) g\big(B^j(\xi{+}k)\big)\Big| \le 2^d |\det A|^{j_0} M\Big((b{+}\delta)/b\Big) \|g\|_\infty^2 I_\Upsilon(\xi), a.e.\ \xi \in \mathbb{R}^d$$

where $\delta = \text{diam}(B^{j_0} I_d), \Upsilon = \bigcup_{j<J_0} B^{-j}(B^{j_0} I_d + \xi_0)$ and $I_d = (-1/2, 1/2)^d$.

Lemma 2.5. Suppose $F, G \in L^2(\mathbb{R}^d)$, and $\operatorname{supp} F,\ \operatorname{supp} G$ are bounded. Then

$$\sum_{k\in\mathbb{Z}^d} \hat{F}(k)\overline{\hat{G}(k)} = \int_{\mathbb{R}^d} \left(\sum_{\ell \in \mathbb{Z}^d} F(\xi + \ell)\right) \overline{G(\xi)}\, d\xi.$$

Definition 2.6. Let $\mathbb{H}$ be a separable Hilbert space. A sequence $\{f_k\}_{k=1}^\infty$ in $\mathbb{H}$ is called a *frame* if there exist constants A and B with $0 < A \le B < \infty$ such that

$$A\big\|f\big\|^2 \le \sum_{k=1}^\infty \big|\langle f, f_\alpha\rangle\big|^2 \le B\big\|f\big\|^2, \quad \text{for all } f \in \mathbb{H}. \tag{2.11}$$

The largest constant A and the smallest constant B satisfying (2.11) are called the *upper* and the *lower frame bound*, respectively. The sequence $\{f_k\}_{k=1}^\infty$ is called a *Bessel sequence* in $\mathbb{H}$ if only the right-hand side inequality in (2.11) holds. The sequence $\{f_k\}_{k=1}^\infty$ is called a *tight frame* for $\mathbb{H}$ if the upper frame bound A and the lower frame bound B coincide. A frame is called *Parseval frame* or *normalized tight frame* if $A = B = 1$ and in this case, every function $f \in \mathbb{H}$ can be written as

$$\sum_{k=1}^\infty \big|\langle f, f_\alpha\rangle\big|^2 = \big\|f\big\|^2. \tag{2.12}$$

The following theorem gives us an elementary characterization of tight frames.

Theorem 2.7. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in a Hilbert space $\mathbb{H}$ such that

(i) $\displaystyle\sum_{k=1}^{\infty} \left|\langle f, f_k\rangle\right|^2 = \|f\|^2, \quad \text{for all } f \in \mathbb{H}$

(ii) $\|f_k\| \geq 1, \text{ for } k \in \mathbb{Z}^+.$

Then, the sequence $\{f_k\}_{k=1}^{\infty}$ forms a Parseval's frame for $\mathbb{H}$.

We will also consider the set $\mathcal{D}$ as a dense subset of $L^2(\mathbb{R}^d)$ defined by

$$\mathcal{D} = \left\{ f \in L^2(\mathbb{R}^d) : \hat{f} \in L^\infty(\mathbb{R}^d), \text{ supp } \hat{f} \text{ for some compact } K \subset \mathbb{R}^d \setminus \{0\} \right\}.$$

3. CHARACTERIZATION OF MULTIWAVELET PACKETS

In this section, we prove our main results concerning the characterization of multiwavelet packets associated with a dilation matrix A by means of the Fourier transform. We begin this section with the lemma which gives necessary condition for the system $\mathcal{F}(\omega_\ell^n)$ given by (2.8) to be a Bessel family.

Lemma 3.1. *Let $\{\omega_\ell^n : n \in \mathbb{Z}^+, 1 \leq \ell \leq L\}$ be the basic multiwavelet packets associated with the scaling functions φ_ℓ. Then, for $f \in \mathcal{D}$ and $m \in \mathbb{Z}$, we have*

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j \leq m} \sum_{k \in \mathbb{Z}^d} \left|\langle f, \omega_{\ell,j,k}^n\rangle\right|^2 < \infty.$$

Moreover,

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \left|\omega_\ell^n\left(B^j\xi\right)\right|^2, \quad \textit{is locally integrable on } \mathbb{R}^d \setminus \{0\} \tag{3.1}$$

if and only if

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j,k \in \mathbb{Z}^d} \left|\langle f, \omega_{\ell,j,k}^n\rangle\right|^2 < \infty \quad \textit{for all } f \in \mathcal{D}. \tag{3.2}$$

Proof. Since $\hat{\omega}_{\ell,j,k}^n(\xi) = |\det A|^{-j/2} \omega_\ell^n(B^{-j}\xi) e^{-2\pi i\langle k, B^{-j}\xi\rangle}$. Therefore, by applying Parseval's formula, we obtain

$$\begin{aligned}
\langle f, \omega_{\ell,j,k}^n\rangle = \langle \hat{f}, \hat{\omega}_{\ell,j,k}^n\rangle &= |\det A|^{-j/2} \int_{\mathbb{R}^d} \hat{f}(\xi)\, \overline{\hat{\omega}_\ell^n\left(B^{-j}\xi\right)}\, e^{2\pi i\langle k, B^{-j}\xi\rangle}\, d\xi \\
&= |\det A|^{-j/2} \int_{\mathbb{R}^d} \hat{f}\left(B^j\xi\right) \overline{\hat{\omega}_\ell^n(\xi)}\, e^{2\pi i\langle k, \xi\rangle} |\det A|^j\, d\xi \\
&= |\det A|^{j/2} \int_{\mathbb{R}^d} \hat{f}\left(B^j\xi\right) \overline{\hat{\omega}_\ell^n(\xi)}\, e^{2\pi i\langle k, \xi\rangle}\, d\xi.
\end{aligned} \tag{3.3}$$

With the help of (3.3), we can write the series as

$$I=\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}\sum_{k\in\mathbb{Z}^d}\left|\langle f,\omega^n_{\ell,j,k}\rangle\right|^2$$

$$=\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}\sum_{k\in\mathbb{Z}^d}|\det A|^j\left|\int_{\mathbb{R}^d}\hat f\big(B^j\xi\big)\,\overline{\hat\omega^n_\ell(\xi)}\,e^{2\pi i\langle k,\xi\rangle}\,d\xi\right|^2. \tag{3.4}$$

For any fixed $j\in\mathbb{Z}$, let $F(\xi)\equiv\hat f(B^j)\,\overline{\hat\omega^n_\ell(\xi)}$; then by Lemma 2.5 when $F=G$, we have

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\int_{\mathbb{R}^d}\hat f\big(B^j\xi\big)\,\overline{\hat\omega^n_\ell(\xi)}\,e^{2\pi i\langle k,\xi\rangle}d\xi\right|^2$$

$$=\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\left\{\int_{\mathbb{R}^d}\hat f\big(B^j\xi\big)\,\hat\omega^n_\ell(\xi)\left(\sum_{k\in\mathbb{Z}^d}\hat f\big(B^j(\xi+k)\big)\overline{\hat\omega^n_\ell(\xi+k)}\right)d\xi\right\}.$$

Hence

$$I=\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}|\det A|^j\int_{\mathbb{R}^d}\left|\hat f\big(B^j\xi\big)\right|^2\left|\hat\omega^n_\ell(\xi)\right|^2d\xi+\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}|\det A|^j$$

$$\times\int_{\mathbb{R}^d}\overline{\hat f\big(B^j\xi\big)}\,\hat\omega^n_\ell(\xi)\left\{\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\hat f\big(B^j(\xi+k)\big)\,\overline{\hat\omega^n_\ell(\xi+k)}\right\}d\xi. \tag{3.5}$$

Note that for any $\ell=1,\dots,L$ and $n\in\mathbb{Z}^+$, we have

$$2\left|\hat\omega^n_\ell(\xi)\hat\omega^n_\ell(\xi+k)\right|\le\left|\hat\omega^n_\ell(\xi)\right|^2+\left|\hat\omega^n_\ell(\xi+k)\right|^2.$$

Therefore, the second sum is absolutely convergent in $L^1(\mathbb{R}^d)$ and, thus absolutely summable for $a.e.\ \xi\in\mathbb{R}^d$ even if we extend the summation over all $j\in\mathbb{Z}$; i.e.,

$$\int_{\mathbb{R}^d}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d\setminus\{0\}}|\det A|^j\left|\hat f\big(B^j\xi\big)\,\hat\omega^n_\ell(\xi)\right|\left|\hat f\big(B^j(\xi+k)\big)\,\hat\omega^n_\ell(\xi+k)\right|d\xi$$

$$\le\frac12\int_{\mathbb{R}^d}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d\setminus\{0\}}|\det A|^j\left[\left|\hat f\big(B^j\xi\big)\hat f\big(B^j(\xi+k)\big)\right|+\left|\hat f\big(B^j(\xi-k)\big)\hat f\big(B^j\xi\big)\right|\right]$$

$$=\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\int_{\mathbb{R}^d}\left|\hat\omega^n_\ell(\xi)\right|^2\left\{\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d\setminus\{0\}}|\det A|^j\left|\hat f\big(B^j\xi\big)\hat f\big(B^j(\xi+k)\big)\right|\right\}d\xi$$

$$\leq C\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\int_{\mathbb{R}^d}\left|\hat{\omega}_\ell^n(\xi)\right|^2 d\xi<\infty, \tag{3.6}$$

where C is the constant appearing in Lemma 2.4 depending on the size and the location of supp $\hat{f}$. Furthermore, the first sum appearing in (3.5) can be estimated crudely by

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\leq m}|\det A|^j\int_{\mathbb{R}^d}\left|\hat{f}\big(B^j\xi\big)\right|^2\left|\hat{\omega}_\ell^n(\xi)\right|^2 d\xi\leq\|\hat{f}\|_\infty^2\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\leq m}|\det A|^j\int_{\mathbb{R}^d}\left|\hat{\omega}_\ell^n(\xi)\right|^2 d\xi$$

$$=\frac{|\det A|^{m+1}}{|\det A|-1}\|\hat{f}\|_\infty^2\|\omega_\ell^n\|^2. \tag{3.7}$$

In order to prove the second part of the theorem, we have

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j,k\in\mathbb{Z}^d}\left|\langle f,\omega_{\ell,j,k}^n\rangle\right|^2=\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}|\det A|^j\int_{\mathbb{R}^d}\left|\hat{f}\big(B^j\xi\big)\right|^2\left|\hat{\omega}_\ell^n(\xi)\right|^2 d\xi$$

$$+\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}|\det A|^j\int_{\mathbb{R}^d}\overline{\hat{f}\big(B^j\xi\big)}\hat{\omega}_\ell^n(\xi)\left\{\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\hat{f}\big(B^j(\xi+k)\big)\overline{\hat{\omega}_\ell^n(\xi+k)}\right\}d\xi,$$

where the second expression in this decomposition is always finite by (3.6). Thus, the first implication follows from the fact that

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}^d}|\det A|^j\int_{\mathbb{R}^d}\left|\hat{f}\big(B^j\xi\big)\right|^2\left|\hat{\omega}_\ell^n(\xi)\right|^2 d\xi=\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\left|\hat{f}(\xi)\right|^2\left|\hat{\omega}_\ell^n\big(B^{-j}\xi\big)\right|^2 d\xi$$

$$\leq\|\hat{f}\|_\infty^2\int_{\text{supp}\hat{f}}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}^d}\left|\hat{\omega}_\ell^n\big(B^{-j}\xi\big)\right|^2 d\xi<\infty,$$

where as the converse implication is simply the consequence of applying the above to $\hat{f}=\chi_K$ for any compact $K\subset\mathbb{R}^d\setminus\{0\}$, since we have equality (instead of inequality) in the above formula. $\square$

Theorem 3.2. *Let* $\{\omega_\ell^n : n\in\mathbb{Z}^+, 1\leq\ell\leq L\}$ *and* $\{\tilde{\omega}_\ell^n : n\in\mathbb{Z}^+, 1\leq\ell\leq L\}$ *be the dual multiwavelet packets associated with the dilation matrix* A. *Then*

$$\lim_{m\to\infty}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\leq m}\sum_{k\in\mathbb{Z}^d}\langle f,\omega_{\ell,j,k}^n\rangle\langle\tilde{\omega}_{\ell,j,k}^n,f\rangle=\|f\|_2^2,\quad \textit{for all } f\in\mathcal{D} \tag{3.8}$$

if and only if

$$\lim_{m\to\infty}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq -m}\hat{\omega}_\ell^n\left(B^j\xi\right)\overline{\hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right)}=1,\quad \textit{weakly in } L^1(K),\ K\subset\mathbb{R}^d\setminus\{0\} \tag{3.9}$$

$$t_s(\xi)=\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right)\overline{\hat{\omega}_\ell^n\left(B^j(\xi+s)\right)}=0,\ a.e.\ \xi\in\mathbb{R}^d,\ s\in\mathbb{S}=\mathbb{Z}^d\setminus B\mathbb{Z}^d. \tag{3.10}$$

Proof. We first show that the series given by (3.8), (3.9) and (3.10) are all absolutely convergent. Since

$$2\left|\langle f,\omega_{\ell,j,k}^n\rangle\langle\tilde{\omega}_{\ell,j,k}^n,f\rangle\right|\leq\left|\langle f,\omega_{\ell,j,k}^n\rangle\right|^2+\left|\langle\tilde{\omega}_{\ell,j,k}^n,f\rangle\right|^2.$$

Therefore, the series in (3.8) is summable by Lemma 3.1. Moreover, by the polarization identity, condition (3.8) is equivalent to

$$\langle f,g\rangle=\lim_{m\to\infty}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\leq m}\sum_{k\in\mathbb{Z}^d}\langle f,\omega_{\ell,j,k}^n\rangle\langle\tilde{\omega}_{\ell,j,k}^n,g\rangle\quad\text{for all } f,g\in\mathcal{D}. \tag{3.11}$$

Thus, for $s\in\mathbb{R}^d$ and $\omega_\ell^n\in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}\int_{\mathbb{R}^d}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq -m}\left|\hat{\omega}_\ell^n\left(B^j(\xi+s)\right)\right|^2d\xi &= \int_{\mathbb{R}^d}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{-j\leq m}|\det A|^{-j}\left|\hat{\omega}_\ell^n\left(\xi+B^js\right)\right|^2d\xi\\ &= \frac{|\det A|^{m+1}}{|\det A|-1}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\int_{\mathbb{R}^d}\left|\hat{\omega}_\ell^n(\xi)\right|^2d\xi<\infty.\end{aligned}$$

Therefore, we have

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq -m}\left|\hat{\omega}_\ell^n\left(B^j(\xi+s)\right)\right|^2<\infty\quad\text{for } a.e.\ \xi. \tag{3.12}$$

Using the above when $s=0$ yields

$$2\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq -m}\left|\hat{\omega}_\ell^n\left(B^j\xi\right)\overline{\hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right)}\right|^2\leq\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq -m}\left|\hat{\omega}_\ell^n\left(B^j\xi\right)\right|^2+\left|\hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right)\right|^2<\infty,\ a.e.\ \xi.$$

Similarly, implementation of (3.12) when $m=0$ implies

$$2\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\left|\hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right)\overline{\hat{\omega}_\ell^n\left(B^j(\xi+s)\right)}\right|^2\leq\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\left|\hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right)\right|^2+\left|\hat{\omega}_\ell^n\left(B^j(\xi+s)\right)\right|^2<\infty.$$

Next, we prove that (3.9) and (3.10) implies (3.8). To do so, let us suppose that $f, g \in \mathcal{D}$. Then, by equation (3.3), we have

$$\langle f, \omega^n_{\ell,j,k}\rangle\langle\tilde{\omega}^n_{\ell,j,k}, g\rangle = |\det A|^j \int_{\mathbb{R}^d} \hat{f}\big(B^j\xi\big)\,\overline{\hat{\omega}^n_\ell(\xi)}\,e^{2\pi i\langle k,\xi\rangle}d\xi \int_{\mathbb{R}^d} \overline{\hat{g}(B^j\xi)}\hat{\tilde{\omega}}^n_\ell(\xi)e^{-2\pi i\langle k,\xi\rangle}d\xi.$$

For any fixed $\ell = 1, \ldots, L$ and $j \in \mathbb{Z}$, let

$$F(\xi) \equiv \hat{f}(B^j\xi)\,\overline{\omega^n_\ell(\xi)}, \quad G(\xi) \equiv \hat{g}(B^j\xi)\,\overline{\tilde{\omega}^n_\ell(\xi)},\ n \in \mathbb{Z}^+.$$

Then, using the Lemma 2.5 and the above fact, we obtain

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^d}\langle f, \omega^n_{\ell,j,k}\rangle\langle\tilde{\omega}^n_{\ell,j,k}, g\rangle = \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\int_{\mathbb{R}^d}\left\{\sum_{k\in\mathbb{Z}^d}\hat{f}\big(B^j(\xi+k)\big)\overline{\omega^n_\ell(\xi+k)}\right\} \times \overline{\hat{g}(B^j\xi)}\hat{\tilde{\omega}}^n_\ell(\xi)\,d\xi. \tag{3.13}$$

Hence

$$I = I(m) = \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}\sum_{k\in\mathbb{Z}^d}\langle f, \omega^n_{\ell,j,k}\rangle\langle\tilde{\omega}^n_{\ell,j,k}, g\rangle = I_1 + I_2 \tag{3.14}$$

where

$$I_1(m) = \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}|detA|^j\int_{\mathbb{R}^d}\hat{f}\big(B^j\xi\big)\,\overline{\hat{g}(B^j\xi)}\,\overline{\hat{\omega}^n_\ell(\xi)}\,\hat{\tilde{\omega}}^n_\ell(\xi)\,d\xi$$

$$I_2(m) = \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}|detA|^j\int_{\mathbb{R}^d}\overline{\hat{g}(B^j\xi)}\,\hat{\tilde{\omega}}^n_\ell(\xi)\left\{\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\hat{f}\big(B^j(\xi+k)\big)\,\overline{\omega^n_\ell(\xi+k)}\right\}d\xi$$

by splitting the sum (3.13) into terms corresponding to $k = 0$ and $k \neq 0$. Moreover, we can interchange the summation and integration in I_1 and I_2, since for $h \in \mathcal{D}$, defined by $\hat{h} = \max(|\hat{f}|, |\hat{g}|)$, we have

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}|detA|^j\int_{\mathbb{R}^d}\left|\hat{h}\big(B^j\xi\big)\right|^2\big|\hat{\omega}^n_\ell(\xi)\hat{\tilde{\omega}}^n_\ell(\xi)\big|d\xi < \infty$$

and

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}|detA|^j\int_{\mathbb{R}^d}\left|\hat{h}\big(B^j\xi\big)\hat{\tilde{\omega}}^n_\ell(\xi)\right|\left\{\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\left|\hat{h}\big(B^j(\xi+k)\big)\hat{\omega}^n_\ell(\xi+k)\right|\right\}d\xi < \infty. \tag{3.15}$$

Now, in order to estimate (3.15), we use (3.6), (3.7) and the fact that

$$2\big|\hat{\omega}^n_\ell(\xi)\hat{\tilde{\omega}}^n_\ell(\xi)\big| \le \big|\hat{\omega}^n_\ell(\xi)\big|^2 + \big|\hat{\tilde{\omega}}^n_\ell(\xi)\big|^2 \quad \text{and} \quad 2\big|\hat{\tilde{\omega}}^n_\ell(\xi)\hat{\omega}^n_\ell(\xi+k)\big| \le \big|\hat{\tilde{\omega}}^n_\ell(\xi)\big|^2 + \big|\hat{\omega}^n_\ell(\xi+k)\big|^2.$$

Therefore, we can manipulate the sums as

$$\begin{aligned}
I_2 &= \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}|detA|^j\int_{\mathbb{R}^d}\overline{\hat{g}(B^j\xi)}\,\hat{\omega}_\ell^n(\xi)\left\{\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\hat{f}\big(B^j(\xi+k)\big)\overline{\hat{\omega}_\ell^n(\xi+k)}\right\}d\xi\\
&= \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}\int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\,\hat{\omega}_\ell^n\big(B^{-j}\xi\big)\left\{\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\hat{f}\big(\xi+B^jk\big)\,\overline{\hat{\omega}_\ell^n(B^{-j}\xi+k)}\right\}d\xi\\
&= \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}\int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\,\hat{\omega}_\ell^n\big(B^{-j}\xi\big)\sum_{r\ge 0}\sum_{s\in\mathbb{S}}\hat{f}\big(\xi+B^jB^rs\big)\,\overline{\hat{\omega}_\ell^n(B^{-j}\xi+B^rs)}d\xi\\
&= \int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{s\in\mathbb{S}}\sum_{r\ge 0}\sum_{j\le m}\hat{\omega}_\ell^n\big(B^r(B^{-r-j}\xi)\big)\hat{f}\big(\xi+B^{j+r}s\big)\overline{\hat{\omega}_\ell^n\big(B^r(B^{-r-j}\xi+s)\big)}d\xi\\
&= \int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{s\in\mathbb{S}}\sum_{r\ge 0}\sum_{p\le m+r}\hat{\omega}_\ell^n\big(B^r(B^{-p}\xi)\big)\overline{\hat{\omega}_\ell^n(B^r(B^{-p}\xi+s))}\,\hat{f}\big(\xi+B^ps\big)\,d\xi\\
&= \int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{s\in\mathbb{S}}\sum_{r\ge 0}\sum_{p\in\mathbb{Z}}\hat{\omega}_\ell^n\big(B^r(B^{-p}\xi)\big)\overline{\hat{\omega}_\ell^n\big(B^r(B^{-p}\xi+s)\big)}\,\hat{f}\big(\xi+B^ps\big)d\xi,
\end{aligned}$$

for m sufficiently large so that $\hat{g}(\xi)\hat{f}(\xi+B^ps)=0$ for all $p\ge m, s\in\mathbb{S}$, i.e., $(\text{supp}\hat{f}-\text{supp}\,\hat{g})\cap B^p\mathbb{S}=\emptyset$ for all $p\ge m$. Now, if we take, $b=\sup\Big\{|\xi|:\xi\in(\text{supp}\,\hat{f}-\text{supp}\,\hat{g})\Big\}$; then, by (2.10) any $m\ge\big[\log_\lambda(b/c)\big]$ works. Therefore, for any $f,g\in\mathcal{D}$ and sufficiently large m, we have

$$I(m)=I_1(m)+I_2(m),$$

where

$$\begin{aligned}
I_1(m) &= \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\ge -m}\int_{\mathbb{R}^d}\hat{f}(\xi)\,\overline{\hat{g}(\xi)}\;\overline{\hat{\omega}_\ell^n(B^j\xi)}\,\hat{\omega}_\ell^n(B^j\xi)\,d\xi\\
I_2(m) &= \int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\sum_{p\in\mathbb{Z}}\sum_{s\in\mathbb{S}}\hat{f}\big(\xi+B^ps\big)\,t_s\big(B^{-p}\xi\big)\,d\xi. \qquad (3.16)
\end{aligned}$$

Here I_1 follows by a simple change of variables, and I_2 does not depend on m. Equation (3.16), combined with assumptions (3.9) and (3.10) immediately implies

$$\lim_{m\to\infty}I(m)=\lim_{m\to\infty}I_1(m)+I_2(m)=\Big\langle\hat{f},\hat{g}\Big\rangle=\langle f,g\rangle.$$

Conversely, we shall prove that (3.8) implies (3.10). For any fixed $s_0\in\mathbb{S}$ and $q>0$, we define

$$\Omega(q)=\Big\{\xi\in\mathbb{R}^d:|\xi|>q,\,|\xi+s_0|>q\Big\}.$$

Now for any $\xi_0\in\Omega(q)$ and $j\ge 0$, define

$\hat{f}_j(\xi) = |B^{-j} I_d|^{-1/2} \arg t_{s_0}(\xi)\chi_{B^{-j}I_d+\xi_0}(\xi)$ and $\hat{g}_j(\xi) = |B^{-j} I_d|^{-1/2}\chi_{B^{-j}I_d+\xi_0+s_0}(\xi)$,

where for the purpose of the proof, we define, for $z \in \mathbb{C}$,

$$\arg z = \begin{cases} z/|z|, & z \neq 0 \\ 1, & z = 0. \end{cases}$$

By separating the term corresponding to $p = 0$ and $s = s_0$ in equation (3.16) for $I_2(m)$, $f = f_j, g = g_j$, from the rest, which we denote by $R(j)$, we have

$$I_2(m) = \frac{1}{|B^{-j} I_d|}\int_{B^{-j}I_d+\xi_0} \left|t_{s_0}(\xi)\right| d\xi + \int_{\mathbb{R}^d} \overline{\hat{g}_j(\xi)} \sum_{\substack{p\in\mathbb{Z}, s\in\mathbb{S} \\ (p,s)\neq(0,s_0)}} \hat{f}_j\big(\xi+B^p s\big)t_s\big(B^{-p}\xi\big)\, d\xi. \tag{3.17}$$

Next, if $|\hat{g}_j(\xi)\hat{f}_j(\xi+B^p s)| \neq 0$ for some $\xi \in \mathbb{R}^d$, then $(B^{-j}I_d+\xi_0)\cap(B^{-j}I_d+\xi_0+s_0-B^p s) \neq \emptyset$, hence $B^{-j}(2I_d)\cap(s_0 - B^p\mathbb{S}) \neq \emptyset$ which means $2I_d \cap (B^j s_0 - B^{p+j}\mathbb{S}) \neq \emptyset$. Also, if $p + j \geq 0$, then $B^j s_0 - B^{p+j}\mathbb{S} \subset \mathbb{Z}^d$, and since $2I_d \cap \mathbb{Z}^d = \{0\}$, $s_0 \notin B^p\mathbb{S}$ for $p \neq 0$, the only nonzero term happens for $p = 0$ and $s = s_0$. Therefore, the other nonzero terms can contribute only if $p + j < 0$, so we can restrict the sum in (3.17) to $p < -j$.

Using the estimate

$$2\left|t_s(\xi)\right| \leq \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{m'\geq 0} \left|\hat{\omega}_\ell^n\big(B^{m'}\xi\big)\right|^2 + \left|\hat{\omega}_\ell^n\big(B^{m'}(\xi+s)\big)\right|^2 \leq T(\xi) + T(\xi+s),$$

where

$$T(\xi) \equiv \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{m'\geq 0} \left|\hat{\omega}_\ell^n\big(B^{m'}\xi\big)\right|^2 + \left|\hat{\omega}_\ell^n\big(B^{m'}\xi\big)\right|^2, \quad \text{is locally integrable on } \mathbb{R}^d.$$

Therefore, we have

$$\begin{aligned} |R(j)| &\leq \frac{1}{2}\int_{\mathbb{R}^d}\sum_{p<-j}\sum_{s\in\mathbb{S}} |\det A|^p \left|\hat{g}_j\big(B^p\xi\big)\right|\left|\hat{f}_j\big(B^p(\xi+s)\big)\right| T(\xi) d\xi \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^d}\sum_{p<-j}\sum_{s\in\mathbb{S}} |\det A|^p \left|\hat{g}_j\big(B^p\xi\big)\right|\left|\hat{f}_j\big(B^p(\xi+s)\big)\right| T(\xi+s) d\xi \\ &= \frac{1}{2}\int_{\mathbb{R}^d}\sum_{p<-j}\sum_{s\in\mathbb{S}} |\det A|^p \left|\hat{g}_j\big(B^p\xi\big)\right|\left|\hat{f}_j\big(B^p(\xi+s)\big)\right| T(\xi) d\xi \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^d}\sum_{p<-j}\sum_{s\in\mathbb{S}} |\det A|^p \left|\hat{g}_j\big(B^p(\xi-s)\big)\right|\left|\hat{f}_j\big(B^p\xi\big)\right| T(\xi) d\xi. \end{aligned} \tag{3.18}$$

Using Lemma 2.4 with the assumptions that $v > 0$, where $v = v(j) = \inf\big\{|\xi| : \xi \in B^{-j} I_d + \xi_0\big\}$, $\delta = \delta(j) = \operatorname{diam}(B^{-j} I_d), \Upsilon = \Upsilon(j) = \bigcup_{p<-j} B^{-p}(B^{-j} I_d + \xi_0)$ and the fact that $\big|\hat{f}_j(\xi)\big| = \big|\hat{g}_j(\xi - s_0)\big|$, we obtain

$$\begin{aligned}
\sum_{p<-j}\sum_{s\in\mathbb{S}} |\det A|^p \big|\hat{g}_j\big(B^p\xi\big)\big|\big|\hat{f}_j\big(B^p(\xi+s)\big)\big| &= \sum_{p<-j}\sum_{s\in\mathbb{S}} |\det A|^p \big|\hat{g}_j\big(B^p\xi\big)\big|\big|\hat{g}_j\big(B^p(\xi+s)-s_0\big)\big| \\
&= \sum_{p<-j}\sum_{s\in\mathbb{S}} |\det A|^p \big|\hat{g}_j\big(B^p\xi\big)\big|\big|\hat{g}_j\big(B^p(\xi+s-B^{-p}s_0)\big)\big| \\
&\le \sum_{p<-j}\sum_{k\in\mathbb{Z}^d\setminus\{0\}} |\det A|^p \big|\hat{g}_j\big(B^p\xi\big)\big|\big|\hat{g}_j\big(B^p(\xi+k)\big)\big| \\
&\le 2^d |\det A|^j M\Big((v+\delta)/v\Big)\|\hat{g}_j\|_\infty^2 \chi_\Upsilon(\xi) \\
&= 2^d M\Big((v+\delta)/v\Big)\chi_\Upsilon(\xi), \qquad (3.19)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\sum_{p<-j}\sum_{s\in\mathbb{S}} |\det A|^p \big|\hat{g}_j\big(B^p(\xi-s)\big)\big|\big|\hat{f}_j\big(B^p(\xi)\big)\big| &\le \sum_{p<-j}\sum_{k\in\mathbb{Z}^d\setminus\{0\}} |\det A|^p \big|\hat{f}_j\big(B^p\xi\big)\big|\big|\hat{f}_j\big(B^p(\xi+k)\big)\big| \\
&\le 2^d |\det A|^j M\Big((v'+\delta)/v'\Big)\|\hat{f}_j\|_\infty^2 \chi_{\Upsilon'}(\xi) \\
&= 2^d M\Big((v'+\delta)/v'\Big)\chi_{\Upsilon'}(\xi), \qquad (3.20)
\end{aligned}$$

by Lemma 2.4, assuming $v' > 0$, where

$$v' = v'(j) = \inf\Big\{|\xi| : \xi \in B^{-j} I_d + \xi_0 + s_0\Big\} \text{ and } \Upsilon' = \Upsilon'(j) = \bigcup_{p<-j} B^{-p}\Big(B^{-j} I_d + \xi_0 + s_0\Big).$$

For any $\varepsilon > 0$, there exists $r > 0$, so that $\int_{|\xi|>r} T(\xi)\, d\xi < \varepsilon$. By (2.10), we can find $j_0 > 0$ so that $\delta(j) < q/2$ and consequently $v(j) > q/2, v'(j) > q/2$ for $j > j_0$. Furthermore, by (2.10) we can choose j_0 large enough so that for all $j > j_0$, we have

$$\inf\{|\xi| : \xi \in \Upsilon(j)\} = \inf\Big\{|\xi| : \xi \in \bigcup_{p>j} B^p(B^{-j} I_d + \xi_0)\Big\} > c\lambda^j q/2 > r, \text{ and}$$

$$\inf\{|\xi| : \xi \in \Upsilon'(j)\} = \inf\Big\{|\xi| : \xi \in \bigcup_{p>j} B^p(B^{-j} I_d + \xi_0 + s_0)\Big\} > c\lambda^j q/2 > r.$$

Substituting (3.19) and (3.20) into (3.18), we obtain

$$|R(j)| \le 2^{d-1} M(2) \int_{\Upsilon(j)} T(\xi) d\xi + 2^{d-1} \int_{\Upsilon'(j)} T(\xi) d\xi \le 2^d M(2) \int_{|\xi|>r} T(\xi) d\xi < 2^d M(2)\varepsilon \qquad (3.21)$$

for $j > j_0$ independent of the choice of $\xi_0 \in \Omega(q)$. Since the supports of $\hat{f}_j$ and $\hat{g}_j$ are disjoint $I_1(j) = 0$; moreover (3.8) (and thus (3.11)) implies

$$0 = \langle f_j, g_j \rangle = \lim_{m\to\infty} I(m) = \lim_{m\to\infty} I_2(m) = I_2.$$

Since $\varepsilon > 0$, is arbitrary, therefore (3.17) and (3.21) yields

$$\lim_{j\to\infty} \sup_{\xi_0\in\Omega(q)} \frac{1}{|B^{-j}I_d|} \int_{B^{-j}I_d+\xi_0} \left|t_{s_0}(\xi)\right| d\xi = 0. \tag{3.22}$$

Consider any ball $B(r)$ with radius $r > 0$ such that $B(r) \subset \Omega(2q)$. Let $\mathrm{Z} = \left\{B^{-j}k : B^{-j}(I_d + k) \cap B(r) \neq \emptyset, k \in \mathbb{Z}^d\right\}$. If j is sufficiently large, then diam$(B^{-j}I_d) < \min(q, r)$, so

$$\tilde{Z} = \bigcup_{\xi_0\in \mathrm{Z}} (B^{-j}I_d + \xi_0) \subset \Omega(q) \cap B(2r).$$

Hence,

$$\begin{aligned}
\int_{B(r)} \left|t_{s_0}(\xi)\right| d\xi &\leq \int_{\tilde{Z}} \left|t_{s_0}(\xi)\right| d\xi \\
&\leq \sum_{\xi_0\in \mathrm{Z}} \int_{B^{-j}I_d+\xi_0} \left|t_{s_0}(\xi)\right| d\xi \\
&\leq \sum_{\xi_0\in \mathrm{Z}} \left|B^{-j}I_d + \xi_0\right|\varepsilon = |\tilde{Z}|\varepsilon = 2^d|B(r)|\varepsilon
\end{aligned}$$

for sufficiently large $j = j(\varepsilon)$ by (3.22). Since $\varepsilon > 0$, is arbitrary so $\int_{B(r)} |t_{s_0}(\xi)|\, d\xi = 0$, for any ball $B(r) \subset \Omega(2q)$. Therefore, $\int_{\Omega(2q)} |t_{s_0}(\xi)|\, d\xi = 0$ and since $q > 0$ is arbitrary $\int_{\mathbb{R}^d} |t_{s_0}(\xi)|\, d\xi = 0$ which implies $t_{s_0}(\xi) = 0$ for a.e. $\xi \in \mathbb{R}^d, s_0 \in \mathbb{S}$.

Finally, (3.8) implies that (3.9). Equation (3.9) follows easily from (3.10) and (3.16) since any function $h \in L^\infty(K)$ can be represented as $h = \overline{\hat{f}\hat{g}}$ for some $f, g \in \mathcal{D}$. $\square$

Theorem 3.3. *Let* $\{\omega_\ell^n : n \in \mathbb{Z}^+, 1 \leq \ell \leq L\}$ *and* $\{\tilde{\omega}_\ell^n : n \in \mathbb{Z}^+, 1 \leq \ell \leq L\}$ *be the dual multiwavelet packets associated with the dilation matrix* A *such that*

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \left|\hat{\omega}_\ell^n\left(B^j\xi\right)\right|^2 \quad \textit{and} \quad \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \left|\hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right)\right|^2, \tag{3.23}$$

are locally integrable on $\mathbb{R}^d \setminus \{0\}$. *Then*

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \langle f, \omega_{\ell,j,k}^n\rangle\langle\tilde{\omega}_{\ell,j,k}^n, f\rangle = \left\|f\right\|_2^2, \quad \textit{for all } f \in \mathcal{D} \tag{3.24}$$

if and only if

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \hat{\omega}_\ell^n\left(B^j\xi\right) \overline{\hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right)} = 1 \qquad a.e.\ \xi \in \mathbb{R}^d \tag{3.25}$$

$$t_s(\xi) \equiv \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right) \overline{\hat{\omega}_\ell^n\left(B^j(\xi+s)\right)} = 0,\ a.e.\ \xi \in \mathbb{R}^d, s \in \mathbb{Z}^d \setminus B\mathbb{Z}^d. \tag{3.26}$$

Proof. By Lemma 3.1 and (3.23), the series in (3.24) is absolutely convergent. Also, by (3.23), the series in (3.25) converges absolutely in $L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ and, hence, is absolutely convergent for $a.e.\ \xi$. Therefore, under the hypothesis, $(3.23), (3.9) \Leftrightarrow (3.24)$ and $(3.10) \Leftrightarrow$ (3.25). Hence, the desired result follows from Theorem 3.2. □

Theorem 3.4. *Let* $\{\omega_\ell^n : n \in \mathbb{Z}^+, 1 \le \ell \le L\}$ *be the basic multiwavelet packets associated with the scaling functions* φ_ℓ*. Then*

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \left|\langle f, \omega_{\ell,j,k}^n \rangle\right|^2 = \left\|f\right\|_2^2, \quad \textit{for all } f \in L^2(\mathbb{R}^d) \tag{3.27}$$

if and only if

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \left|\omega_\ell^n\left(B^j\xi\right)\right|^2 = 1, \quad a.e.\ \xi \in \mathbb{R}^d \tag{3.28}$$

$$t_s(\xi) \equiv \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \hat{\omega}_\ell^n\left(B^j\xi\right) \overline{\hat{\omega}_\ell^n\left(B^j(\xi+s)\right)} = 0,\ a.e.\ \xi \in \mathbb{R}^d, s \in \mathbb{S} = \mathbb{Z}^d \setminus B\mathbb{Z}^d. \tag{3.29}$$

In particular, the system $\mathcal{F}(\omega_\ell^n)$ *given by* (2.8) *forms Parseval's frame for* $L^2(\mathbb{R}^d)$ *if and only if* (3.28), (3.29) *hold and* $\|\omega_\ell^n\|_2 = 1,$ *for* $n \in \mathbb{Z}^+, \ell = 1, ..., L.$

Proof. Using Lemma 3.1 and (3.27), we have

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \left|\hat{\omega}_\ell^n\left(B^j\xi\right)\right|^2 \in L^1_{loc}(\mathbb{R}^d \setminus \{0\}).$$

Therefore, we can apply Theorem 3.3 with $\omega_\ell^n = \tilde{\omega}_\ell^n \in L^2(\mathbb{R}^d)$ to obtain (3.28) and (3.29). Conversely, assume (3.28) and (3.29); then again by Theorem 3.3, we have

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \left|\langle f, \omega_{\ell,j,k}^n \rangle\right|^2 = \left\|f\right\|_2^2, \quad \text{for all } f \in \mathcal{D}.$$

By Theorem 2.7, we have the above for all $f \in L^2(\mathbb{R}^d)$. Furthermore, the system $\mathcal{F}(\omega_\ell^n)$ forms Parseval's frame for $L^2(\mathbb{R}^d)$ if $\|\omega_\ell^n\|_2 \ge 1$ for $n \in \mathbb{Z}^+, \ell = 1, \dots, L$. □

Acknowledgement

The author was partially supported by the University of Kashmir, under seed money grant scheme, letter no. F(Seed Money Grant)Res/KU; dated: March 26, 2013.

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