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A MODIFIED ITERATIVE ALGORITHM FOR PREŠIĆ TYPE NONEXPANSIVE OF MAPPING IN HADAMARD SPACES

PATCHARAPUND KHAJORNPHET[1], TIWABHORN KHANPANAK[2], SUPANSA NOINAKORN[2] AND NUTTAPOL PAKKARANANG[2,*]

[1] Mathematics Program, Faculty of Science and Technology, Phetchabun Rajabhat University, Phetchabun 67000, Thailand.

[2] Mathematics and Computing Science Program, Faculty of Science and Technology, Phetchabun Rajabhat University, Phetchabun 67000, Thailand.

ABSTRACT. In this paper, we introduce a modified iterative method tailored for Prešić nonexpansive mappings in Hadamard spaces. Furthermore, we establish a Δ-convergence theorem aimed at approximating fixed points through the proposed iterative algorithm under mild conditions. Our findings not only enhance existing results in the field but also offer a broader applicability within the literature.

KEYWORDS: Prešić type nonexpansive mapping, fixed point, S-type iterative algorithm, Convergence theorem, CAT(0) spaces.
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1. Introduction

Let (X, d) be a metric space and C be a nonempty subset of X. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y)$$

for all $x, y \in C$. A point $x \in C$ is called a *fixed point* of T if $x = Tx$ and, forward, $F(T)$ denotes the set of fixed points of the mapping T. A sequence $\{x_n\}$ in C is called the *approximate fixed point sequence* for T if

$$\lim_{n\to\infty} (x_n, Tx_n) = 0.$$

Next, we will introduce an interesting generalization of Banach principle has been discovered in 1965 by authors by Prešić [21]

* *Corresponding author.*
Email address : st631102132110@pcru.ac.th (P. Khajornphet), mathiproof@gmail.com (T. Khanpanak), supansa.noi@pcru.ac.th (S. Noinakorn), nuttapol.pak@pccru.ac.th (N. Pakkaranang).
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Theorem 1.1. *([21]) Let (X,d) be a complete metric space, k a positive integer $\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_k \in \mathbb{R}^+, \Sigma_{i=1}^{k}\alpha_i = \alpha < 1$ and $f : X^k \to X$ a mapping satisfying $d(f(x_0,x_1,x_2,\ldots,x_{k-1}), f(x_1,x_2,x_3,\ldots,x_{k-1}) \leq \Sigma_{i=0}^{k-1}\alpha_{i+1}d(x_i,x_{i+1})$ for all $x_0,x_1,x_2,\ldots,x_{k-1} \in X$ for all $x_0,x_1,x_2,\ldots,x_{k-1} \in X$. Then f has a unique fixed point x^* , that is , there exists a unique $x^* \in X$ such that $f(x^*,x^*,\ldots,x^*) = x^*$ and the sequence defined by $x_{n+1} = f(x_{n-k+1},\ldots,x_n), n = k-1,k,k+1,\ldots$ converges to x^* for any $x_0,x_1,x_2,\ldots,x_{k-1} \in X$.*

We note that Theorem 1.1 collapses into the classical Banach contraction principle for $k = 1$. When $\alpha = 1$, T is called nonexpansive.

Some generalization of mapping of Theorem 1.1 have been obtained in [7, 22] (See also [19, 20]). Nonexpansive type mappings do not inherit much from contraction mappings.

On the other hand, in 1953, Mann [17] introduced the following iteration $\{x_n\}$ for approximating a fixed point of a nonexpansive mapping T in a Hilbert space H, which is defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n \end{cases} \tag{1.1}$$

for all $n \geq 1$, where $\{\alpha_n\}$ is a appropriate sequence in $(0,1)$. In 1974, Ishikawa [12] introduced the following iteration $\{x_n\}$ for approximating a fixed point of a nonexpansive mapping T in a Hilbert space H, which is defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n \end{cases} \tag{1.2}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0,1)$ with some conditions. This Ishikawa iteration reduces to the Mann iteration when $\beta_n = 0$ for all $n \geq 1$.

In 2007, Agarwal et al. [2] introduced the following S-iteration $\{x_n\}$ for a nearly asymptotically nonexpansive mapping T in a Banach space E, which is defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T(y_n), \\ y_n = (1-\beta_n)x_n + \beta_n T(x_n) \end{cases} \tag{1.3}$$

for all $n \geq 1$, where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$ with some conditions. Note that this iteration is independent and converges faster than both of the Ishikawa and Mann iterations.

Inspired and motivated by the S-type iterative algorithm x_n introduced by Agarwal et al., we propose a modification of the S-iteration x_n for Prešić nonexpansive mappings in Hadamard spaces. Additionally, we establish a Δ-convergence theorem to approximate fixed points using the proposed iterative algorithm under mild conditions.

2. Preliminaries

Let (x,d) be a metric space. A geodesic path from x to y in a metric space (X,d) is a mapping $c : [0,d(x,y)] \to X$ such that $c(0) = x, c(d(x,y)) = y$, and $d(c(t),c(t')) = |t-t'|$ for all $t,t' \in [0,d(x,y)]$. The image of c is called a geodesic segment. The space X is known as uniquely geodesic space if every two points of X are joined by a unique geodesic segment. A geodesic triangle $\triangle\ (x_1,x_2,x_3)$ in a geodesic metric space X consists of three points $x_1,x_2,x_3 \in X$ and a geodesic segment between each pair of these points. A comparison triangle for a geodesic triangle $\triangle\ (x_1,x_2,x_3)$ in X is triangle $\overline{\triangle}(x_1,x_2,x_3) :=\triangle\ (\overline{x_1},\overline{x_2},\overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j})$ for all $i, j = 1, 2, 3$. A geodesic space X is a $CAT(0)$ space if the following inequality $d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y})$ holds for all $x, y \in \triangle$ in X and $\overline{x}, \overline{y} \in \overline{\triangle}$ in $\mathbb{R}^2$. Since a $CAT(0)$ space (X, d) is uniquely geodesic, we use $[x, y]$ to denote the geodesic segment between x and y and $(1-\alpha)x \oplus \alpha y$ to denote the unique point $z \in [x, y]$ such that $d(z, x) = \alpha d(x, y)$ and $d(z, y) = (1-\alpha)d(x, y)$. A subset C of a $CAT(0)$ space in convex if $[x, y] \subset C$ for all $x, y \in C$. Some examples of $CAT(0)$ spaces are :

(i) Hilbert spaces - the only Banach spaces which are $CAT(0)$;
(ii) $\mathbb{R}$- trees : a metric space X is and $\mathbb{R}$- tree if for $x, y \in X$,
(iii) Classical hyperbolic spaces $\mathbb{H}^n$;
(iv) Complete simply connected Riemannian manifolds with nonpositive sectional curvature;
(v) Euclidean buildings. These examples illustrate that the class of $CAT(0)$ spaces encompasses both smooth and singular objects.

There are several ways to construct new $CAT(0)$ spaces from known ones. Clearly, a convex subset of a $CAT(0)$ space, endowed with the induced metric, is itself a $CAT(0)$ spaces. Fixed point theorems in $CAT(0)$ spaces (specially in $\mathbb{R}$ - trees) are applicable in biology, computer science and graph theory theory (see e.g.,[6, 9]). Kirk [16] proved the existence of fixed points for nonexpansive mappings in complete $CAT(0)$ spaces as follows: If C is a bounded, closed and convex subset of a complete $CAT(0)$ space X, then a nonexpansive mapping $T : C \to C$ has a fixed point. On the other hand,the iterative construction of fixed points of nonlinear mappings is itself a fascinating filed of research (see [3]) . The fixed point problems for nonexpansive mappings and their generalizations have been studied extensively on linear as well as nonlinear domains (see [1, 12, 13, 14, 15, 17]).

For a bounded sequence $x_n \subset X$, we define a function $r(., \{x_n\}) : X \to \mathbb{R}^+$ by $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$, for all $x \in X$.

The asymptotic radius $r_c(\{x_n\})$ of $\{x_n\}$ with respect to a subset $C \subseteq X$ is defined as $r_c(\{x_n\}) = \inf_{x\in C} r(x, \{x_n\})$, while the asymptotic center $A_c(\{x_n\})$ of $(\{x_n\})$ with respect to C is, by definition, the set $A_c(\{x_n\}) = \{y \in C : r(y, \{x_n\}) = r_c(\{x_n\})$.

A sequence $\{x_n\}$ $\triangle$ - converges to $x \in C$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$ (see [9]; in this case, we write x as $\triangle$-limit of $\{x_n\}$, i.e., $\triangle \lim_{n\to\infty} x_n = x)$.

In the sequel, we shall need the following well known results to prove in main results section.

Lemma 2.1. [8] *if X is a $CAT(0)$ space, then $d(z, \alpha x \oplus (1-\alpha)y) \leq \alpha d(z, x) + (1-\alpha)d(z, y)$ for any $\alpha \in [0, 1]$ and $x, y, z \in X$.*

Lemma 2.2. [8] *A geodesic space X is a $CAT(0)$ space if and only if $d(z, \alpha x \oplus (1-\alpha)y)^2 \leq \alpha d(z, x)^2 + (1-\alpha)d(z, y)^2 - (1-\alpha)d(x, y)^2$ for any $\alpha \in [0, 1]$ and $x, y, z \in X$.*

Lemma 2.3. [10] *Let C be a nonempty, closed and convex subset of a complete $CAT(0)$ space X. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to C.*

Lemma 2.4. [10] *Let C be a nonempty closed and convex subset of a $CAT(0)$ space X. Let $\{x_n\}$ be a bounded sequence in C such that $A_c(\{x_n\}) = \{x\}$ and $r_c(\{x_n\}) = \rho$. If $\{y_n\}$ is another sequence in C such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = x$.*

Lemma 2.5. [11] *Let C be a bounded, closed and convex subset of complete $CAT(0)$ spaces X. Let $f : C^k \longrightarrow C$ be a Prešic nonexpansive mapping and k be a positive integer. Then f has a fixed point, that is, there exists $x^* \in X$ such that $f(x^*, x^*, ..., x^*) = x^*$*

3. Main Results

Let C be a closed convex subset of complete CAT(0) spaces. Let $f : C^k \longrightarrow C$ be a Prešic nonexpansive mapping and k be a positive integer.

Initial with $x_1 \in C$, we introduce the following iterative algorithms.

(i) The Mann iterative algorithm [17] in complete CAT(0) is defined by

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n f(x_n, x_n, ..., x_n);\ n \geq 1 \tag{3.1}$$

where $\{\alpha_n\}$ be a sequence in [0,1]

(ii) The Ishikawa iterative algorithm [12] is defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n \oplus \alpha_n f(y_n, y_n, ..., y_n) \\ y_n &= (1 - \beta_n)X_n \oplus \beta_n f(x_n, x_n, ..., x_n) \end{aligned} \tag{3.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence in [0,1]

In this paper, we construct S-type to approximate fixed point for Prešić nonexpansive mapping in Hadamard spaces as follows:

$$\begin{aligned} y_n &= (1 - \alpha_n)x_n \oplus \alpha_n f(x_n, x_n, ..., x_n) \\ x_{n+1} &= (1 - \beta_n)f(x_n, x_n, ..., x_n) \oplus \beta_n f(y_n, y_n, ..., y_n) \end{aligned} \tag{3.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1].

Lemma 3.1. *Let C be a closed convex subset of complete CAT(0) spaces. Let $f : C^k \longrightarrow C$ be a Prešic nonexpansive mapping and k be a positive integer. Then sequence $\{x_n\}$ in 3.3, $\lim_{n\longrightarrow\infty} d(x_n, m)$ exists for all $m \in F(f)$*

Proof. Define $T : C \longrightarrow C$ by

$$T(z) = f(z, z, ..., z), z \in C$$

□

Hence by 3.2 T is nonexpansive with $F(T) = f(f)$. It follows from algorithm 3.3 then we have

$$\begin{aligned} y_n &= (1 - \alpha_n)x_n \oplus \alpha_n T x_n \\ x_{n+1} &= (1 - \beta_n)T x_n \oplus \beta_n T y_n \end{aligned} \tag{3.4}$$

Proof. We will prove $\lim_{n\longrightarrow\infty} d(x_n, m)$ exists for all $m \in F(f)$ By using Lemma 3.1 in the algorithm 3.4 we get

$$\begin{aligned}
d(y_n,m) &= d((1-\alpha_n)x_n \oplus \alpha_n Tx_n, m)\\
&\leq (1-\alpha_n)d(x_n,m)+\alpha_n d(Tx_n,m)\\
&\leq (1-\alpha_n)d(x_n,m)+\alpha_n d(x_n,m)\\
&= ((1-\alpha_n)+\alpha_n)d(x_n,m)\\
&= d(x_n,m)\\
d(x_{n+1},m) &= d((1-\beta_n)T(x_n)\oplus \beta_n T(y_n),m)\\
&\leq (1-\beta_n)d(Tx_n,m)+\beta_n d(Ty_n,m)\\
&\leq (1-\beta_n)d(x_n,m)+\beta_n d(y_n,m)\\
&\leq (1-\beta_n)d(x_n,m)+\beta_n d(x_n,m)\\
&= ((1-\beta_n)+\beta_n)d(x_n,m)\\
&= d(x_n,m)
\end{aligned}$$

So, we get $d(x_{n+1},m) \leq d(x_n,m)$ then the sequence $\{x_n\}$ bounded and convergent. Therefore, $\lim_{n\longrightarrow\infty} d(x_n,m)$ exists for all $m \in F(f)$ □

Lemma 3.2. *Let C be a closed convex subset of complete CAT(0) spaces. Let $f : C^k \longrightarrow C$ be a Prešic nonexpansive mapping and k be a positive integer. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy either of the following conditions:*

- *(i)* $\sum_{n=1}^{\infty} \alpha_1(1-\alpha_n) = \infty$ *and* $\limsup_{n\longrightarrow\infty} \beta_n < 1$
- *(ii)* $\sum_{n=1}^{\infty} \beta_1(1-\alpha_n) = \infty$ *and* $\liminf_{n\longrightarrow\infty} \alpha_n > 0$

If $F(f) \neq \emptyset$ then for the sequece $\{x_n\}$ in (3.4),
we have that $\liminf_{n\longrightarrow\infty} d(x_n, f(x_n,...,x_n)) = 0$

Proof. Assume that the condition (i) holds. Since $\limsup_{n\longrightarrow\infty} \beta_n < 1$, there exists $b \in (0,1)$ such that $\beta_n < b$ for all $n \geq 1$ Define $T : C \longrightarrow C$ similar with the proof of Lemma 3.1
for all $m \in F(f) = F(T)$ by using Lemma 2.2 with iteartive algorithm (3.4) we get

$$\begin{aligned}
d(x_{n+1},m)^2 &= d((1-\beta_n)Tx_n \oplus \beta_n Ty_n, m)^2\\
&\leq (1-\beta_n)d(Tx_n,m)^2+\beta_n d(Ty_n,m)^2-\beta_n(1-\beta_n)d(Tx_n,Ty_n)^2\\
&\leq (1-\beta_n)d(x_n,m)^2+\beta_n d(y_n,m)^2-\beta_n(1-\beta_n)d(Tx_n,Ty_n)^2\\
&\leq (1-\beta_n)d(x_n,m)^2+\beta_n d(x_n,m)^2-\beta_n(1-\beta_n)d(Tx_n,Ty_n)^2\\
&\leq d(x_n,m)^2-\beta_n d(x_n,m)^2+\beta_n d(x_n,m)^2-\beta_n(1-\beta_n)d(Tx_n,Ty_n)^2\\
&\leq d(x_n,m)^2-\beta_n(1-\beta_n)d(Tx_n,Ty_n)^2\\
&\leq d(x_n,m)^2-\beta_n(1-\beta_n)d(x_n,y_n)^2\\
&\leq d(x_n,m)^2-\beta_n(1-\beta_n)d(x_n,(1-\alpha)x_n\oplus\alpha_n Tx_n)^2\\
&\leq d(x_n,m)^2-\beta_n(1-\beta_n)[(1-\alpha_n)d(x_n.x_n)^2+\alpha_n d(x_n,Tx_n)^2\\
&\quad +\alpha_n(1-\alpha_n)d(x_n,Tx_n)^2]\\
&\leq d(x_n,m)^2-\beta_n(1-\beta_n)[\alpha_n d(x_n,Tx_n)^2-\alpha_n d(x_n,Tx_n)^2\\
&\quad +\alpha_n^2 d(x_n,Tx_n)^2]\\
&\leq d(x_n,m)^2-\beta_n(1-\beta_n)[\alpha_n^2 d(x_n,Tx_n)^2]
\end{aligned}$$

That is,

$$\beta_n(1-\beta_n)\alpha_n^2 d(x_n,Tx_n)^2 \leq d(x_n,m)^2 - d(x_{n+1},m)^2$$

and, so $\sum_{n=1}^{\infty} \beta_n(1-\beta_n)\alpha_n^2 d(x_n,Tx_n)^2 < \infty$

By the condition $\sum_{n=1}^{\infty} \beta_n(1-\beta_n) = \infty$, we get that $\liminf_{n\longrightarrow\infty} d(x_n,Tx_n) = 0$ that is,

$$\liminf_{n\longrightarrow\infty} d(x_n, f(x_n,...,x_n)) = 0$$

□

Theorem 3.3. *Let C be a closed convex subset of complete CAT(0) spaces. Let $f : C^k \longrightarrow C$ be a Prešic nonexpansive mapping and k be a positive integer. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy either of the following conditions:*

- *(i)* $\sum_{n=1}^{\infty} \alpha_1(1-\alpha_n) = \infty$ *and* $\limsup_{n\longrightarrow\infty} \beta_n < 1$
- *(ii)* $\sum_{n=1}^{\infty} \beta_1(1-\alpha_n) = \infty$ *and* $\liminf_{n\longrightarrow\infty} \alpha_n > 0$

Then for sequence $\{x_n\}$ *in* (3.2)*, there is a subsequence* $\{z_n\}$ *of* $\{x_n\}$ Δ*-convergess to a fixed point of* f

Proof. In Lemma 3.2 we have proved that $\liminf_{n\longrightarrow\infty} d(x_n, f(x_n, x_n, ..., x_n)) = 0$ for the sequence $\{x_n\}$ in (3.4). Hence there is a subsequence $\{z_n\}$ of $\{x_n\}$ sunch that $\lim_{n\longrightarrow\infty} d(z_n, f(z_n, z_n, ..., z_n)) = 0$. It follows from Lemma 3.1 that $\{z_n\}$ is bonded in C
By using Lemma 2.3 the sequence $\{z_n\}$ has a unique asymptotic center, that is, $A_C(\{z_n\}) = \{z\}$. Let $\{u_n\}$ be any subsequence of $\{z_n\}$ such that $A_C(\{u_n\}) = \{u\}$
By using Lemma 3.2, we have

$$\lim_{n\longrightarrow\infty} d(z_n, f(u_n, u_n, ..., u_n)) = 0$$

We next prove that $u \in F(f)$. Define a sequence of $\{z_k\}$ in C given by $z_k = f^k(u, u, ...u) = T^k u$ and we observe that

$$d(z_k, u_n) \leq d(T^k u, T^k u_n) + \sum_{j=1}^{k} d(T^j u_n, T^{j-1} u_n) \leq d(u, u_n) + kd(Tu_n, u_n)$$

Therefore, we have

$$r(z_k, \{u_n\}) = \limsup_{n\longrightarrow\infty} d(z_k, u_n) \leq \limsup_{n\longrightarrow\infty} d(u, u_n) = r0(u, \{u_n\})$$

This implies that $|r(z_k, \{u_n\}) - r(u, \{u_n\})| \longrightarrow 0$ as $k \longrightarrow \infty$. It follows from 2.4 that $\lim_{k\longrightarrow\infty} T^k u = u$. Since C is closed, so $\lim_{k\longrightarrow\infty} T^k u = u \in C$ and $\lim_{k\longrightarrow\infty} T^{k+1} u = Tu$. That is, $Tu = u$. Therefore $u \in F(f)$ If $z \neq u$ then by the uniqueness of asymptotic centres and the fact that $\lim_{n\longrightarrow\infty} d(x_n, m)$ exists for all $m \in F(f)$, we have

$$\begin{aligned}
\limsup_{n\longrightarrow\infty} d(u_n, u) &< \limsup_{n\longrightarrow\infty} d(u_n, z) \\
&\leq \limsup_{n\longrightarrow\infty} d(z_n, z) \\
&< \limsup_{n\longrightarrow\infty} (z_n, u) \\
&= \limsup_{n\longrightarrow\infty} (u_n, u)
\end{aligned}$$

This is a contradiction and hence $z = u$, which implies that, the swquence $\{z_n\}$ Δ-converges to $z \in F(f)$. □

4. Conclusion

In this work, we introduce S-type iterative algorithm and prove Δ-convergence theorem of the proposed algorithm for Prešić type nonexpansive of mapping in complete CAT(0) spaces under some mild conditions.

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