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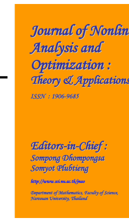

CONE METRIC TYPE SPACE AND NEW COUPLED FIXED POINT THEOREMS

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ABSTRACT. The notion of coupled fixed point was initiated in 2006 by Bhaskar and Lakshmikantham. On the other hand, Radenović and Kadelburg [S. Radenović, Z. Kadelburg, Quasi-contractions on symmetric and cone symmetric spaces, Banach J. Math. Anal. 5 (1) (2011) 38-50] defined cone metric type space and proved several fixed point theorems. In this paper we introduce the concept of a coupled fixed point for a contractive condition in cone metric type space and prove some coupled fixed point theorems.

KEYWORDS : Cone metric type space; Coupled fixed point; Cone metric space.
AMS Subject Classification: 47H10.

1. INTRODUCTION AND PRELIMINARIES

The symmetric space, as metric-like spaces lacking the triangle inequality was introduced in 1931 by Wilson [31]. Recently, a new type of spaces which they called metric type spaces are defined by Khamsi and Hussain [16] and Boriceanu [6]. Also, Jovanović et al. [14], Rahimi and Soleimani Rad [24], Bota et al. [7], Pavlović et al. [20], Singh et al. [28] and Hussain et al. [11] generalized some fixed point theorems of metric spaces by considering metric type space.

On the other hand, the cone metric space was initiated in 2007 by Huang and Zhang [12] and several fixed and common fixed point results in cone metric spaces were introduced in [2, 3, 13, 22, 23, 25, 30] and the references contained therein. In the sequel, analogously with definition of metric type space, Ćvetković et al. [9] and Radenović and Kadelburg [21] defined cone metric type space and proved several fixed and common fixed point theorems (See [17]).

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In 2006, Bhaskar and Lakshmikantham [5] introduced the concept of coupled fixed point theorem in partially ordered metric spaces. Then, some other authors generalized this concept and proved several common coupled fixed point and coupled fixed point theorems in ordered metric and ordered cone metric spaces (See [1, 4, 8, 15, 18, 19, 26, 27, 29] and the references contained therein).

In this paper we define the concept of coupled fixed point in a cone metric type space and prove some coupled fixed point theorems. Our results generalize, extend and unify several well known comparable results in the literature.

Let us start by defining some important definitions.

Definition 1.1. (See [31]). Let X be a nonempty set and the mapping $D : X \times X \to [0, \infty)$ satisfies

$$(S1) \qquad D(x, y) = 0 \Longleftrightarrow x = y;$$
$$(S2) \qquad D(x, y) = D(y, x),$$

for all $x, y \in X$. Then D is called a symmetric on X and (X, D) is called a symmetric space.

Definition 1.2. (See [10, 12]). Let E be a real Banach space and P be a subset of E. Then P is called a cone if and only if
(a) P is closed, non-empty and $P \neq \{\theta\}$;
(b) $a, b \in R, a, b \geq 0, x, y \in P$ imply that $ax + by \in P$;
(c) if $x \in P$ and $-x \in P$, then $x = \theta$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by

$$x \preceq y \Longleftrightarrow y - x \in P.$$

We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in intP$ (where $intP$ is the interior of P). The cone P is named normal if there is a number $K > 0$ such that for all $x, y \in E$, we have

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|.$$

The least positive number satisfying the above is called the normal constant of P.

Definition 1.3. (See [12]). Let X be a nonempty set and the mapping $d : X \times X \to E$ satisfies
$(d1)$ $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
$(d2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;
$(d3)$ $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
Then, d is called a cone metric on X and (X, d) is called a cone metric space.

Example 1.4. (See [12, 22]).
(i) Let $E = \mathbf{R}^2$, $P = \{(x, y) \in E | x, y \geq 0\} \subset \mathbf{R}^2$, $X = \mathbf{R}$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.
(ii) Let $X = [0, 1]$, $E = C^2_{\mathbf{R}}[0, 1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, $P = \{f \in E | f \geq 0\}$ and $d(x, y)(t) = |x - y|2^t$. Then (X, d) is a cone metric space with non-normal solid cone.

Definition 1.5. (See [16]). Let X be a nonempty set, and $K \geq 1$ be a real number. Suppose the mapping $D_m : X \times X \to [0, \infty)$ satisfies
$(D1)$ $D_m(x, y) = 0$ if and only if $x = y$;
$(D2)$ $D_m(x, y) = D_m(y, x)$ for all $x, y \in X$;
$(D3)$ $D_m(x, z) \leq K(D_m(x, y) + D_m(y, z))$ for all $x, y, z \in X$.

(X, D_m, K) is called metric type space. Obviously, for $K = 1$, metric type space is a metric space.

Definition 1.6. (See [9, 21]). Let X be a nonempty set, $K \geq 1$ be a real number and E a real Banach space with cone P. Suppose that the mapping $D : X \times X \to E$ satisfies
$(cd1)$ $\theta \preceq D(x, y)$ for all $x, y \in X$ and $D(x, y) = \theta$ if and only if $x = y$;
$(cd2)$ $D(x, y) = D(y, x)$ for all $x, y \in X$;
$(cd3)$ $D(x, z) \preceq K(D(x, y) + D(y, z))$ for all $x, y, z \in X$.
(X, D, K) is called cone metric type space. Obviously, for $K = 1$, cone metric type space is a cone metric space.

Example 1.7. (See [9]). Let $B = \{e_i | i = 1, \cdots, n\}$ be orthonormal basis of $\mathbf{R}^n$ with inner product $(.,.)$ and $p > 0$. Define

$$X_p = \Big\{[x] | x : [0, 1] \to \mathbf{R}^n, \int_0^1 |(x(t), e_j)|^p dt \in \mathbf{R}, j = 1, 2, \cdots, n\Big\},$$

where $[x]$ represents class of element x with respect to equivalence relation of functions equal almost everywhere. Let $E = \mathbf{R}^n$ and

$$P_B = \big\{y \in \mathbf{R}^n | (y, e_i) \geq 0, i = 1, 2, \cdots, n\big\}$$

be a solid cone. Define $d : X_p \times X_p \to P_B \subset \mathbf{R}^n$ by

$$d(f, g) = \sum_{i=1}^n e_i \int_0^1 |((f - g)(t), e_i)|^p dt, \qquad f, g \in X_p.$$

Then (X_p, d, K) is cone metric type space with $K = 2^{p-1}$.

Similarly, we define convergence in cone metric type spaces.

Definition 1.8. (See [9, 21]). Let (X, D, K) be a cone metric type space, $\{x_n\}$ a sequence in X and $x \in X$.
(i) $\{x_n\}$ converges to x if for every $c \in E$ with $\theta \ll c$ there exist $n_0 \in \mathbf{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$, and we write $\lim_{n\to\infty} x_n = x$.
(ii) $\{x_n\}$ is called a Cauchy sequence if for every $c \in E$ with $\theta \ll c$ there exist $n_0 \in \mathbf{N}$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$.

Lemma 1.9. (*See* [9, 21]). *Let (X, D, K) be a cone metric type space over ordered real Banach space E. Then the following properties are often used, particularly when dealing with cone metric type spaces in which the cone need not be normal.*
(P_1) *If $u \preceq v$ and $v \ll w$, then $u \ll w$.*
(P_2) *If $\theta \preceq u \ll c$ for each $c \in intP$, then $u = \theta$.*
(P_3) *If $u \preceq \lambda u$ where $u \in P$ and $0 \leq \lambda < 1$, then $u = \theta$.*
(P_4) *Let $x_n \to \theta$ in E and $\theta \ll c$. Then there exists positive integer n_0 such that $x_n \ll c$ for each $n > n_0$.*

2. MAIN RESULTS

At the first, we define the concept of the coupled fixed point in a cone metric type space. Then, we prove some fixed point theorems as generalization of Sabetghadam et al.'s works in [26] and Bhaskar and Lakshmikantham's results in [5].

Definition 2.1. Let (X, D, K) be a cone metric type space with constant $K \geq 1$. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Note that if (x, y) is a coupled fixed point of F then (y, x) is coupled fixed point of F too.

Theorem 2.2. *Let (X, D, K) be a complete cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$D(F(x, y), F(x^*, y^*)) \preceq \alpha D(x, x^*) + \beta D(y, y^*), \tag{2.1}$$

where α, β are nonnegative constants with $\alpha + \beta < 1/K$. Then F has a unique coupled fixed point.

Proof. Let $x_0, y_0 \in X$ and set

$$x_1 = F(x_0, y_0), y_1 = F(y_0, x_0), \cdots, x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n).$$

From (2.1), we have

$$D(x_n, x_{n+1}) \preceq \alpha D(x_{n-1}, x_n) + \beta D(y_{n-1}, y_n), \tag{2.2}$$

and

$$D(y_n, y_{n+1}) \preceq \alpha D(y_{n-1}, y_n) + \beta D(x_{n-1}, x_n). \tag{2.3}$$

Let $D_n = D(x_n, x_{n+1}) + D(y_n, y_{n+1})$. From (2.2) and (2.3), we get

$$\begin{aligned} D_n &\preceq (\alpha + \beta)\big(D(x_{n-1}, x_n) + D(y_{n-1}, y_n)\big) \\ &= \lambda D_{n-1}, \end{aligned}$$

where $\lambda = \alpha + \beta < 1/K$. Thus, for all n,

$$\theta \preceq D_n \preceq \lambda D_{n-1} \preceq \lambda^2 D_{n-2} \preceq \cdots \preceq \lambda^n D_0. \tag{2.4}$$

If $D_0 = \theta$ then (x_0, y_0) is a coupled fixed point of F. Now, let $D_0 > \theta$. If $m > n$, we have

$$\begin{aligned} D(x_n, x_m) &\preceq K[D(x_n, x_{n+1}) + D(x_{n+1}, x_m)] \\ &\preceq KD(x_n, x_{n+1}) + K^2[D(x_{n+1}, x_{n+2}) + D(x_{n+2}, x_m)] \\ &\preceq \cdots \preceq KD(x_n, x_{n+1}) + K^2 D(x_{n+1}, x_{n+2}) + \cdots \\ &\quad +K^{m-n-1} D(x_{m-2}, x_{m-1}) + K^{m-n} D(x_{m-1}, x_m), \end{aligned} \tag{2.5}$$

and similarly,

$$\begin{aligned} D(y_n, y_m) &\preceq KD(y_n, y_{n+1}) + K^2 D(y_{n+1}, y_{n+2}) + \cdots \\ &\quad +K^{m-n-1} D(y_{m-2}, y_{m-1}) + K^{m-n} D(y_{m-1}, y_m). \end{aligned} \tag{2.6}$$

Adding up (2.5) and (2.6) and using (2.4). Since $\lambda < 1/K$, we have

$$\begin{aligned} D(x_n, x_m) + D(y_n, y_m) &\preceq KD_n + K^2 D_{n+1} + \cdots + K^{m-n} D_{m-1} \\ &\preceq [K\lambda^n + K^2 \lambda^{n+1} + \cdots + K^{m-n} \lambda^{m-1}] D_0 \\ &\preceq \frac{K\lambda^n}{1 - K\lambda} D_0 \to \theta \quad as \quad n \to \infty. \end{aligned}$$

Now, by (P_1) and (P_4), it follows that for every $c \in intP$ there exist positive integer N such that $D(x_n, x_m) + D(y_n, y_m) \ll c$ for every $m > n > N$, so $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is a complete cone metric type space, there exist $x', y' \in X$ such that $x_n \to x'$ and $y_n \to y'$ as $n \to \infty$. Now, we prove that $F(x', y') = x'$ and $F(y', x') = y'$. From $(cd3)$ and (2.1), we have

$$\begin{aligned} D(F(x', y'), x') &\preceq K[D(F(x', y'), x_{n+1}) + D(x_{n+1}, x')] \\ &\preceq K\alpha D(x', x_n) + K\beta D(y', y_n) + KD(x_{n+1}, x'). \end{aligned}$$

Since $x_n \to x'$ and $y_n \to y'$, by using Lemma 1.9 we have $D(F(x',y'),x') = \theta$; that is, $F(x',y') = x'$. Similarly, we can get $D(F(y',x'),y') = \theta$; that is, $F(y',x') = y'$. Therefore, (x',y') is a coupled fixed point of F. Now, if (x'',y'') is another coupled fixed point of F, then

$$D(x',x'') + D(y',y'') \preceq \lambda\big(D(x',x'') + D(y',y'')\big). \tag{2.7}$$

Since $\lambda = \alpha+\beta < \frac{1}{K}$ and $K \geq 1$, (2.7) and (P_2) imply that $D(x',x'')+D(y',y'') = \theta$. Thus, $(x',y') = (x'',y'')$. This completes the proof. □

Corollary 2.3. *Let (X,D,K) be a complete cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$D(F(x,y),F(x^*,y^*)) \preceq \frac{\gamma}{2}[D(x,x^*) + D(y,y^*)], \tag{2.8}$$

where $\gamma \in [0, \frac{1}{K})$ is a constant. Then F has a unique coupled fixed point.

Proof. Corollary 2.3 follows from Theorem 2.2 by setting $\alpha = \beta = \frac{\gamma}{2}$. □

Theorem 2.4. *Let (X,D,K) be a complete cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$D(F(x,y),F(x^*,y^*)) \preceq \alpha D(F(x,y),x) + \beta D(F(x^*,y^*),x^*), \tag{2.9}$$

where α, β are nonnegative constants with $K\alpha + \beta < 1$. Then F has a unique coupled fixed point.

Proof. Let $x_0, y_0 \in X$ and set

$$x_1 = F(x_0,y_0), y_1 = F(y_0,x_0), \cdots, x_{n+1} = F(x_n,y_n), y_{n+1} = F(y_n,x_n).$$

From (2.9), we have

$$\begin{aligned} D(x_n,x_{n+1}) &\preceq \alpha D(F(x_{n-1},y_{n-1}),x_{n-1}) + \beta D(F(x_n,y_n),x_n) \\ &= \alpha D(x_n,x_{n-1}) + \beta D(x_{n+1},x_n) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} D(y_n,y_{n+1}) &\preceq \alpha D(F(y_{n-1},x_{n-1}),y_{n-1}) + \beta D(F(y_n,x_n),y_n) \\ &= \alpha D(y_n,y_{n-1}) + \beta D(y_{n+1},y_n). \end{aligned} \tag{2.11}$$

From (2.10) and (2.11), we have

$$\begin{aligned} D(x_n,x_{n+1}) &\preceq \lambda D(x_{n-1},x_n), \\ D(y_n,y_{n+1}) &\preceq \lambda D(y_{n-1},y_n), \end{aligned}$$

where $\lambda = \alpha/(1-\beta) < 1/K$. By the analogous arguments as in Theorem 2.2 we conclude that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is a complete cone metric type space, there exist $x', y' \in X$ such that $x_n \to x'$ and $y_n \to y'$ as $n \to \infty$. Now, we prove that $F(x',y') = x'$ and $F(y',x') = y'$. From $(cd3)$ and (2.9), we have

$$\begin{aligned} D(F(x',y'),x') &\preceq K[D(F(x',y'),x_{n+1}) + D(x_{n+1},x')] \\ &\preceq K\alpha D(F(x',y'),x') + K\beta D(F(x_n,y_n),x_n) + KD(x_{n+1},x'). \end{aligned}$$

Since $x_n \to x'$ and $y_n \to y'$, by using Lemma 1.9 we have $D(F(x',y'),x') = \theta$; that is, $F(x',y') = x'$. Similarly, we can get $D(F(y',x'),y') = \theta$; that is, $F(y',x') = y'$.

Therefore, (x', y') is a coupled fixed point of F. Now, if (x'', y'') is another coupled fixed point of F, then

$$D(x', x'') \preceq \alpha D(F(x', y'), x') + \beta D(F(x'', y''), x'').$$

Therefore, $D(x', x'') = \theta$; that is $x' = x''$. Similarly, we have $y' = y''$. Thus $(x', y') = (x'', y'')$. This completes the proof. □

Corollary 2.5. *Let (X, D, K) be a complete cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$D(F(x, y), F(x^*, y^*)) \preceq \frac{\gamma}{2}[D(F(x, y), x) + D(F(x^*, y^*), x^*)], \tag{2.12}$$

where $\gamma \in [0, \frac{2}{K+1})$ is a constant. Then F has a unique coupled fixed point.

Proof. Similar to Corollary 2.3, Corollary 2.5 follows from Theorem 2.4 by setting $\alpha = \beta = \frac{\gamma}{2}$. □

Theorem 2.6. *Let (X, D, K) be a complete cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$D(F(x, y), F(x^*, y^*)) \preceq \alpha D(F(x, y), x^*) + \beta D(F(x^*, y^*), x), \tag{2.13}$$

where α, β are nonnegative constants with $\alpha + \beta < 2/(K(K + 1))$. Then F has a unique coupled fixed point.

Proof. Let $x_0, y_0 \in X$ and set

$$x_1 = F(x_0, y_0), y_1 = F(y_0, x_0), \cdots, x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n).$$

From (2.13), we have

$$\begin{aligned} D(x_n, x_{n+1}) &\preceq \alpha D(F(x_{n-1}, y_{n-1}), x_n) + \beta D(F(x_n, y_n), x_{n-1}) \\ &\preceq K\beta[D(x_n, x_{n-1}) + D(x_{n+1}, x_n)] \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} D(x_{n+1}, x_n) &\preceq \alpha D(F(x_n, y_n), x_{n-1}) + \beta D(F(x_{n-1}, y_{n-1}), x_n) \\ &\preceq K\alpha[D(x_n, x_{n-1}) + D(x_{n+1}, x_n)]. \end{aligned} \tag{2.15}$$

Adding up (2.14) and (2.15), we have

$$D(x_n, x_{n+1}) \preceq \lambda D(x_{n-1}, x_n),$$

where $\lambda = \frac{K(\alpha+\beta)}{2-K(\alpha+\beta)} < \frac{1}{K}$.
Similarly,

$$D(y_n, y_{n+1}) \preceq \lambda D(y_{n-1}, y_n),$$

where $\lambda = \frac{K(\alpha+\beta)}{2-K(\alpha+\beta)} < \frac{1}{K}$. By the same arguments as in Theorem 2.2 we conclude that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is a complete cone metric type space, there exist $x', y' \in X$ such that $x_n \to x'$ and $y_n \to y'$ as $n \to \infty$. Now, we prove that $F(x', y') = x'$ and $F(y', x') = y'$. From $(cd3)$ and (2.13), we have

$$\begin{aligned} D(F(x', y'), x') &\preceq K[D(F(x', y'), x_{n+1}) + D(x_{n+1}, x')] \\ &\preceq K\alpha D(F(x', y'), x_n) + K\beta D(F(x_n, y_n), x') + KD(x_{n+1}, x'). \end{aligned}$$

Since $x_n \to x'$ and $y_n \to y'$, by using Lemma 1.9 we have $D(F(x', y'), x') = \theta$; that is, $F(x', y') = x'$. Similarly, we can get $D(F(y', x'), y') = \theta$; that is, $F(y', x') = y'$. Therefore, (x', y') is a coupled fixed point of F. By the same arguments as in Theorem 2.2 we conclude that (x', y') is unique. □

Corollary 2.7. *Let (X, D, K) be a complete cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$D(F(x,y), F(x^*,y^*)) \preceq \frac{\gamma}{2}[D(F(x,y), x^*) + D(F(x^*,y^*), x)], \tag{2.16}$$

where $\gamma \in [0, 2/(K^2+K))$ is a constant. Then F has a unique coupled fixed point.

Remark 2.8.
(i) The Theorems 2.2, 2.4 and 2.6 generalized some fixed point theorems of cone metric spaces of Sabeghadam et al.'s works in [26] by considering cone metric type spaces.
(ii) Choosing $K = 1$ from the Corollaries 2.3, 2.5 and 2.7 we get the Theorems 2.1, 2.2 and 2.4 from Bhaskar and Lakshmikantham's results in a cone metric space.

Example 2.9. Let $E = \mathbf{R}$, $P = [0, \infty)$, $X = [0, 1]$ and $D : X \times X \to [0, \infty)$ be defined by $D(x, y) = |x - y|^2$. Then (X, D) is a cone metric type space, but it is not a cone metric space since the triangle inequality is not satisfied. Starting with Minkowski inequality, we get $|x - z|^2 \leq 2(|x - y|^2 + |y - z|^2)$. Here $K = 2$.
Define the mapping $F : X \times X \to X$ by $F(x, y) = (x + y)/4$. Therefore, F satisfies the contractive condition (2.8) for $\gamma = 1/4 \in [0, 1/K)$ with $K = 2 \geq 1$; that is,

$$D(F(x,y), F(x^*,y^*)) \preceq \frac{1}{8}[D(x, x^*) + D(y, y^*)].$$

According to Corollary 2.3, F has a unique coupled fixed point. $(0, 0)$ is a unique coupled fixed point of F

Remark 2.10. Similar to previous example, one can get many examples of other coupled fixed point theorems in cone metric type spaces.

3. GENERAL APPROACH

We start with following Lemma.

Lemma 3.1. (1) *Suppose that (X, D, K) is a cone metric type space with $K \geq 1$. Then, (X^2, D_1, K) is a cone metric type space with*

$$D_1((x,y),(u,v)) = D(x,u) + D(y,v). \tag{3.1}$$

Further, (X, D, K) is complete if and only if (X^2, D_1, K).
(2) *Mapping $F : X^2 \to X$ has a coupled fixed point if and only if mapping $T_F : X^2 \to X^2$ defined by $T_F(x, y) = (F(x, y), F(y, x))$ has a fixed point in X^2.*

Proof. (1) Similar to cone metric version, one can check $(cd1)$ and $(cd2)$ conditions. Thus, we only prove $(cd3)$ condition for (X^2, D_1, K). Since (X, D, K) is a cone metric type space, we have

$$D(x,u) \preceq K(D(x,z) + D(z,u)) \tag{3.2}$$

for all $x, z, u \in X$ and

$$D(y,v) \preceq K(D(y,w) + D(w,v)) \tag{3.3}$$

for all $y, v, w \in X$. Adding up (3.2) and (3.3), we get

$$\begin{aligned} D_1((x,y),(u,v)) &= D(x,u) + D(y,v) \\ &\preceq K(D(x,z) + D(z,u)) + K(D(y,w) + D(w,v)) \\ &= K(D(x,z) + D(y,w)) + K(D(z,u) + D(w,v)) \\ &= K\big[D_1((x,y),(z,w)) + D_1((z,w),(u,v))\big]. \end{aligned}$$

Thus, (X^2, D_1, K) is a cone metric type space. The completeness proof is easy and is left to the reader.
(2) Let (x, y) be a coupled fixed point of F. In this case, $F(x, y) = x$ and $F(y, x) = y$. Thus,

$$T_F(x, y) = (F(x, y), F(y, x)) = (x, y).$$

Therefore, $(x, y) \in X^2$ is a fixed point of T_F. Conversely, suppose that $(x, y) \in X^2$ is a fixed point of T_F, then

$$T_F(x, y) = (x, y).$$

Consequently, $F(x, y) = x$ and $F(y, x) = y$. □

Now, we prove a general version of Theorem 2.2.

Theorem 3.2. *Let (X, D, K) be a complete cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$D(F(x, y), F(x^*, y^*)) + D(F(y, x), F(y^*, x^*)) \preceq \lambda[D(x, x^*) + D(y, y^*)], \tag{3.4}$$

where λ is a nonnegative constant with $\lambda < 1/K$. Then F has a unique coupled fixed point.

Proof. According to (3.1) and Lemma 3.1(2), the contractive condition (3.4) for all $Y = (x, y), V = (u, v) \in X^2$ become

$$D_1(T_F(Y), T_F(V)) \preceq \lambda D_1(Y, V).$$

Since $\lambda < 1/K$, the proof further follows by ([14], Theorem 3.3). □

Remark 3.3. Now, we can get Theorem 2.2 such as the result of Theorem 3.2. Also, one can prove some other theorems for general contractive version and get Theorems 2.4 and 2.6.

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