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STRONG CONVERGENCE THEOREMS FOR SYSTEM OF ITERATIVE METHODS OF STRONGLY NONLINEAR NONCONVEX VARIATIONAL INEQUALITIES

KIATTISAK RATTANASEEHA[1] AND ISSARA INCHAN[*,2]

[1]Faculty of Science and Technology, Loei Rajabhat University, Loei, THAILAND
[2]Faculty of Science and Technology, Uttaradit Rajabhat University, Uttaradit, THAILAND

ABSTRACT. Drawing upon the concept of prox-regularity, we introduce and validate the convergence of a modified algorithm designed for systems of strongly nonlinear, nonconvex variational inequality problems. Through this approach, we derive known results within this domain as particular cases, further contributing to the advancement of this field.

KEYWORDS: Lipschitz continuous; strongly monotone mapping; Nonconvex; Uniformly prox regular; strongly nonlinear nonconvex variational inequalities.
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1. Introduction

The theory of variational inequalities, first introduced by Stampacchia [2], provides a simple and unified framework for studying a wide range of problems in finance, economics, transportation, network and structural analysis, elasticity, and optimization. Recent years have seen a surge in research papers focusing on both the theory and applications of this field, as evidenced by works such as [4, 7, 8] and their respective references.

Traditionally, the investigation of variational inequalities has been centered around convex sets, owing to the reliance on properties of the projection operator inherent in convex sets. However, recent advancements have led to the generalization of convex sets in various ways. Notably, uniformly prox-regular sets emerge as a direct consequence of this generalization, encompassing nonconvex sets and including convex sets as a special case.

In the early 2000s, researchers such as Bounkhel [3], Noor [10], Moudafi [9], and Pang et al. [15] began exploring variational inequality problems over these

* *Corresponding author.*
Email address : kiattisakrat@live.com, peissara@uru.ac.th.
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nonconvex sets. They proposed and analyzed projection-type iterative algorithms utilizing prox-regular and auxiliary principle techniques.

Building on this foundation, Noor [12] introduced and studied new classes of variational and Wiener-Hiof equations in 2009, establishing equivalence between general nonconvex variational inequalities and fixed point problems, as well as the Wiener-Hopf equation, through projection techniques. Noor also presented new projection iterative methods for solving nonconvex variational inequalities and proved the convergence of these methods under suitable conditions.

In the same year, Moudafi [9] introduced the convergence of two-step projection methods for systems of nonconvex variational inequality problems, assuming a mapping T is γ-strongly monotone and L-Lipschitz continuous.

More recently, in 2013, Al-Shemas [1] introduced strongly nonlinear general nonconvex variational inequalities and proved the convergence of the predictor-corrector method, requiring only pseudomonotonicity, a weaker condition than monotonicity.

Motivated by the works of Moudafi [9] and Al-Shemas [1], we introduce and study the convergence of a modified algorithm for systems of strongly nonlinear nonconvex variational inequality problems, assuming two mappings satisfy strong monotonicity and Lipschitz continuity. This work extends and improves upon some known results in the field.

2. Preliminaries

Let C be a closed subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ respectively. Let us recall the following well-known definitions and some auxiliary results of nonlinear convex analysis and nonsmooth analysis.

Definition 2.1. Let $u \in H$ be a point not lying in C. A point $v \in C$ is called a closest point or a projection of u onto C if $d_C(u) = \|u - v\|$ when d_C is a usual distance. The set of all such closest points is denoted by $P_C(u)$; that is,

$$P_C(u) = \{v \in C : d_C(u) = \|u - v\|\}. \tag{2.1}$$

Definition 2.2. Let C be a subset of H. The proximal normal cone to C at x is given by

$$N_C^P(x) = \{z \in H : \exists \rho > 0; x \in P_C(x + \rho z)\}. \tag{2.2}$$

The following characterization of $N_C^P(x)$ can be found in [5].

Lemma 2.3. *Let C be a closed subset of a Hilbert space H. Then*

$$z \in N_C^P(x) \text{ if and only if } \exists \sigma > 0, \langle z, y - x\rangle \leq \sigma \|y - x\|^2, \quad \forall y \in C. \tag{2.3}$$

Clark et al. [6] and Poliquin et al. [14] have introduced and atudied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class or uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems, and differential inclusions.

Definition 2.4. For a given $r \in (0, +\infty]$, a subset C of H is said to be uniformly r-prox-regular with respect to r if, for all $\overline{x} \in C$ and for all $0 \neq z \in N_C^P(x)$, one has

$$\langle \frac{z}{\|z\|}, x - \overline{x} \rangle \leq \frac{1}{2r}\|x - \overline{x}\|^2, \quad \forall x \in C. \tag{2.4}$$

It is well known that a closed subset of a Hilbert space is convex if and only if it is proximally smooth of radius $r > 0$. Thus, in Definition 2.4, in the case of $r = \infty$, the uniform r-prox-regularity C is equivalent to convexity of C. Then, it is clear that the class of uniformly prox-regular sets is sufficiently large to include the class

p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets, and many other nonconvex sets; see [6, 14].

In this work let C be a closed subset of a real Hilbert space H with is uniformly r-prox-regular(nonconvex), set $C_r := \{x \in H : d(x, C) < r\}$. For given nonlinear mappings $T_1, T_2 : C_r \to H$, we consider the problem of finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T_1 y^* + x^* - y^*, x - x^* \rangle + \lambda \|x - x^*\|^2 \geq \langle Ay^*, x - x^* \rangle, \forall x \in C_r, \rho > 0 \\ &\langle \eta T_2 x^* + y^* - x^*, y - y^* \rangle + \lambda \|y - y^*\|^2 \geq \langle Ax^*, y - y^* \rangle, \forall y \in C_r, \eta > 0, \end{aligned} \tag{2.5}$$

which is called the *system of strongly nonlinear nonconvex variational inequalities*(SSNNVI).

If $A(x^*) \equiv 0,\ A(y^*) \equiv 0$ and $T_1 = T_2 = T$ then the problem (2.5) is equivalent to finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T y^* + x^* - y^*, x - x^* \rangle + \lambda \|x - x^*\|^2 \geq 0, \forall x \in C_r, \rho > 0 \\ &\langle \eta T x^* + y^* - x^*, y - y^* \rangle + \lambda \|y - y^*\|^2 \geq 0, \forall y \in C_r, \eta > 0, \end{aligned} \tag{2.6}$$

which is called the *system of nonconvex variational inequalities*(SNVI). We known that the inequalities (2.6) is equivalent as follows:

$$\begin{aligned} &y^* - x^* - \rho T y^* \in N^P_{C_r} x^*, \\ &x^* - y^* - \eta T x^* \in N^P_{C_r} y^*. \end{aligned} \tag{2.7}$$

Which is introduce by Moudafi [9].

If $A(x^*) \equiv 0, A(y^*) \equiv 0,\ T_1 = T_2 = T$ and $\lambda = 0$, then the problem (2.5) is equivalent to finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T y^* + x^* - y^*, x - x^* \rangle \geq 0, \forall x \in C_r, \rho > 0 \\ &\langle \eta T x^* + y^* - x^*, y - y^* \rangle \geq 0, \forall y \in C_r, \eta > 0. \end{aligned} \tag{2.8}$$

Which is called *system of variational inequalities*(SVI), introduced by Verma [16].

If $T_1 = T_2 = T,\ x^* = y^*$ and $\rho = \eta = 1$, then the problem (2.5) is equivalent to finding $x^* \in C_r$ such that

$$\langle T x^*, x - x^* \rangle + \lambda \|x - x^*\|^2 \geq \langle Ax^*, x - x^* \rangle, \forall x \in C_r \tag{2.9}$$

which is known as the strongly nonlinear nonconvex variational inequality and studied by Noor [17].

In inequalities (2.9) if we let $A(x^*) \equiv 0$, we have to finding $x^* \in C_r$ such that

$$\langle T x^*, x - x^* \rangle + \lambda \|x - x^*\|^2 \geq 0, \forall x \in C_r \tag{2.10}$$

which is called the *nonconvex variational inequalities*(NVI), introduced and studied by Bounkhel et. al.[3] and Noor [10, 11].

It is worth mentioning that if $C_r = C$ is convex set, then problem (2.10) is equivalent to finding $x^* \in C$ such that

$$\langle T x^*, x - x^* \rangle \geq 0, \forall x \in C, \tag{2.11}$$

which is known as *variational inequalities*, introduced and studied by Stamphacia [2].

Now, if C_r is a nonconvex(uniform r-prox regular) set, then problem (2.5) is equivalent to finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &0 \in \rho T_1 y^* + x^* - y^* - Ay^* + N^P_{C_r} x^*, \\ &0 \in \eta T_2 x^* + y^* - x^* - Ax^* + N^P_{C_r} y^*, \end{aligned} \tag{2.12}$$

which is $N^P_{C_r}u$ denote the normal cone of C_r at u. The problem (2.12) is called the *the system of nonconvex variational inclusion problem associated with nonconvex variational inequalities.*

We now recall the well-known lemmas of the uniform prox-regular sets.

Lemma 2.5. *Let C be a nonempty closed subset of H, $r \in (0, +\infty]$ and set $C_r ;= \{x \in H : d(x, C) < r\}$. If C is uniform r-uniformly prox-regular, then the following hold:*

(1) for all $x \in C_r, P_C(x) \neq \emptyset$,

(2) for all $s \in (0, r), P_C$ is Lipschitz continuous with constant $t_s = \frac{r}{r-s}$ on C_s,

(3) the proximal normal cone is closed as a set-valued mapping.

Let H be a real Hilbert space. A mapping $T : H \to H$ is called $\gamma -$ *strongly monotone* if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \gamma\|x - y\|^2, \tag{2.13}$$

for all $x, y \in H$. A mapping T is called $\mu-$*Lipschitz* if there exists a constant $\mu > 0$ such that

$$\|Tx - Ty\| \leq \mu\|x - y\|, \tag{2.14}$$

for all $x, y \in H$.

Lemma 2.6. *In a real Hilbert space H, there holds the inequality*

(i) $\|x+y\|^2 \leq \|x\|^2+2\langle y, x+y\rangle \quad x, y \in H$ *and* $\|x-y\|^2 = \|x\|^2-2\langle x, y\rangle+\|y\|^2$,

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \forall t \in [0, 1]$.

3. Main Results

In this section we first establish the equivalent between the system of nonconvex variational inequalities (2.6) with the projection technique.

Lemma 3.1. *For given $x^*, y^* \in C_r$ are solution of system of strongly nonlinear general nonconvex variational inequalities (2.5), if and only if*

$$\begin{aligned} x^* &= P_{C_r}[y^* - \rho T_1 y^* + Ay^*], \\ y^* &= P_{C_r}[x^* - \eta T_2 x^* + Ax^*], \end{aligned} \tag{3.1}$$

where $P_{C_r} = (I + N^P_{C_r})^{-1}$ is the projection of H onto the uniformly prox-regular set C_r.

Proof. Let $x^*, y^* \in C_r$ be a solution of (2.5), for a constant $\rho > 0$, we have

$$\langle \rho T_1 y^* + x^* - y^*, x - x^*\rangle + \lambda\|x - x^*\|^2 \geq \langle Ay^*, x - x^*\rangle$$

if and only if

$$\langle Ay^* - \rho T_1 y^* - x^* + y^*, x - x^*\rangle \leq \lambda\|x - x^*\|^2.$$

Then,

$$Ay^* - \rho T_1 y^* - x^* + y^* \in N^P_{C_r}x^*$$

it implies that

$$\begin{aligned} 0 &\in \rho T_1 y^* + x^* - y^* - Ay^* + N^P_{C_r}x^* = (I + N^P_{C_r})x^* - (y^* - \rho T_1 y^* + Ay^*) \\ &\Leftrightarrow (I + N^P_{C_r})x^* = (y^* - \rho T_1 y^* + Ay^*) \\ &\Leftrightarrow x^* = P_{C_r}[y^* - \rho T_1 y^* + Ay^*] \end{aligned}$$

where we have used the well-known fact that $P_{C_r} = (I + N^P_{C_r})^{-1}$.

Similarly, we obtain

$$y^* = P_{C_r}[x^* - \eta T_2 x^* + Ax^*].$$

This prove our assertions. ☐

Algorithm 3.2. *For arbitrarily chosen initial points $x_0 \in C_r$, the sequence $\{x_n\}$ and $\{y_n\}$ in the following way:*

$$\begin{aligned} y_n &= P_{C_r}[x_n - \eta T_2 x_n + A x_n], \eta > 0 \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n P_{C_r}[y_n - \rho T_1 y_n + A y_n], \rho > 0, \end{aligned} \tag{3.2}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$.

Now, we suggest and analyze the following explicit projection method (3.2) for solving the system of nonconvex variational inequalities (2.6).

Theorem 3.3. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T_1, T_2, A : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ_1-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous and γ_2-strongly monotone mapping and A is a β-Lipschitz continuous. If there exists constant $\rho, \eta > 0$ such that*

$$\begin{aligned} |\rho - \frac{\gamma_1}{\mu_1^2}| &< \frac{\sqrt{\gamma_1^2 t_s^2 - t_s\mu_1^2(t_s + t_s\beta - 1)}}{t_s\mu_1^2}, \ t_s\sqrt{t_s^2 - 1} < \frac{\gamma_1}{\mu_1} \\ |\eta - \frac{\gamma_2}{\mu_2^2}| &< \frac{\sqrt{\gamma_2^2 t_s^2 - t_s\mu_2^2(t_s + t_s\beta - 1)}}{t_s\mu_2^2}, \ t_s\sqrt{t_s^2 - 1} < \frac{\gamma_2}{\mu_2}, \end{aligned} \tag{3.3}$$

where $t_s = \frac{r}{r-s}$ for some $s \in (0, r)$. If the sequence of positive real number $\alpha_n \in [0,1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = 0$, then the sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 3.2 converge to a solution of the system of nonconvex variational inequalities (2.6).

Proof. Let $x^*, y^* \in C_r$ be a solution of (2.6) and from Lemma 3.1, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n P_C[y_n - \rho T_1 y_n + A y_n] - x^*\| \\ &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(P_C[y_n - \rho T_1 y_n + A y_n] \\ &\qquad - P_C[y^* - \rho T_1 y^* + A y^*])\| \\ &\le (1-\alpha_n)\|x_n - x^*\| \\ &\quad + \alpha_n t_s\|(y_n - \rho T_1 y_n + A y_n) - (y^* - \rho T_1 y^* + A y^*)\| \\ &= (1-\alpha_n)\|x_n - x^*\| \\ &\quad + \alpha_n t_s\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*) + (A y_n - A y^*)\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n t_s[\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\| \\ &\quad + \|A y_n - A y^*\|] \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n t_s[\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\| \\ &\quad + \beta\|y_n - y^*\|]. \end{aligned} \tag{3.4}$$

Since T_1 are both μ_1-Lipschitz continuous and γ_1-strongly monotone mapping and from Lemma 2.6, we consider

$$\begin{aligned} \|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\|^2 &= \|y_n - y^*\|^2 - 2\rho\langle y_n - y^*, T_1 y_n - T_1 y^*\rangle \\ &\quad + \rho^2\|T_1 y_n - T_1 y^*\|^2 \\ &\le \|y_n - y^*\|^2 - 2\rho\gamma_1\|y_n - y^*\|^2 + \rho^2\mu_1^2\|y_n - y^*\|^2 \\ &= (1 - 2\rho\gamma_1 + \rho^2\mu_1^2)\|y_n - y^*\|^2. \end{aligned}$$

It follows that

$$\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\| \le \sqrt{1 - 2\rho\gamma_1 + \rho^2\mu_1^2}\|y_n - y^*\|. \tag{3.5}$$

Replace (3.5) into (3.4), we have

$$\|x_{n+1} - x^*\| \le (1-\alpha_n)\|x_n - x^*\| + \alpha_n t_s(\beta + \sqrt{1 - 2\rho\gamma_1 + \rho^2\mu_1^2})\|y_n - y^*\|. \quad (3.6)$$

On the other hand, we can compute that

$$\begin{aligned} \|y_n - y^*\| &= \|P_C[x_n - \eta T_2 x_n + Ax_n] - y^*\| \\ &= \|P_C[x_n - \eta T_2 x_n + Ax_n] - P_C[x^* - \eta T_2 x^* + Ax^*]\| \\ &\le t_s\|(x_n - \eta T_2 x_n + Ax_n) - (x^* - \eta T_2 x^* + Ax^*)\| \\ &\le t_s[\|(x_n - x^*) - \eta(T_2 x_n - T_2 x^*)\| + \|Ax_n - Ax^*\|] \\ &\le t_s[\|(x_n - x^*) - \eta(T_2 x_n - T_2 x^*)\| + \beta\|x_n - x^*\|]. \end{aligned} \quad (3.7)$$

Similarly, from T_2 are both μ_2-Lipschitz continuous and γ_2-strongly monotone mapping, we have

$$\begin{aligned} \|(x_n - x^*) - \eta(T_2 x_n - T_2 x^*)\|^2 &= \|x_n - x^*\|^2 - 2\eta\langle x_n - x^*, T_2 x_n - T_2 x^*\rangle \\ &\quad + \eta^2\|T_2 x_n - T_2 x^*\|^2 \\ &\le \|x_n - x^*\|^2 - 2\eta\gamma_2\|x_n - x^*\|^2 + \eta^2\mu_2^2\|x_n - x^*\|^2 \\ &= (1 - 2\eta\gamma_2 + \eta^2\mu_2^2)\|x_n - x^*\|^2. \end{aligned}$$

It follows that

$$\|(x_n - x^*) - \eta(T_2 x_n - T_2 x^*)\| \le \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_2^2}\|x_n - x^*\|. \quad (3.8)$$

Replace (3.8) into (3.7), we have

$$\|y_n - y^*\| \le t_s(\beta + \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_2^2})\|x_n - x^*\|. \quad (3.9)$$

Moreover, from (3.6) and (3.9) we put $\theta_1 = t_s(\beta + \sqrt{1 - 2\rho\gamma_1 + \rho^2\mu_1^2}), \theta_2 = t_s(\beta + \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_2^2})$, it follows that

$$\begin{aligned} \|x_{n+1} - x^*\| &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\theta_2\|x_n - x^*\| \\ &= (1 - (1-\theta_1\theta_2)\alpha_n)\|x_n - x^*\| \\ &\le \prod_{i=0}^{n}(1 - (1-\theta_1\theta_2)\alpha_i)\|x_0 - x^*\|. \end{aligned} \quad (3.10)$$

Since $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ and (3.14), we obtain

$$\lim_{n\to\infty}\prod_{i=0}^{n}(1 - (1-\theta_1\theta_2)\alpha_i) = 0. \quad (3.11)$$

It follows from (3.11) and (3.10), we have

$$\lim_{n\to\infty}\|x_n - x^*\| = 0. \quad (3.12)$$

From (3.9) and (3.12), we have

$$\lim_{n\to\infty}\|y_n - y^*\| = 0. \quad (3.13)$$

Which is $x^*, y^* \in C_r$ satisfying the system of nonconvex variational inequalities (2.6). This completes the proof. □

Corollary 3.4. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T : C \to H$ be such that T are both μ-Lipschitz continuous and γ-strongly monotone mapping. If there exists constant $\rho, \eta > 0$ such that*

$$\frac{\gamma}{t_s^2\mu^2} - \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2-1)}}{t_s^2\mu^2} < \rho, \eta < \frac{\gamma}{t_s^2\mu^2} + \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2-1)}}{t_s^2\mu^2}, \ t_s\sqrt{t_s^2-1} < \frac{\gamma}{\mu}, \quad (3.14)$$

where $t_s = \frac{r}{r-s}$ *for some* $s \in (0, r)$*. If the sequence of positive real number* $\alpha_n \in [0, 1]$ *with* $\Sigma_{n=0}^{\infty} \alpha_n = 0$*, then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by for arbitrarily chosen initial points* $x_0, y_0 \in C_r$

$$\begin{aligned} y_n &= P_C[x_n - \eta T x_n], \eta > 0 \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n P_C[y_n - \rho T y_n], \rho > 0, \end{aligned} \tag{3.15}$$

converge to a solution of the system of nonconvex variational inequalities (2.8).

Proof. From Theorem 3.3, if $T_1 = T_2 = T$ we have a result. □

Corollary 3.5. *Let* C *be a uniformly* r*-prox-regular closed subset of a Hilbert space* H*, and let* $T : C \to H$ *be such that* T *are both* μ*-Lipschitz continuous and* γ*-strongly monotone mapping. If there exists constant* $\rho, \eta > 0$ *such that*

$$\frac{\gamma}{t_s^2\mu^2} - \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2 - 1)}}{t_s^2\mu^2} < \rho, \eta < \frac{\gamma}{t_s^2\mu^2} + \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2 - 1)}}{t_s^2\mu^2}, \; t_s\sqrt{t_s^2 - 1} < \frac{\gamma}{\mu}, \tag{3.16}$$

where $t_s = \frac{r}{r-s}$ *for some* $s \in (0, r)$*. Then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by for arbitrarily chosen initial points* $x_0, y_0 \in C_r$

$$\begin{aligned} y_n &= P_C[x_n - \eta T x_n], \eta > 0 \\ x_{n+1} &= P_C[y_n - \rho T y_n], \rho > 0, \end{aligned} \tag{3.17}$$

converge to a solution of the system of nonconvex variational inequalities (2.8).

Proof. From Theorem 3.3, if $T_1 = T_2 = T$ and $\alpha_n = 1$ for any $n \geq 1$, we have a result. □

4. Applications

In this section, we can applied Theorem 3.3 to the system of general nonconvex variational inequalities, for given nonlinear mappings $T, g : C_r \to H$, we consider the problem of finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T_1 g(y^*) + g(x^*) - g(y^*), x - g(x^*)\rangle \geq 0, \forall x \in C_r, \rho > 0 \\ &\langle \eta T_2 g(x^*) + g(y^*) - g(x^*), x - g(y^*)\rangle \geq 0, \forall x \in C_r, \eta > 0, \end{aligned} \tag{4.1}$$

which is called the *system of general nonconvex variational inequalities.*

If $T_1 = T_2 = T$, then the problem (4.1) is equivalent to finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T g(y^*) + g(x^*) - g(y^*), x - g(x^*)\rangle \geq 0, \forall x \in C_r, \rho > 0 \\ &\langle \eta T g(x^*) + g(y^*) - g(x^*), x - g(y^*)\rangle \geq 0, \forall x \in C_r, \eta > 0. \end{aligned} \tag{4.2}$$

Now, similarly of the proof of Lemma 3.1, we have the result.

Lemma 4.1. *For given* $x^*, y^* \in C_r$ *is a solution of system of nonconvex variational inequalities (4.1), if and only if*

$$\begin{aligned} g(x^*) &= P_C[g(y^*) - \rho T_1 g(y^*)], \\ g(y^*) &= P_C[g(x^*) - \eta T_2 g(x^*)], \end{aligned} \tag{4.3}$$

where P_C *is the projection of* H *onto the uniformly prox-regular set* C_r*.*

Theorem 4.2. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T_1, T_2 : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ_1-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous and γ_2-strongly monotone mapping. If there exists constant $\rho, \eta > 0$ such that*

$$|\rho - \frac{\gamma_1}{t_s^2\mu_1^2}| < \frac{\sqrt{\gamma_1^2 - t_s^2\mu_1^2(t_s^2-1)}}{t_s^2\mu_1^2},\ t_s\sqrt{t_s^2-1} < \frac{\gamma_1}{\mu_1}$$
$$|\eta - \frac{\gamma_2}{t_s^2\mu_2^2}| < \frac{\sqrt{\gamma_2^2 - t_s^2\mu_2^2(t_s^2-1)}}{t_s^2\mu_2^2},\ t_s\sqrt{t_s^2-1} < \frac{\gamma_2}{\mu_2}, \tag{4.4}$$

where $t_s = \frac{r}{r-s}$ for some $s \in (0, r)$. If the sequence of positive real number $\alpha_n \in [0, 1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = 0$, then the sequence $\{x_n\}$ and $\{y_n\}$ is generated by for $x_0, y_0 \in C_r$,

$$\begin{aligned} g(y_n) &= P_C[g(x_n) - \eta T_2 g(x_n)], \eta > 0 \\ g(x_{n+1}) &= (1-\alpha_n)g(x_n) + \alpha_n P_C[g(y_n) - \rho T_1 g(y_n)], \rho > 0, \end{aligned} \tag{4.5}$$

strongly converge to a solution of the system of nonconvex variational inequalities (4.1).

Proof. Similar the proof in Theorem 3.3, let $x^*, y^* \in C_r$ be a solution of (4.1) and from Lemma 4.1, we can compute that

$$\|g(x_{n+1}) - g(x^*)\| \le \prod_{i=0}^{n}(1-(1-\theta_1\theta_2)\alpha_i)\|g(x_0) - g(x^*)\|. \tag{4.6}$$

where $\theta_1 = t_s\sqrt{1-2\rho\gamma_1+\rho^2\mu_1^2}, \theta_2 = t_s\sqrt{1-2\eta\gamma_2+\eta^2\mu_2^2}$. From $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ and conditions (4.4), we obtain

$$\lim_{n\to\infty}\prod_{i=0}^{n}(1-(1-\theta_1\theta_2)\alpha_i) = 0. \tag{4.7}$$

It follows from (4.6) and (4.7), we have

$$\lim_{n\to\infty}\|g(x_n) - g(x^*)\| = 0. \tag{4.8}$$

And we can compute that

$$\|g(y_n) - g(y^*)\| \le \theta_2\|g(x_n) - g(x^*)\|, \tag{4.9}$$

it follows that

$$\lim_{n\to\infty}\|g(y_n) - g(y^*)\| = 0. \tag{4.10}$$

From g is injective mapping, we have $\lim_{n\to\infty}\|x_n - x^*\| = 0$ and $\lim_{n\to\infty}\|y_n - y^*\| = 0$ satisfying the system of general nonconvex variational inequalities (4.1). This complete the proof. □

Corollary 4.3. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T : C \to H$ be such that T are both μ-Lipschitz continuous and γ-strongly monotone mapping. If there exists constant $\rho, \eta > 0$ such that*

$$\frac{\gamma}{t_s^2\mu^2} - \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2-1)}}{t_s^2\mu^2} < \rho, \eta < \frac{\gamma}{t_s^2\mu^2} + \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2-1)}}{t_s^2\mu^2},\ t_s\sqrt{t_s^2-1} < \frac{\gamma}{\mu}, \tag{4.11}$$

where $t_s = \frac{r}{r-s}$ for some $s \in (0, r)$. Then the sequence $\{x_n\}$ and $\{y_n\}$ is generated by for $x_0, y_0 \in C_r$,

$$\begin{aligned} g(y_n) &= P_C[g(x_n) - \eta T g(x_n)], \eta > 0 \\ g(x_{n+1}) &= P_C[g(y_n) - \rho T g(y_n)], \rho > 0, \end{aligned} \tag{4.12}$$

strongly converge to a solution of the system of nonconvex variational inequalities (4.2).

Proof. From Theorem 4.2, if $T_1 = T_2 = T$ and $\alpha_n = 1$ for any $n \geq 0$, we have a result. □

5. Conclusion

In this manuscript, we present a novel algorithm tailored for addressing systems of strongly nonlinear, nonconvex variational inequality problems. Leveraging the concept of prox-regularity, our proposed method exhibits convergence properties, providing a robust solution framework for this challenging class of problems. Through rigorous analysis and validation, we establish the convergence of our algorithm, thus offering a valuable tool for tackling real-world optimization challenges. Furthermore, our findings unveil connections to existing results in the field, underscoring the broader applicability and significance of our approach in advancing the state-of-the-art in variational inequality theory.

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