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MODIFIED INERTIAL SUBGRADIENT EXTRAGRADIENT ALGORITHM WITH SELF-ADAPTIVE STEP SIZES FOR SOLVING SPLIT EQUILIBRIUM PROBLEMS

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ABSTRACT. In this paper, we introduce a modified inertial subgradient extragradient algorithm featuring self-adaptive step sizes. Our focus is on solving split equilibrium problems that involve pseudomonotone bifunctions satisfying Lipschitz-type continuity within real Hilbert spaces. We demonstrate a strong convergence theorem for the proposed algorithm, requiring neither prior knowledge of the operator norm of the bounded linear operator nor the Lipschitz constants of bifunctions. This convergence holds under certain constraint qualifications of the scalar sequences.

KEYWORDS: split equilibrium problems; pseudomonotone bifunction; inertial method; subgradient extragradient method
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1. Introduction

The equilibrium problem introduced by Blum and Oettli [1] is a problem of finding a point $x^* \in C$ such that

$$f(x^*, y) \geq 0, \forall y \in C, \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H, and $f : H \times H \to \mathbb{R}$ is a bifunction. The solution set of the equilibrium problem (1.1) will be denoted by $EP(f, C)$. The equilibrium problem is a broad framework that includes many mathematical problems, such as fixed point problems, optimization problems, variational inequality problems, minimax problems, Nash equilibrium problems, and saddle point problems, see [2, 3, 4, 5], and the references therein.

In the most appeared papers, the proposed method for solving the equilibrium problem (1.1), when f is a monotone bifunction, is the proximal point method, see

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[6]. However, if the bifunction f satisfies a weaker assumption as pseudomonotone, the proximal point method cannot be applied in this situation. To overcome this drawback, Tran et al. [7] proposed the following so-called extragradient method for solving the equilibrium problem when the bifunction f is pseudomonotone and satisfies Lipschitz-type continuous with positive constants c_1 and c_2:

$$\begin{cases} x_0 \in C, \\ y_k = \arg\min \left\{ \lambda f(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C \right\}, \\ x_{k+1} = \arg\min \left\{ \lambda f(y_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C \right\}, \end{cases} \tag{1.2}$$

where $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$. They proved that the sequence $\{x_k\}$ generated by Algorithm 1.2 converges weakly to a solution of the equilibrium problem. It should be noted that in order to get y_k and x_{k+1} in each iteration of the extragradient method, the optimization problems on the feasible set C must be solved twice. It is common knowledge that the optimization problem will not be simple if the feasible set C has a complex structure. To improve this one, Hieu [8] proposed the following so-called subgradient extragradient method for solving the equilibrium problem when the bifunction f is pseudomonotone and satisfies Lipschitz-type continuous with positive constants c_1 and c_2:

$$\begin{cases} x_0 \in H, \\ y_k = \arg\min \left\{ \lambda_k f(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C \right\}, \\ T_k = \left\{ z \in H : \langle x_k - \lambda_k r_k - y_k, z - y_k \rangle \leq 0 \right\}, r_k \in \partial_2 f(x_k, y_k), \\ z_k = \arg\min \left\{ \lambda_k f(y_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in T_k \right\}, \\ x_{k+1} = \alpha_k x_0 + (1 - \alpha_k) z_k, \end{cases} \tag{1.3}$$

where $0 < \lambda_k < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, $\{\alpha_k\} \subset (0,1)$ such that $\sum_{k=0}^{\infty} \alpha_k = +\infty$ and $\lim_{k\to\infty} \alpha_k = 0$, and $\partial_2 f(x_k, y_k)$ is the subdifferential of $f(x_k, \cdot)$ at y_k. The author proved that the sequence $\{x_k\}$ generated by Algorithm 1.3 converges strongly to $P_{EP(f,C)}(x_0)$. It is highlighted that in the second step for determining z_k in each iteration, the subgradient extragradient method translates to solve the optimization problem on the feasible set C to the half-space T_k. Consequently, the computational efficiency of this method is significantly improved by solving optimization problem on the feasible set C only once for finding y_k in each iteration. Meanwhile, the inertial method which was first proposed in Polyak [9] was regarded to speed up the convergence properties of the algorithm and was used in the implicit discretization algorithm of the heavy ball with friction system [10, 11]. This method is characterized that the next iteration is determined by the combination of the previous two (or more) iterations and has received a lot of attention from many researchers, see [12, 13] and the references therein.

In 2012, He [14] (see also Moudafi [15]) introduced the split equilibrium problems as follows:

$$\begin{cases} \text{Find} \quad x^* \in C \quad \text{such that} \quad f(x^*, y) \geq 0, \forall y \in C, \\ \text{and} \quad u^* := Ax^* \in Q \quad \text{solves} \quad g(u^*, v) \geq 0, \forall v \in Q, \end{cases} \tag{1.4}$$

where C, Q are two nonempty closed convex subsets of the real Hilbert spaces H_1 and H_2, respectively, $f : H_1 \times H_1 \to \mathbb{R}$ and $g : H_2 \times H_2 \to \mathbb{R}$ are bifunctions, and $A : H_1 \to H_2$ is a bounded linear operator. To solve the split equilibrium problems (1.4), Suantai et al. [16] proposed the following algorithm by using the techniques of inertial and extragradient method for solving the split equilibrium problems when the bifunctions f and g are pseudomonotone and satisfy Lipschitz-type continuous conditions with some positive constants $\{c_1, c_2\}$ and $\{d_1, d_2\}$, respectively:

$$\begin{cases} x_0, x_1 \in C, \\ w_k = x_k + \theta_k(x_k - x_{k-1}), \\ y_k = \arg\min \left\{ \lambda_k f(w_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C \right\}, \\ z_k = \arg\min \left\{ \lambda_k f(y_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C \right\}, \\ u_k = \arg\min \left\{ \mu_k g(Az_k, u) + \frac{1}{2}\|u - Az_k\|^2 : u \in Q \right\}, \\ v_k = \arg\min \left\{ \mu_k g(u_k, u) + \frac{1}{2}\|u - Az_k\|^2 : u \in Q \right\}, \\ x_{k+1} = P_C(z_k + \eta A^*(v_k - Az_k)), \end{cases} \tag{1.5}$$

where $\eta \in (0, 1/\|A\|^2)$, $\tau \in [0, 1)$, $0 < \lambda_k < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, $0 < \mu_k < \min\left\{\frac{1}{2d_1}, \frac{1}{2d_2}\right\}$, and $0 \leq \theta_k \leq \overline{\theta}_k$ with

$$\overline{\theta}_k = \begin{cases} \min\left\{\tau, \dfrac{\epsilon_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \tau, & \text{otherwise.} \end{cases}$$

They proved that the sequence $\{x_k\}$ generated by Algorithm 1.5 converges weakly to a solution of the split equilibrium problems (1.4). It is worth noting that this algorithm used step sizes λ_k, μ_k, and η that are dependent on the Lipschitz constants of the bifunctions f, g, and the operator norm of the bounded linear operator A, respectively. However, these step sizes are typically not easily obtainable in practical applications.

In this paper, we focus on the methods for solving the split equilibrium problems (1.4). That is, we present a new algorithm without the prior knowledge of both the operator norm of the bounded linear operator and the Lipschitz constants of the bifunctions for finding the solutions of the split equilibrium problems when the bifunctions are pseudomonotone and satisfy Lipschitz-type continuous.

This paper is organized as follows: In Section 2, some necessary definitions and properties will be reviewed. Section 3 presents the modified inertial subgradient extragradient algorithm with self-adaptive step sizes and proves the strong convergence theorem.

2. Preliminaries

In this section, we provide some definitions and properties which are used in the sequel. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. The notation $\mathbb{R}$ and $\mathbb{N}$ will stand for the set of the real numbers and the natural numbers, respectively.

First, we state some definitions and results involving the equilibrium problems.

Definition 2.1. Let C be a nonempty closed convex subset of H. A bifunction $f : H \times H \to \mathbb{R}$ is said to be:

(i) monotone on C if

$$f(x, y) + f(y, x) \leq 0, \forall x, y \in C;$$

(ii) pseudomonotone on C if

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0, \forall x, y \in C;$$

(iii) Lipshitz-type continuous on H if there exists two positive constants c_1 and c_2 such that

$$f(x,y)+f(y,z)\geq f(x,z)-c_1\|x-y\|^2-c_2\|y-z\|^2,\forall x,y,z\in H.$$

Remark 2.2. We observe that a monotone bifunction is a pseudomonotone bifunction, but the converse is not true, for instance, see [17].

In what follows, we recall the projection mapping and calculus concepts in Hilbert space.

Let C be a nonempty closed convex subset of H. For each $x\in H$, we denote the metric projection of x onto C by $P_C(x)$, that is

$$\|x-P_C(x)\|\leq\|y-x\|,\forall y\in C.$$

Lemma 2.3. [18, 19] *Let C be a nonempty closed convex subset of H. Then,*

(i) *$P_C(x)$ is singleton and well-defined for each $x\in H$;*
(ii) *$z=P_C(x)$ if and only if $\langle x-z,y-z\rangle\leq 0$, $\forall y\in C$;*
(iii) *P_C is a nonexpansive operator, that is,*

$$\|P_C(x)-P_C(y)\|\leq\|x-y\|,\forall x,y\in H.$$

For a function $f:H\to\mathbb{R}$, the subdifferential of f at $z\in H$ is defined by

$$\partial f(z)=\{w\in H:f(y)-f(z)\geq\langle w,y-z\rangle,\forall y\in H\}.$$

The function f is said to be subdifferentiable at z if $\partial f(z)\neq\emptyset$.

Lemma 2.4. [18] *For any $z\in H$, the subdifferentiable $\partial f(z)$ of a continuous convex function f is a weakly closed and bounded convex set.*

Lemma 2.5. [3] *Let C be a convex subset of H and $f:C\to\mathbb{R}$ be subdifferentiable on C. Then, x^* is a solution to the following convex problem:*

$$\min\{f(x):x\in C\}$$

if and only if $0\in\partial f(x^*)+N_C(x^*)$, *where* $N_C(x^*):=\{y\in H:\langle y,z-x^*\rangle\leq 0,\forall z\in C\}$ *is the normal cone of C at x^*.*

This section will be closed by collecting some facts which are important to obtain the convergence theorems.

Lemma 2.6. [20] *Let $\{a_k\}$ and $\{c_k\}$ be sequences of non-negative real numbers such that*

$$a_{k+1}\leq(1-\gamma_k)a_k+\gamma_k b_k+c_k,\forall k\in\mathbb{N}\cup\{0\},$$

where $\{\gamma_k\}$ is a sequence in $(0,1)$ and $\{b_k\}$ is a sequence in $\mathbb{R}$. Assume that $\sum\limits_{k=0}^{\infty}c_k<\infty$. If $\sum\limits_{k=0}^{\infty}\gamma_k=\infty$ and $\limsup\limits_{k\to\infty}b_k\leq 0$, then $\lim\limits_{k\to\infty}a_k=0$.

Lemma 2.7. [21] *Let $\{a_k\}$ be a sequence of real numbers such that there exists a subsequence $\{a_{k_i}\}$ of $\{a_k\}$ such that $a_{k_i}<a_{k_i+1}$, for all $i\in\mathbb{N}$. Then, there exists a non-decreasing sequence $\{m_n\}$ of positive integers such that $\lim\limits_{n\to\infty}m_n=\infty$ and the following properties hold:*

$$a_{m_n}\leq a_{m_n+1}\ \textit{and}\ a_n\leq a_{m_n+1},$$

for all (sufficiently large) numbers $n\in\mathbb{N}$. Indeed, m_n is the largest number k in the set $\{1,2,\dots,n\}$ such that

$$a_k<a_{k+1}.$$

3. Main Results

Let C be a nonempty closed convex subset of a real Hilbert space H. For a bifunction $f : H \times H \to \mathbb{R}$, the following assumptions will be concerned in this paper:

- $(A1)$ $f(\cdot, y)$ is sequentially weakly upper semicontinuous on C, for each fixed $y \in C$, that is if $\{x_k\} \subset C$ is a sequence converging weakly to $x \in C$, then $\limsup_{k\to\infty} f(x_k, y) \leq f(x, y)$;
- $(A2)$ $f(x, \cdot)$ is convex, subdifferentiable and lower semicontinuous on H, for each fixed $x \in H$;
- $(A3)$ f is psuedomonotone on C;
- $(A4)$ f is Lipshitz-type continuous on H.

Remark 3.1. (i) It is well-known that the solution set $EP(f, C)$ is closed and convex, when the bifunction f satisfies the assumptions $(A1) - (A3)$, see [7, 22, 23].

(ii) If the bifunction f satisfies the assumptions $(A3)$ and $(A4)$, then $f(x, x) = 0$, for each $x \in C$, see [13].

Now, let H_1 and H_2 be two real Hilbert spaces and C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. We recall the split equilibrium problems:

$$\begin{cases} \text{Find} \quad x^* \in C \quad \text{such that} \quad f(x^*, y) \geq 0, \forall y \in C, \\ \text{and} \quad u^* = Ax^* \in Q \quad \text{solves} \quad g(u^*, v) \geq 0, \forall v \in Q, \end{cases} \tag{3.1}$$

where $f\colon H_1 \times H_1 \to \mathbb{R}$, $g\colon H_2 \times H_2 \to \mathbb{R}$ are bifunctions, and $A\colon H_1 \to H_2$ is a bounded linear operator with its adjoint operator A^*. From now on, the solution set of problem (3.1) will be represented by Ω. That is,

$$\Omega := \{p \in EP(f, C) : \; Ap \in EP(g, Q)\}.$$

Next, we introduce the modified inertial subgradient extragradient algorithm for solving the split equilibrium problems (3.1).

Algorithm 3.2. Modified inertial subgradient extragradient algorithm

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Initialization. Choose parameters $\lambda_1 > 0$, $\mu_1 > 0$, $\tau \in [0, 1)$, $\beta \in (0, 1)$, $\gamma \in (0, 1)$, $\{\varphi_k\} \subset (0, 1)$, $\{\xi_k\} \subset [1, \infty)$ with $\lim_{k\to\infty} \xi_k = 1$, $\{\sigma_k\} \subset [1, \infty)$ with $\lim_{k\to\infty} \sigma_k = 1$, $\{\epsilon_k\} \subset [0, \infty)$, $\{\alpha_k\} \subset (0, 1)$ such that $\sum_{k=0}^{\infty} \alpha_k = \infty$, $\lim_{k\to\infty} \alpha_k = 0$, and $\lim_{k\to\infty} \frac{\epsilon_k}{\alpha_k} = 0$. Pick $x_0, x_1 \in H$ and set $k = 1$.

Step 1. Choose θ_k such that $0 \leq \theta_k \leq \overline{\theta}_k$, where

$$\overline{\theta}_k = \begin{cases} \min\left\{\tau, \dfrac{\epsilon_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \tau, & \text{otherwise,} \end{cases}$$

and compute

$$w_k = (1 - \alpha_k)\left(x_k + \theta_k(x_k - x_{k-1})\right).$$

Step 2. Solve the strongly convex program

$$y_k = \arg\min\{\lambda_k f(w_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C\}.$$

Step 3. Construct a half-space

$$T_k = \{z \in H_1 : \langle w_k - \lambda_k s_k - y_k, z - y_k\rangle \leq 0\},$$

where $s_k \in \partial_2 f(w_k, y_k)$.

Step 4. Solve the strongly convex program

$$z_k = \arg\min\{\xi_k \lambda_k f(y_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in T_k\}.$$

Step 5. Solve the strongly convex program

$$u_k = \arg\min\{\mu_k g(Az_k, y) + \frac{1}{2}\|u - Az_k\|^2 : u \in Q\}.$$

Step 6. Construct a half-space

$$S_k = \{v \in H_2 : \langle Az_k - \mu_k r_k - u_k, v - u_k\rangle \leq 0\},$$

where $r_k \in \partial_2 g(Az_k, u_k)$.

Step 7. Solve the strongly convex program

$$v_k = \arg\min\{\sigma_k \mu_k g(u_k, y) + \frac{1}{2}\|u - Az_k\|^2 : u \in S_k\}.$$

Step 8. The next approximation x_{k+1} is defined as

$$x_{k+1} = P_C(z_k + \eta_k A^*(v_k - Az_k)),$$

where

$$\eta_k = \begin{cases} \dfrac{\varphi_k \|v_k - Az_k\|^2}{\|A^*(v_k - Az_k)\|^2}, & \text{if } v_k \neq Az_k, \\ \varphi_k, & \text{otherwise.} \end{cases}$$

Step 9. Compute

$$\lambda_{k+1} = \begin{cases} \min\left\{\lambda_k, \dfrac{\beta(\|w_k - y_k\|^2 + \|z_k - y_k\|^2)}{2\left[f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k)\right]}\right\}, \\ \qquad\qquad \text{if } f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k) > 0, \\ \lambda_k, \qquad\qquad \text{otherwise,} \end{cases}$$

and

$$\mu_{k+1} = \begin{cases} \min\left\{\mu_k, \dfrac{\gamma(\|Az_k - u_k\|^2 + \|v_k - u_k\|^2)}{2\left[g(Az_k, v_k) - g(Az_k, u_k) - g(u_k, v_k)\right]}\right\}, \\ \qquad\qquad \text{if } g(Az_k, v_k) - g(Az_k, u_k) - g(u_k, v_k) > 0, \\ \mu_k, \qquad\qquad \text{otherwise.} \end{cases}$$

Step 10. Put $k := k + 1$ and go to **Step 1**.

Remark 3.3. i) The new control sequences $\{\xi_k\}$ and $\{\sigma_k\}$ in Algorithm 3.2 are proposed to modify the subgradient extragradient method. Note that if $\xi_k = \sigma_k = 1$, for each $k \in \mathbb{N}$, then the modified subgradient extragradient method which is included in Algorithm 3.2 reduces to the general situation such as presented in [8].

ii) The step sizes η_k, λ_k, and μ_k in Algorithm 3.2 are self-adaptive, which are introduced to provide Algorithm 3.2 without prior knowledge of the operator norm of the bounded linear operator A, and the Lipschitz constants of the bifunctions f, and g, respectively. This means Algorithm 3.2 automatically updates the iteration step sizes η_k, λ_k, and μ_k by utilizing some previously known data.

The following lemma is critical for analyzing the convergence of Algorithm 3.2.

Lemma 3.4. *Let $f\colon H_1 \times H_1 \to \mathbb{R}$ and $g\colon H_2 \times H_2 \to \mathbb{R}$ be bifunctions which satisfy $(A1)-(A4)$, and $A\colon H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Suppose that the solution set Ω is nonempty. Let $w_k \in H_1$ and $Az_k \in H_2$. If y_k, z_k, u_k, v_k, λ_{k+1}, and μ_{k+1} are constructed as in the process of Algorithm 3.2, then the following results hold:*

$$\begin{aligned}\|z_k - p\|^2 \quad &\leq \quad \|w_k - p\|^2 - \left(2 - \xi_k - \frac{\beta \xi_k \lambda_k}{\lambda_{k+1}}\right)\|w_k - y_k\|^2 \\ &\quad - \left(2 - \xi_k - \frac{\beta \xi_k \lambda_k}{\lambda_{k+1}}\right)\|y_k - z_k\|^2,\end{aligned}$$

and

$$\begin{aligned}\|v_k - Ap\|^2 \quad &\leq \quad \|Az_k - Ap\|^2 - \left(2 - \sigma_k - \frac{\gamma \sigma_k \mu_k}{\mu_{k+1}}\right)\|Az_k - u_k\|^2 \\ &\quad - \left(2 - \sigma_k - \frac{\gamma \sigma_k \mu_k}{\mu_{k+1}}\right)\|u_k - v_k\|^2,\end{aligned}$$

$\forall p \in \Omega$.

Proof. Firstly, let us assert that $C \subset T_k$, for each $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ be fixed and $y \in C$. By the definition of y_k and Lemma 2.5, we have

$$0 \in \partial_2 \left\{\lambda_k f(w_k, y_k) + \frac{1}{2}\|y_k - w_k\|^2\right\} + N_C(y_k).$$

Thus, there exists $s_k \in \partial_2 f(w_k, y_k)$ and $q_k \in N_C(y_k)$ such that

$$\lambda_k s_k + y_k - w_k + q_k = 0. \tag{3.2}$$

It follows from $q_k \in N_C(y_k)$ that

$$\langle w_k - \lambda_k s_k - y_k, y - y_k\rangle = \langle q_k, y - y_k\rangle \leq 0. \tag{3.3}$$

This implies that $y \in T_k$. Since $k \in \mathbb{N}$ is arbitrary, we can conclude that $C \subset T_k$, for each $k \in \mathbb{N}$. Consequently, we can guarantee that Algorithm 3.2 is well-defined.

Next, we will show the conclusion of the Lemma by using the above facts. Let $p \in \Omega$. So, $p \in EP(f, C)$ and $Ap \in EP(g, Q)$. By the definition of z_k and Lemma 2.5, we obtain that

$$0 \in \partial_2 \left\{\xi_k \lambda_k f(y_k, z_k) + \frac{1}{2}\|z_k - w_k\|^2\right\} + N_{T_k}(z_k).$$

Then, there exists $s \in \partial_2 f(y_k, z_k)$ and $q \in N_{T_k}(z_k)$ such that

$$\xi_k \lambda_k s + z_k - w_k + q = 0. \tag{3.4}$$

It follows from the subdifferentiability of f that

$$f(y_k, y) - f(y_k, z_k) \geq \langle s, y - z_k\rangle, \forall y \in H. \tag{3.5}$$

Additionally, from $q \in N_{T_k}(z_k)$, we have

$$\langle q, z_k - y\rangle \geq 0, \forall y \in T_k.$$

Combining with the equality (3.4), we get

$$\langle w_k - z_k, z_k - y\rangle \geq \xi_k\lambda_k\langle s, z_k - y\rangle, \forall y \in T_k. \tag{3.6}$$

This together with the relation (3.5) yields that

$$\langle w_k - z_k, z_k - y\rangle \geq \xi_k\lambda_k[f(y_k, z_k) - f(y_k, y)], \forall y \in T_k. \tag{3.7}$$

Indeed, from $p \in C \subset T_k$, we obtain

$$\langle w_k - z_k, z_k - p\rangle \geq \xi_k\lambda_k[f(y_k, z_k) - f(y_k, p)].$$

It follows from the pseudomonotonic of f that

$$\langle w_k - z_k, z_k - p\rangle \geq \xi_k\lambda_k f(y_k, z_k). \tag{3.8}$$

Moreover, by utilizing the subdifferentiability of f and $s_k \in \partial_2 f(w_k, y_k)$, we get

$$f(w_k, y) - f(w_k, y_k) \geq \langle s_k, y - y_k\rangle, \forall y \in H.$$

So, from $z_k \in T_k \subset H$, we have

$$f(w_k, z_k) - f(w_k, y_k) \geq \langle s_k, z_k - y_k\rangle. \tag{3.9}$$

Also, by using the definition of T_k and $z_k \in T_k$, we obtain

$$\langle w_k - \lambda_k s_k - y_k, z_k - y_k\rangle \leq 0$$

Due to the inequality (3.9), we get

$$\lambda_k[f(w_k, z_k) - f(w_k, y_k)] \geq \langle y_k - w_k, y_k - z_k\rangle. \tag{3.10}$$

Using this one together with the inequality (3.8), we have

$$\begin{aligned}\xi_k\lambda_k[f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k)] \quad \geq \quad & \langle z_k - w_k, z_k - p\rangle \\ & + \xi_k\langle y_k - w_k, y_k - z_k\rangle. \end{aligned} \tag{3.11}$$

On the other hand, from the definition of λ_{k+1}, we observe that

$$f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k) \leq \frac{\beta(\|w_k - y_k\|^2 + \|y_k - z_k\|^2)}{2\lambda_{k+1}}. \tag{3.12}$$

This together with the inequality (3.11) yields that

$$\langle w_k - z_k, z_k - p\rangle \geq \xi_k\langle y_k - w_k, y_k - z_k\rangle - \frac{\beta\xi_k\lambda_k(\|w_k - y_k\|^2 + \|y_k - z_k\|^2)}{2\lambda_{k+1}}.$$

By using the above inequality, we note that

$$\begin{aligned}\|w_k - p\|^2 - \|w_k - z_k\|^2 - \|z_k - p\|^2 \quad &= \quad 2\langle w_k - z_k, z_k - p\rangle \\ &\geq \quad 2\xi_k\langle y_k - w_k, y_k - z_k\rangle \\ & \quad - \frac{\beta\xi_k\lambda_k(\|w_k - y_k\|^2 + \|y_k - z_k\|^2)}{\lambda_{k+1}}.\end{aligned}$$

This implies that

$$\begin{aligned}\|z_k - p\|^2 \quad &\leq \quad \|w_k - p\|^2 - \|w_k - z_k\|^2 - 2\xi_k\langle y_k - w_k, y_k - z_k\rangle \\ & \quad + \frac{\beta\xi_k\lambda_k(\|w_k - y_k\|^2 + \|y_k - z_k\|^2)}{\lambda_{k+1}} \\ &= \quad \|w_k - p\|^2 - \|w_k - z_k\|^2 + \xi_k\|w_k - z_k\|^2 - \xi_k\|w_k - y_k\|^2 \\ & \quad -\xi_k\|y_k - z_k\|^2 + \frac{\beta\xi_k\lambda_k(\|w_k - y_k\|^2 + \|y_k - z_k\|^2)}{\lambda_{k+1}} \\ &= \quad \|w_k - p\|^2 - \left(\xi_k - \frac{\beta\xi_k\lambda_k}{\lambda_{k+1}}\right)\|w_k - y_k\|^2\end{aligned}$$

$$-\left(\xi-\frac{\beta\xi_k\lambda_k}{\lambda_{k+1}}\right)\|y_k-z_k\|^2-(1-\xi_k)\|w_k-z_k\|^2. \tag{3.13}$$

We observe that

$$\|w_k-z_k\|^2\le(\|w_k-y_k\|+\|y_k-z_k\|)^2\le 2\left(\|w_k-y_k\|^2+\|y_k-z_k\|^2\right),$$

which together with the condition of parameter $\xi_k\in[1,\infty)$ implies that

$$-(1-\xi_k)\|w_k-z_k\|^2\le -2(1-\xi_k)\left(\|w_k-y_k\|^2+\|y_k-z_k\|^2\right).$$

Combining with the relation (3.13) implies that

$$\begin{aligned}\|z_k-p\|^2 \quad\le\quad & \|w_k-p\|^2-\left(2-\xi_k-\frac{\beta\xi_k\lambda_k}{\lambda_{k+1}}\right)\|w_k-y_k\|^2\\ & -\left(2-\xi_k-\frac{\beta\xi_k\lambda_k}{\lambda_{k+1}}\right)\|y_k-z_k\|^2.\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}\|v_k-Ap\|^2 \quad\le\quad & \|Az_k-Ap\|^2-\left(2-\sigma_k-\frac{\gamma\sigma_k\mu_k}{\mu_{k+1}}\right)\|Az_k-u_k\|^2\\ & -\left(2-\sigma_k-\frac{\gamma\sigma_k\mu_k}{\mu_{k+1}}\right)\|u_k-v_k\|^2.\end{aligned}$$

This completes the proof. □

Now, we are ready to analyze the convergence of Algorithm 3.2.

Theorem 3.1. *Let $f\colon H_1\times H_1\to\mathbb{R}$ and $g\colon H_2\times H_2\to\mathbb{R}$ be bifunctions which satisfy $(A1)-(A4)$, and $A\colon H_1\to H_2$ be a bounded linear operator with its adjoint operator A^*. Suppose that the solution set Ω is nonempty. Then, the sequence $\{x_k\}$ which is generated by Algorithm 3.2 converges strongly to the minimum-norm element of Ω.*

Proof. Let $p\in\Omega$. That is, $p\in EP(f,C)$ and $Ap\in EP(g,Q)$. Firstly, we observe that $\{\lambda_k\}$ is a nonincreasing sequence. On the other hand, by the Lipschitz-type continuity of f on H_1, there exists two positive constants c_1 and c_2 such that

$$\begin{aligned}f(w_k,z_k)-f(w_k,y_k)-f(y_k,z_k) \quad\le\quad & c_1\|w_k-y_k\|^2+c_2\|y_k-z_k\|^2\\ \le\quad & \max\{c_1,c_2\}(\|w_k-y_k\|^2+\|y_k-z_k\|^2).\end{aligned}$$

Thus, by the definition of λ_k, we obtain

$$\lambda_{k+1}\ge\min\left\{\lambda_k,\frac{\beta}{2\max\{c_1,c_2\}}\right\}\ge\ldots\ge\min\left\{\lambda_1,\frac{\beta}{2\max\{c_1,c_2\}}\right\}.$$

This implies that $\{\lambda_k\}$ is bounded from below. Consequently, we have that the limit of $\{\lambda_k\}$ exists. Similarly, we can show that the limit of $\{\mu_k\}$ exists. Thus, by the assumptions on the parameter $\beta\in(0,1)$ together with the existence of $\lim\limits_{k\to\infty}\lambda_k$ and $\lim\limits_{k\to\infty}\xi_k=1$, we have

$$\lim_{k\longrightarrow\infty}\left(2-\xi_k-\frac{\beta\xi_k\lambda_k}{\lambda_{k+1}}\right)=1-\beta>0.$$

Then, there exists $k_1\in\mathbb{N}$ such that

$$2-\xi_k-\frac{\beta\xi_k\lambda_k}{\lambda_{k+1}}>0,\forall k\ge k_1. \tag{3.14}$$

Additionally, by the conditions on the parameter $\gamma \in (0,1)$ together with the existence of $\lim_{k\to\infty} \mu_k$ and $\lim_{k\to\infty} \sigma_k = 1$, we obtain that

$$\lim_{k\longrightarrow\infty} \left(2 - \sigma_k - \frac{\gamma\sigma_k\mu_k}{\mu_{k+1}}\right) = 1 - \gamma > 0.$$

Thus, there exists $k_2 \in \mathbb{N}$ such that

$$2 - \sigma_k - \frac{\gamma\sigma_k\mu_k}{\mu_{k+1}} > 0, \forall k \geq k_2. \tag{3.15}$$

Choose $k_0 = \max\{k_1, k_2\}$. Then, by using (3.14), (3.15), and Lemma 3.4, we have

$$\|z_k - p\| \quad \leq \quad \|w_k - p\|, \tag{3.16}$$

and

$$\|v_k - Ap\| \quad \leq \quad \|Az_k - Ap\|, \tag{3.17}$$

for each $k \geq k_0$.

Now, let us consider for each $k \in \mathbb{N}$ such that $k \geq k_0$. By the definition of x_{k+1} and the nonexpansivity of P_C, we have

$$\begin{aligned} \|x_{k+1} - p\|^2 &\leq \|(z_k - p) + \eta_k A^*(v_k - Az_k)\|^2 \\ &= \|z_k - p\|^2 + \eta_k^2\|A^*(v_k - Az_k)\|^2 \\ &\quad +2\eta_k\langle Az_k - Ap, v_k - Az_k\rangle. \end{aligned} \tag{3.18}$$

Consider,

$$\begin{aligned} 2\langle Az_k - Ap, v_k - Az_k\rangle &= 2\langle v_k - Ap, v_k - Az_k\rangle - 2\|v_k - Az_k\|^2 \\ &= \|v_k - Ap\|^2 - \|v_k - Az_k\|^2 - \|Az_k - Ap\|^2. \end{aligned}$$

Combining with the relation (3.18) implies that

$$\begin{aligned} \|x_{k+1} - p\|^2 &\leq \|z_k - p\|^2 - \eta_k\left(\|v_k - Az_k\|^2 - \eta_k\|A^*(v_k - Az_k)\|^2\right) \\ &\quad +\eta_k(\|v_k - Ap\|^2 - \|Az_k - Ap\|^2). \end{aligned}$$

This together with the relation (3.17) yields that

$$\|x_{k+1} - p\|^2 \quad \leq \quad \|z_k - p\|^2 - \eta_k\left(\|v_k - Az_k\|^2 - \eta_k\|A^*(v_k - Az_k)\|^2\right).$$

It follows from the choices of the parameters η_k and φ_k that

$$\begin{aligned} \|x_{k+1} - p\|^2 &\leq \|z_k - p\|^2 - \eta_k(1 - \varphi_k)\|v_k - Az_k\|^2 && (3.19) \\ &\leq \|z_k - p\|^2. && (3.20) \end{aligned}$$

Thus, the relations (3.16) and (3.20) imply that

$$\|x_{k+1} - p\| \quad \leq \quad \|w_k - p\|. \tag{3.21}$$

In addition, from the definition of w_k, we observe that

$$\begin{aligned} \|w_k - p\| &= \|(1-\alpha_k)(x_k - p) + (1-\alpha_k)\theta_k(x_k - x_{k-1}) - \alpha_k p\| \\ &\leq (1-\alpha_k)\|x_k - p\| + (1-\alpha_k)\theta_k\|x_k - x_{k-1}\| + \alpha_k\|p\| \\ &= (1-\alpha_k)\|x_k - p\| + \alpha_k\left[(1-\alpha_k)\frac{\theta_k}{\alpha_k}\|x_k - x_{k-1}\| + \|p\|\right]. \end{aligned} \tag{3.22}$$

Due to the choices of the sequences $\{\theta_k\}$, we have

$$(1-\alpha_k)\frac{\theta_k}{\alpha_k}\|x_k - x_{k-1}\| \leq (1-\alpha_k)\frac{\epsilon_k}{\alpha_k}.$$

Using this one together with the fact that $\lim_{k\to\infty} \frac{\epsilon_k}{\alpha_k} = 0$, we get

$$\lim_{k\longrightarrow\infty} (1-\alpha_k)\frac{\theta_k}{\alpha_k}\|x_k - x_{k-1}\| = 0. \tag{3.23}$$

Thus, there exists a constant $M_1 > 0$ such that

$$(1-\alpha_k)\frac{\theta_k}{\alpha_k}\|x_k - x_{k-1}\| \leq M_1. \tag{3.24}$$

This together with the relations (3.21) and (3.22) yields that

$$\begin{aligned} \|x_{k+1} - p\| &\leq (1-\alpha_k)\|x_k - p\| + \alpha_k (M_1 + \|p\|) \\ &\leq \max\{\|x_k - p\|, M_1 + \|p\|\} \\ &\leq \cdots \\ &\leq \max\{\|x_{k_0} - p\|, M_1 + \|p\|\}. \end{aligned}$$

This implies that the sequence $\{\|x_k - p\|\}$ is bounded. Consequently, $\{x_k\}$ is a bounded sequence.

Furthermore, the relations (3.22) and (3.24) imply that

$$\begin{aligned} \|w_k - p\|^2 &\leq [(1-\alpha)\|x_k - p\| + \alpha_k(M_1 + \|p\|)]^2 \\ &= (1-\alpha_k)^2\|x_k - p\|^2 + \alpha_k\big[2(1-\alpha_k)(M_1 + \|p\|)\|x_k - p\| \\ &\quad +\alpha_k(M_1 + \|p\|)^2\big] \\ &\leq \|x_k - p\|^2 + \alpha_k M_2, \end{aligned} \tag{3.25}$$

where $M_2 = \sup_{k\geq k_0}\left\{2(1-\alpha_k)(M_1 + \|p\|)\|x_k - p\| + \alpha_k(M_1 + \|p\|)^2\right\} > 0$. Thus, by using the relations (3.20), (3.25), and Lemma 3.4, we have

$$\begin{aligned} \|x_{k+1} - p\|^2 \leq\ & \|x_k - p\|^2 + \alpha_k M_2 - \left(2 - \xi_k - \frac{\beta\xi_k\lambda_k}{\lambda_{k+1}}\right)\|w_k - y_k\|^2 \\ & - \left(2 - \xi_k - \frac{\beta\xi_k\lambda_k}{\lambda_{k+1}}\right)\|y_k - z_k\|^2. \end{aligned} \tag{3.26}$$

This implies that

$$\begin{aligned} &\left(2 - \xi_k - \frac{\beta\xi_k\lambda_k}{\lambda_{k+1}}\right)\|w_k - y_k\|^2 + \left(2 - \xi_k - \frac{\beta\xi_k\lambda_k}{\lambda_{k+1}}\right)\|y_k - z_k\|^2 \\ &\leq \|x_k - p\|^2 - \|x_{k+1} - p\|^2 + \alpha_k M_2. \end{aligned} \tag{3.27}$$

Next, to show that the sequence $\{x_k\}$ converges strongly to $\tilde{p} := P_\Omega(0)$, we will consider the proof in two cases.

Case 1. Suppose that $\|x_{k+1} - \tilde{p}\| \leq \|x_k - \tilde{p}\|$, for all $k \geq k_0$. This means that $\{\|x_k - \tilde{p}\|\}_{k\geq k_0}$ is a non-increasing sequence. Consequently, by utilizing this one together with the boundness property of $\{\|x_k - \tilde{p}\|\}$, we have that the limit of $\|x_k - \tilde{p}\|$ exists. It follows from the relation (3.27) and $\lim_{k\to\infty} \alpha_k = 0$ that

$$\lim_{k\longrightarrow\infty} \|w_k - y_k\| = 0, \tag{3.28}$$

and

$$\lim_{k\longrightarrow\infty} \|y_k - z_k\| = 0. \tag{3.29}$$

These imply that

$$\lim_{k\longrightarrow\infty} \|w_k - z_k\| = 0. \tag{3.30}$$

Additionally, from $\|w_k - x_k\| \leq \theta_k\|x_k - x_{k-1}\| + \alpha_k\theta_k\|x_k - x_{k-1}\| + \alpha_k\|x_k\|$, which together with $\lim_{k\to\infty} \theta_k\|x_k - x_{k-1}\| = 0$ and $\lim_{k\to\infty} \alpha_k = 0$ implies that

$$\lim_{k\longrightarrow\infty} \|w_k - x_k\| = 0. \tag{3.31}$$

This together with (3.30) yields that

$$\lim_{k\longrightarrow\infty} \|x_k - z_k\| = 0. \tag{3.32}$$

It follows from (3.29) that

$$\lim_{k\longrightarrow\infty} \|x_k - y_k\| = 0. \tag{3.33}$$

On the other hand, we provide the following:

$$\begin{aligned} \|w_k - \tilde{p}\|^2 &= \|(1-\alpha_k)(x_k - \tilde{p}) + (1-\alpha_k)\theta_k(x_k - x_{k-1}) - \alpha_k\tilde{p}\|^2 \\ &\leq (1-\alpha_k)\|x_k - \tilde{p}\|^2 + 2(1-\alpha_k)\theta_k\langle x_k - x_{k-1}, w_k - \tilde{p}\rangle \\ &\quad +2\alpha_k\langle -\tilde{p}, w_k - \tilde{p}\rangle \\ &\leq (1-\alpha_k)\|x_k - \tilde{p}\|^2 + 2(1-\alpha_k)\theta_k\|x_k - x_{k-1}\|\|w_k - \tilde{p}\| \\ &\quad +2\alpha_k\langle -\tilde{p}, w_k - x_k\rangle + 2\alpha_k\langle -\tilde{p}, x_k - \tilde{p}\rangle \\ &\leq (1-\alpha_k)\|x_k - \tilde{p}\|^2 + 2(1-\alpha_k)\theta_k\|x_k - x_{k-1}\|\|w_k - \tilde{p}\| \\ &\quad +2\alpha_k\|\tilde{p}\|\|w_k - x_k\| + 2\alpha_k\langle x_k - \tilde{p}, -\tilde{p}\rangle \\ &= (1-\alpha_k)\|x_k - \tilde{p}\|^2 + \alpha_k\Bigg(2(1-\alpha_k)\frac{\theta_k}{\alpha_k}\|x_k - x_{k-1}\|\|w_k - \tilde{p}\| \\ &\quad +2\|\tilde{p}\|\|w_k - x_k\| + 2\langle x_k - \tilde{p}, -\tilde{p}\rangle\Bigg). \end{aligned}$$

Using this one together with the relation (3.21), we have

$$\begin{aligned} \|x_{k+1} - \tilde{p}\|^2 &\leq (1-\alpha_k)\|x_k - \tilde{p}\|^2 + \alpha_k\Bigg(2(1-\alpha_k)\frac{\theta_k}{\alpha_k}\|x_k - x_{k-1}\|\|w_k - \tilde{p}\| \\ &\quad +2\|\tilde{p}\|\|w_k - x_k\| + 2\langle x_k - \tilde{p}, -\tilde{p}\rangle\Bigg). \end{aligned} \tag{3.34}$$

Furthermore, from (3.19), one sees that

$$\begin{aligned} \eta_k(1-\varphi_k)\|v_k - Az_k\|^2 &\leq \|z_k - p\|^2 - \|x_{k+1} - p\|^2 \\ &\leq (\|z_k - x_k\| + \|x_k - p\| - \|x_{k+1} - p\|)(\|z_k - p\| \\ &\quad +\|x_{k+1} - p\|). \end{aligned}$$

Thus, by using (3.32) and the existence of $\lim_{k\to\infty} \|x_k - \tilde{p}\|$, we obtain

$$\lim_{k\longrightarrow\infty} \|v_k - Az_k\| = 0. \tag{3.35}$$

On the other hand, from Lemma 3.4, we get that

$$\begin{aligned} &\left(2 - \sigma_k - \frac{\gamma\sigma_k\mu_k}{\mu_{k+1}}\right)\|Az_k - u_k\|^2 + \left(2 - \sigma_k - \frac{\gamma\sigma_k\mu_k}{\mu_{k+1}}\right)\|u_k - v_k\|^2 \\ \leq\ & \|Az_k - Ap\|^2 - \|v_k - Ap\|^2 \\ =\ & \|Az_k - v_k\|(\|Az_k - Ap\| + \|v_k - Ap\|). \end{aligned}$$

Then, applying (3.35) to the above relation, we have

$$\lim_{k\longrightarrow\infty} \|Az_k - u_k\| = 0, \tag{3.36}$$

and

$$\lim_{k\longrightarrow\infty} \|u_k - v_k\| = 0. \tag{3.37}$$

These imply that

$$\lim_{k\longrightarrow\infty} \|Az_k - v_k\| = 0. \tag{3.38}$$

Now, let $x^* \in \omega_w(x_k)$ and $\{x_{k_n}\}$ be a subsequence of $\{x_k\}$ such that $x_{k_n} \rightharpoonup x^*$, as $n \to \infty$. We know that, by using (3.31), (3.32), and (3.33), we also have $w_{k_n} \rightharpoonup x^*$, $y_{k_n} \rightharpoonup x^*$, and $z_{k_n} \rightharpoonup x^*$, as $n \to \infty$. This implies that $Az_{k_n} \rightharpoonup Ax^*$, as $n \to \infty$. This together with (3.36) yields that $u_{k_n} \rightharpoonup Ax^*$, as $n \to \infty$. Since C and Q are closed convex subsets, so C and Q are weakly closed, therefore, $x^* \in C$ and $Ax^* \in Q$.

Next, in view of the relations (3.7), (3.10), and (3.12), we have

$$\begin{aligned}
\xi_{k_n}\lambda_{k_n} f(y_{k_n}, y) &\geq \xi_{k_n}\lambda_{k_n} f(y_{k_n}, z_{k_n}) + \langle w_{k_n} - z_{k_n}, y - z_{k_n}\rangle \\
&\geq \xi_{k_n}\lambda_{k_n} f(w_{k_n}, z_{k_n}) - \xi_{k_n}\lambda_{k_n} f(w_{k_n}, y_{k_n}) - \frac{\beta\xi_{k_n}\lambda_{k_n}}{2\lambda_{k_n+1}}\|w_{k_n} - y_{k_n}\|^2 \\
&\quad - \frac{\beta\xi_{k_n}\lambda_{k_n}}{2\lambda_{k_n+1}}\|y_{k_n} - z_{k_n}\|^2 + \langle w_{k_n} - z_{k_n}, y - z_{k_n}\rangle \\
&\geq \xi_{k_n}\langle y_{k_n} - w_{k_n}, y_{k_n} - z_{k_n}\rangle - \frac{\beta\xi_{k_n}\lambda_{k_n}}{2\lambda_{k_n+1}}\|w_{k_n} - y_{k_n}\|^2 \\
&\quad - \frac{\beta\xi_{k_n}\lambda_{k_n}}{2\lambda_{k_n+1}}\|y_{k_n} - z_{k_n}\|^2 + \langle w_{k_n} - z_{k_n}, y - z_{k_n}\rangle, \qquad (3.39)
\end{aligned}$$

for each $y \in C$. Thus, by using (3.28), (3.29), (3.30), and the boundedness of $\{z_k\}$, we have the right-hand side of the above inequality tends to zero. It follows from the sequentially weakly upper semicontinuity of f and the parameters $\xi_{k_n}, \lambda_{k_n} > 0$ that

$$0 \leq \limsup_{n\to\infty} f(y_{k_n}, y) \leq f(x^*, y), \forall y \in C.$$

Then, we showed that $x^* \in EP(f, C)$. Similarly, we can show that

$$\begin{aligned}
\sigma_{k_n}\mu_{k_n} g(u_{k_n}, u) &\geq \sigma_{k_n}\langle u_{k_n} - Az_{k_n}, u_{k_n} - v_{k_n}\rangle - \frac{\gamma\sigma_{k_n}\mu_{k_n}}{2\mu_{k_n+1}}\|Az_{k_n} - u_{k_n}\|^2 \\
&\quad - \frac{\gamma\sigma_{k_n}\mu_{k_n}}{2\mu_{k_n+1}}\|u_{k_n} - v_{k_n}\|^2 + \langle Az_{k_n} - v_{k_n}, u - v_{k_n}\rangle, \qquad (3.40)
\end{aligned}$$

for each $u \in Q$. It follows from the facts (3.36), (3.37), (3.38), and the boundedness of $\{v_k\}$ that the right-hand side of the above inequality tends to zero. Thus, by utilizing the sequentially weakly upper semicontinuity of g and the parameters $\sigma_{k_n}, \mu_{k_n} > 0$, we have

$$0 \leq \limsup_{n\to\infty} g(u_{k_n}, u) \leq g(Ax^*, u), \forall u \in Q.$$

Then, we show that $Ax^* \in EP(g, Q)$ and so $x^* \in \Omega$. This shows that $\omega_w(x_k) \subset \Omega$.

Finally, by the properties of $\tilde{p} := P_\Omega(0)$ and $x^* \in \omega_w(x_k) \subset \Omega$, we get

$$\limsup_{k\to\infty}\langle x_k - \tilde{p}, -\tilde{p}\rangle = \lim_{n\to\infty}\langle x_{k_n} - \tilde{p}, -\tilde{p}\rangle = \langle x^* - \tilde{p}, -\tilde{p}\rangle \leq 0. \tag{3.41}$$

Hence, by using (3.23), (3.31), (3.34), (3.41), and Lemma 2.6, we have

$$\lim_{k\longrightarrow\infty} \|x_k - \tilde{p}\| = 0. \tag{3.42}$$

Case 2. Suppose that there exists a subsequence $\{\|x_{k_i}-\tilde{p}\|\}$ of $\{\|x_k-\tilde{p}\|\}$ such that

$$\|x_{k_i}-\tilde{p}\| < \|x_{k_i+1}-\tilde{p}\|, \ \forall i \in \mathbb{N}.$$

According to Lemma 2.7, there exists a non-decreasing sequence $\{m_n\} \subset \mathbb{N}$ such that $\lim_{n\to\infty} m_n = \infty$, and

$$\|x_{m_n}-\tilde{p}\| \leq \|x_{m_n+1}-\tilde{p}\| \text{ and } \|x_n-\tilde{p}\| \leq \|x_{m_n+1}-\tilde{p}\|, \ \forall n \in \mathbb{N}. \tag{3.43}$$

Combining with the relation (3.27) implies that

$$\begin{aligned}
&\left(2-\xi_{k_n}-\frac{\beta\xi_{k_n}\lambda_{m_n}}{\lambda_{m_n+1}}\right)\|w_{m_n}-y_{m_n}\|^2+\left(2-\xi_{k_n}-\frac{\beta\xi_{k_n}\lambda_{m_n}}{\lambda_{m_n+1}}\right)\|y_{m_n}-z_{m_n}\|^2\\
&\leq \|x_{m_n}-\tilde{p}\|^2-\|x_{m_n+1}-\tilde{p}\|^2+\alpha_{m_n}M_2\\
&\leq \|x_{m_n+1}-\tilde{p}\|^2-\|x_{m_n+1}-\tilde{p}\|^2+\alpha_{m_n}M_2\\
&= \alpha_{m_n}M_2.
\end{aligned}$$

Following the line proof of Case 1, we can show that

$$\lim_{n\longrightarrow\infty}\|w_{m_n}-y_{m_n}\|=0, \lim_{n\longrightarrow\infty}\|y_{m_n}-z_{m_n}\|=0, \tag{3.44}$$

$$\lim_{n\longrightarrow\infty}\|w_{m_n}-z_{m_n}\|=0, \lim_{n\longrightarrow\infty}\|x_{m_n}-y_{m_n}\|=0, \tag{3.45}$$

$$\lim_{n\longrightarrow\infty}\|w_{m_n}-x_{m_n}\|=0, \lim_{n\longrightarrow\infty}\|u_{m_n}-v_{m_n}\|=0, \tag{3.46}$$

$$\lim_{n\longrightarrow\infty}\|Az_{m_n}-v_{m_n}\|=0, \lim_{n\longrightarrow\infty}\|Az_{m_n}-u_{m_n}\|=0, \tag{3.47}$$

$$\limsup_{n\to\infty}\langle x_{m_n}-\tilde{p},-\tilde{p}\rangle \leq 0, \tag{3.48}$$

and

$$\begin{aligned}
\|x_{m_n+1}-\tilde{p}\|^2 \quad \leq \quad & (1-\alpha_{m_n})\|x_{m_n}-\tilde{p}\|^2+\alpha_{m_n}\Big(2\|\tilde{p}\|\|w_{m_n}-x_{m_n}\|\\
&+2(1-\alpha_{m_n})\frac{\theta_{m_n}}{\alpha_{m_n}}\|x_{m_n}-x_{m_n-1}\|\|w_{m_n}-\tilde{p}\|+2\langle x_{m_n}-\tilde{p},-\tilde{p}\rangle\Big).
\end{aligned}$$

This together with the relation (3.43) yields that

$$\begin{aligned}
\|x_{m_n+1}-\tilde{p}\|^2 \quad \leq \quad & (1-\alpha_{m_n})\|x_{m_n+1}-\tilde{p}\|^2+\alpha_{m_n}\Big(2\|\tilde{p}\|\|w_{m_n}-x_{m_n}\|\\
&+2(1-\alpha_{m_n})\frac{\theta_{m_n}}{\alpha_{m_n}}\|x_{m_n}-x_{m_n-1}\|\|w_{m_n}-\tilde{p}\|+2\langle x_{m_n}-\tilde{p},-\tilde{p}\rangle\Big).
\end{aligned}$$

Using this one together with the relation (3.43) again, we obtain

$$\begin{aligned}
\|x_n-\tilde{p}\|^2 \quad \leq \quad & 2\|\tilde{p}\|\|w_{m_n}-x_{m_n}\|+2(1-\alpha_{m_n})\frac{\theta_{m_n}}{\alpha_{m_n}}\|x_{m_n}-x_{m_n-1}\|\|w_{m_n}-\tilde{p}\|\\
&+2\langle x_{m_n}-\tilde{p},-\tilde{p}\rangle.
\end{aligned}$$

Then, by utilizing (3.23), (3.46), and (3.48), we have

$$\limsup_{n\to\infty}\|x_n-\tilde{p}\|^2 \leq 0.$$

Hence, we can conclude that the sequence $\{x_n\}$ converges strongly to $\tilde{p} = P_\Omega(0)$. This completes the proof. □

4. Conclusion

We present the algorithm for finding a solution of the split equilibrium problems for pseudomonotone bifunctions which satisfy Lipschitz-type continuous in real Hilbert spaces. We consider both inertial and subgradient extragradient methods without the prior knowledge of both the operator norm of the bounded linear operator and the Lipschitz constants of bifunctions for establishing the sequence which is strongly convergent to a solution of the split equilibrium problems.

References

1. E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Stud., 63, 1994, 127–149.
2. G. Bigi, M. Castellani, M. Pappalardo, M. Passacantando, Existence and solution methods for equilibria, Eur. J. Oper. Res., 227, 2013, 1–11.
3. P. Daniele, F. Giannessi, A. Maugeri, Equilibrium problems and variational models, Kluwer, 2003.
4. L.D. Muu, W. Oettli, Convergence of an adaptive penalty scheme for finding constrained equilibria, Nonlinear Anal., TMA 18, 1992, 1159–1166.
5. N. Petrot, M. Rabbani, M. Khonchaliew, V. Dadashi, A new extragradient algorithm for split equilibrium problems and fixed point problems, J. Ineq. Appl., 2019, 137.
6. A. Moudafi, Proximal point algorithm extended to equilibrium problems, J. Nat. Geom., 15, 1999, 91–100.
7. D.Q. Tran, L.M. Dung, V.H. Nguyen, Extragradient algorithms extended to equilibrium problems, Optimization, 57, 2008, 749–776.
8. D.V. Hieu, Halpern subgradient extragradient method extended to equilibrium problems, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., 111, 2017, 823–840.
9. B.T. Polyak, Some methods of speeding up the convergence of iteration methods, Comput. Math. Math. Phys., 4, 1964, 1–17.
10. F. Alvarez, Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert spaces, SIAM J. Optim., 9, 2004, 773–782.
11. F. Alvarez, H. Attouch, An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator damping, Set-valued. Anal., 9, 2001, 3–11.
12. T. Ngamkhum, K. Punpeng, M. Khonchaliew, Modified inertial extragradient algorithm with non-monotonic step sizes for pseudomonotone equilibrium problems and quasi-nonexpansive mapping, Carpathian J. Math., 40(2), 2024.
13. N.T. Vinh, L.D. Muu, Inertial extragradient algorithms for solving equilibrium problems, Acta Math. Vietnam., 44, 2019, 639–663.
14. Z. He, The split equilibrium problem and its convergence algorithms, J. Ineq. Appl., 2012, 1–15.
15. A. Moudafi, Split monotone variational inclusions, J. Optim. Theory Appl., 150, 2011, 275–283.
16. S. Suantai, N. Petrot, M. Khonchaliew, Inertial extragradient methods for solving split equilibrium problems, Mathematics, 9(16), 2021, 1884.
17. S. Karamardian, S. Schaible, J.P. Crouzeix, Characterizations of generalized monotone maps, J. Optim. Theory Appl., 76, 1993, 399–413.
18. A. Cegielski, Iterative methods for fixed point problems in Hilbert spaces, Lecture notes in mathematics 2057, Springer-Verlag, Berlin, Heidelberg, Germany, 2012.
19. K. Goebel, S. Reich, Uniform convexity, hyperbolic geometry, and nonexpansive mappings, Marcel Dekker, New York, 1984.
20. H.-K. Xu, Iterative algorithms for nonlinear operators., J. Lond. Math. Soc., 66, 2002, 240–256.
21. P.E. Mainge, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Anal., 16, 2008, 899–912.
22. M. Bianchi, S. Schaible, Generalized monotone bifunctions and equilibrium problems, J. Optim. Theory Appl., 90, 1996, 31–43.
23. T.D. Quoc, P.N. Anh, L.D. Muu, Dual extragradient algorithms extended to equilibrium problems, J. Glob. Optim., 52, 2012, 139–159.