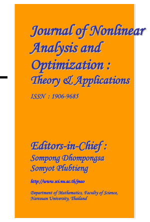
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 5, No. 2, (2014), 91-103
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



KKT OPTIMALITY CONDITIONS FOR INTERVAL VALUED OPTIMIZATION PROBLEMS

DEEPAK SINGH[1,*], BILAL AHMAD DAR[2], ANIL GOYAL[2]

[1] Department of Applied Sciences, NITTTR (under Ministry of HRD, Govt. of India), Bhopal, M.P., India.

[2] Department of Mathematics, Rajiv Gandhi Proudyogiki Vishwavidyalaya (state technological university of M.P.), Bhopal, M.P., India.

ABSTRACT. In the present paper we study the class of convex optimization problems in uncertain environment. The objective and constraint functions are assumed to be interval valued. Solution concepts are proposed under two order relations on the set of all closed intervals. Weakly continuously differentiability is employed in order to derive necessary and sufficient conditions for KKT optimality conditions. These theoretical developments are illustrated through a numerical example.

KEYWORDS : Interval valued functions; weak differentiability; KKT-optimality conditions; Type-I and Type-II solutions.
AMS Subject Classification: 90C29, 90C30.

1. INTRODUCTION

For solutions of the optimization problems the main components are theory and methods of mathematical modelling. In practice, it is usually difficult to determine the real valued coefficients of objective and/or constraint functions involved. There are two deterministic optimization models to deal with uncertain data viz. robust optimization Ben-tal et al. [2], and another is interval valued optimization Ben-Israel and Robers [1]. Many approaches have been developed to deal with these problems. Birge and Louveaux [14], Vajda [24], Stanchu-Minasian [12], Prekopa [4] provide various techniques for solving stochastic optimization problems. On the other hand the collection of papers on fuzzy optimization edited by Slowinski [21] and Delgado et al. [16] gives the main stream to the topic. Lai and Hwang [25, 26] also give useful survey. Inuiguchi and Ramik [17] give a review of fuzzy optimization and a comparison with stochastic optimization in portfolio selection problems. Slowinski and Teghem [22] provide comparison between two types of the

* *Corresponding author.*
Email address : dk.singh1002@gmail.com(Deepak Singh) and sahilbilal99in@gmail.com(Bilal Ahmad Dar).
Article history : Received January 05, 2014, Accepted May 15, 2014.

optimization problems for multiobjective programming problems.

Charnes et al. [3] considered the linear programming problems in which the right-hand sides of linear inequality constraints were taken as closed intervals. In a paper Stancu-Minasian and Tigan [13] obtained solutions for interval valued optimization problems. Steuer [20] proposed three algorithms, called the F-cone algorithm, E-cone algorithm and all emanating algorithm to solve the linear programming problems with interval objective functions. Ishibuchi and Tanaka [7] proposed the ordering relation between two closed intervals by considering the maximization and minimization problems separately. Mraz [6] proved algorithms to compute the exact upper bound and lower bound for linear programming problems with interval coefficients. Chanas and Kuchta [23] presented an approach to unify the solution methods proposed in Ishibuchi and Tanaka [7] and in Rommelfanger and Hanuscheck [8]. Oliveria and Antunes [5] provided an overview of multiobjective linear programming problems with interval coefficients by illustrating many numerical examples. Lai and Huang et al. [25] proposed an interval parameter fuzzy nonlinear optimization model for stream water quality management under uncertainty.

The Karush-Kuhn-Tucker optimality conditions play an important role in the area of optimization theory and have been studied for over a century. For interval valued optimization problems, the KKT optimality conditions are also studied in many recent publications. Recently Wu [10, 11] have studied KKT optimality conditions for interval valued optimization problems. Also Chalco-Cano et al. [27] studied the KKT optimality conditions of interval valued optimization problem via generalised derivative. Moreover Zhang et al. [15] derived the KKT optimality conditions for non-convex programming problems with interval valued objective functions. This paper focuses on nonlinear programming problems in which objective and constraint functions are interval valued. The main motivation for considering interval valued constraints is that the uncertainty that is imposed on objective functions is likely also to be imposed on constraints. The remaining paper is organised as: In section 2, we introduce some preliminaries of interval arithmetic and the concept of weak differentiability for intervals valued functions. Moreover by using the concept of order relations " $\preceq_{LU}$ " and " $\preceq_{UC}$ " the solution concepts for interval valued optimization problems are given. Also the concept of LU-convexity and UC-convexity are provided. In section 3, KKT optimality conditions are derived for optimization problems with interval valued objective and interval valued constraint functions. Also by invoking pseudoconvexity the same is derived. An example is also given in order to illustrate our main result. Finally section 4 is devoted to the conclusion.

2. PRELIMINARIES AND NOTATIONS

The values of objective and constraint functions in our model are closed intervals, we need to compare the closed intervals. Let us denote by I, the class of all closed and bounded intervals in R. Throughout this paper, when A is a closed interval, then it is also bounded. We also adopt the notation $A = [a^L, a^U]$, where a^L and a^U are the lower and upper end points of A, respectively.

Let $A, B \in I$. Then $A + B$ is defined by $A + B = \{a + b : a \in A$ and $b \in B\} = [a^L + b^L, a^U + b^U]$. And $-A$ is defined by $-A = -a : a \in A$. And

$A - B = A + (-B) = [a^L - b^U, a^U - b^L]$.

Further for any real number k,

$$kA = \begin{cases} [ka^L, ka^U] \text{ if } k \geq 0, \\ [ka^U, ka^L] \text{ if } k < 0; \end{cases}$$

and for $h > 0$,

$$\frac{A}{h} = \left[\frac{a^L}{h}, \frac{a^U}{h}\right].$$

The function $f : R^n \longrightarrow I$, is called an interval valued function, i.e., $f(x) = f(x_1, x_2, \cdots, x_n)$ is a closed interval in R for each $x \in R^n$. Clearly $f(x) = [f^L(x), f^U(x)]$, where f^L and f^U are real valued functions defined on R^n and satisfy $f^L(x) \leq f^U(x)$ for every $x \in R^n$.

Definition 2.1. [10] Let X be open in R^n and let $x_0 = (x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}) \in X$ be fixed. An interval valued function $f : R^n \longrightarrow I$ with $f(x) = [f^L(x), f^U(x)]$ is said to be

(i) weakly differentiable at $x_0 \in X$ if the real valued functions f^L and f^U are differentiable at x_0 (in the usual sense i.e., all of the partial derivatives $\left(\frac{\partial f^L}{\partial x_i}\right)$ and $\left(\frac{\partial f^U}{\partial x_i}\right)$ exist at x_0 for $i = 1, 2, \cdots, n$).

(ii) weakly continuously differentiable at x_0 if the real valued functions f^L and f^U are continuously differentiable at x_0 (i.e., all of the partial derivatives of f^L and f^U exist on some neighborhoods of x_0 and are continuous at x_0).

Wu [10] has formulated two solution concepts for interval valued optimization problem. We may follow similar solution concepts as that used in [10]. Consider the following interval valued optimization problem.

$(IVP1)$

$$\text{Min } f(x) = [f^L(x_1, x_2, \cdots, x_n), f^U(x_1, x_2, \cdots, x_n)] = [f^L(x), f^U(x)]$$
$$\text{Subject to } x = (x_1, x_2, \cdots, x_n) \in X \subseteq R^n.$$

Since the objective function $f(x)$ is a closed interval, we need to make clear the meaning of minimization problem $(IVP1)$. Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two closed intervals in R. We write $A \preceq_{LU} B$ if and only if $a^L \leq b^L$ and $a^U \leq b^U$, and $A \prec_{LU} B$ if and only if $A \preceq_{LU} B$ and $A \neq B$. Equivalently, $A \prec_{LU} B$ if and only if

$$\begin{cases} a^L \leq b^L \\ a^U < b^U; \end{cases} or \begin{cases} a^L < b^L \\ a^U < b^U; \end{cases} or \begin{cases} a^L < b^L \\ a^U \leq b^U. \end{cases} \tag{2.1}$$

Definition 2.2. [10] Let x^* be a feasible solution of $(IVP1)$. We say that x^* is type-I solution of $(IVP1)$ if there exists no $\bar{x} \in X$, such that $f(\bar{x}) \prec_{LU} f(x^*)$.

Another solution concept follows from Ishibuchi and Tanaka [7]. Let $A = [a^L, a^U]$ be the closed interval in R. Then we can calculate the centre $a^C = \frac{1}{2}[a^L + a^U]$ and half width $a^W = \frac{1}{2}[a^U - a^L]$ of A. In this case we can use the notation $\langle a^C, a^W \rangle$ for A. i.e., $A = \langle a^C, a^W \rangle$. Ishibuchi and Tanaka [7] have proposed the ordering relation between closed intervals A and B by using minimization and maximization problem separately.

(i) For maximization we write, $A \preceq_{CW} B$ iff $a^C \leq b^C$ and $a^W \geq b^W$. i.e., the interval with higher centre and lower half width (i.e., less uncertainty) is preferred for maximization problem.
(ii) For minimization we write, $A \preceq_{CW} B$ iff $a^C \leq b^C$ and $a^W \leq b^W$. i.e., the interval with lower centre and lower half width (i.e., less uncertainty) is preferred for minimization problem.

Also we write $A \prec_{CW} B$ iff $A \preceq_{CW} B$ and $A \neq B$.

Ishibuchi and Tanaka [7] proved that

(i) $A \preceq_{UC} B$ Iff $A \preceq_{LU} B$ or $A \preceq_{CW} B$.
(ii) $A \prec_{UC} B$ Iff $A \prec_{LU} B$ or $A \prec_{CW} B$.

Definition 2.3. Let x^* be a feasible solution of $(IVP1)$. We say that x^* is type-II solution of $(IVP1)$ if there exists no $\bar{x} \in X$. s.t., $f(\bar{x}) \prec_{UC} f(x^*)$.

Remark 2.4. [10] Let x^* be a feasible solution of $(IVP1)$. If x^* is a type-I solution of $(IVP1)$ then x^* is also a type-II solution of $(IVP1)$.

For our on-going discussion we consider the following definition of convexity for interval valued functions.

Definition 2.5. [10] Let $f(x) = [f^L(x), f^U(x)]$ be an interval valued function defined on convex set $X \subseteq R^n$. We say that F is LU-convex or simply convex at x^* if

$$f(\lambda x^* + (1-\lambda)x) \preceq_{LU} \lambda f(x^*) + (1-\lambda)f(x) \tag{2.2}$$

for each $\lambda \in (0,1)$ and for each $x \in X$. f is said to be LU-convex on X if it is LU-convex on each point of X. Similarly we can define UC-convexity by using relation '$\preceq_{UC}$'.

Proposition 2.6. [10] *Let X be a convex subset of R and f be an interval valued function defined on X. Then we have the following properties.*

(i) *f is LU-convex at x^* iff f^L and f^U are convex at x^*.*
(ii) *f is UC-convex at x^* iff f^U and f^C are convex at x^*.*
(iii) *If f is LU-convex at x^*, then f is UC-convex at x^*.*

3. THE KARUSH-KUHN-TUCKER OPTIMALITY CONDITIONS

Consider the optimization problem.
$(IVP2)$

$$\text{Min } f(x) = [f^L(x), f^U(x)]$$
$$\text{Subject to } g_j(x) \leq 0, j = 1, 2, \cdots, m.$$

Where $X = \{x \in R^n : g_j(x) \leq 0, j = 1, 2, \cdots, m\}$ be feasible set of problem $(IVP2)$ and let $J(x^*) = \{j : g_j(x^*) = 0, j = 1, 2, \cdots, m\}$ be the index set of active constraints. We say that the real valued constraint functions $g_j, j = 1, 2, \cdots, m$ satisfy the Kuhn-Tucker constraint qualification at x^* when, if, $\nabla g_j(x^*)^T d \leq 0$ for all $j \in J(x^*)$, where $d \in R^n$, then there exists an n-dimensional vector function $a : [0,1] \longrightarrow R^n$ such that a is right-differentiable at $0, a(0) = x^*, a(t) \in X$ for all $t \in [0,1]$, and there exists a real number $\alpha > 0$ with $a'_+(0) = \alpha d$ Wu [9]. Also let $x^* \in X$, We say that $g_j, j = 1, 2, \cdots, m$ satisfy KKT-assumptions at x^* if g_j is convex on R^n and continuously differentiable at x^* for $j = 1, 2, \cdots, m$ Wu [10]. The KKT optimality conditions for $(IVP2)$ obtained in Wu [10] are as follows.

Theorem 3.1. [10] *Suppose that the real valued constraint functions $g_j : R^n \longrightarrow R, j = 1, 2, \cdots, m$ satisfies KKT-assumptions at x^* and the interval valued objective functions $f : R^n \longrightarrow I$ is LU-convex and weakly continuously differentiable at x^*. If there exist (lagrange) multipliers $\lambda = (\lambda^L, \lambda^U) > 0$ and $\mu_j \geq 0, j = 1, 2, \cdots, m$ in R, such that*

(i) $\lambda^L \nabla f^L(x^*) + \lambda^U \nabla f^U(x^*) + \sum_{j=1}^{m} \mu_j \nabla g_j(x^*) = 0$

(ii) $\mu_j g_j(x^*) = 0$*for* $j = 1, 2, \cdots, m.$

Then x^* is type-I and type-II solution of $(IVP2)$.

Next in this section we are going to obtain KKT optimality conditions for optimization problem $(IVP1)$. For this we consider the optimization problem $(IVP1)$ with feasible set $X = \{x \in R^n : g_j(x) \preceq_{LU} [0,0], j = 1, 2, ..., m\}$, where $g_j(x) = [g_j^L(x), g_j^U(x)]$ are interval valued functions for $j = 1, 2, ..., m$, defined on R^n. That is
$(IVP3)$

$$\text{Min } f(x) = [f^L(x), f^U(x)]$$
$$\text{Subject to } g_j(x) \preceq_{LU} [0,0], j = 1, 2, ..., m.$$

Where f and $g_j, j = 1, 2, ..., m$ are interval valued functions.

Next we define the following.

Definition 3.2. Let x^* be the feasible solution of $(IVP3)$. We say that the interval valued constraint functions $g_j, j = 1, 2, ..., m$ satisfy the Kuhn-Tucker constraint qualification at x^* if g_j^L and $g_j^U, j = 1, 2, ..., m$ satisfy the Kuhn-Tucker constraint qualification at x^*.

In the proof of the following theorem, the Motzkins theorem of alternative is required. It states that, given matrices $A \neq 0$ and C, exactly one of the following system has a solution:

System 1: $Ax < 0, Cx \leq 0$ for some $x \in R^n$.
System 2: $A^T\lambda + C^T\mu = 0$ for some $\mu \geq 0$ and $\lambda \geq 0$ with $\lambda \neq 0$.

In the following theorem we obtain necessary conditions for type-I solution.

Theorem 3.3. *(KKT optimality conditions) Suppose that x^* is type-I solution of problem $(IVP3)$ and the interval valued functions f and $g_j, j = 1, 2, \cdots, m$ are weakly differentiable at x^*. Also assume that the interval valued constraint function $g_j, j = 1, 2, \cdots, m$ satisfy Kuhn-Tucker constraint qualification at x^*. Then there exist (Lagrange) multipliers $\mu_j^L, \mu_j^U \geq 0, j = 1, 2, \cdots, m$ and $\xi^L, \xi^U > 0$ in R , such that*

$$\xi^L \nabla f^L(x^*) + \xi^U \nabla f^U(x^*) + \sum_{j=1}^{m} \mu_j^L \nabla g_j^L(x^*) + \sum_{j=1}^{m} \mu_j^U \nabla g_j^U(x^*) = 0;$$

$$\mu_j^L g_j^L(x^*) = 0 = \mu_j^U g_j^U(x^*), j = 1, 2, \cdots, m.$$

Proof. Since f is weakly differentiable at x^*, by Definition 2.1 f^L and f^U are differentiable at x^*. Let there exists $d \in R^n$, such that

$$\begin{cases} \nabla f^L(x^*)^T d < 0, \\ \nabla f^U(x^*)^T d < 0, \\ \nabla g_j^L(x^*)^T d \leq 0, j \in J(x^*), \\ \nabla g_j^U(x^*)^T d \leq 0, j \in J(x^*). \end{cases} \tag{3.1}$$

Since $g_j, j = 1, 2, \cdots, m$ satisfy Kuhn-Tucker constraint qualification at x^* and f^L is differentiable at x^*, we have

$$\begin{aligned} f^L(a(t)) &= f^L(x^*) + \nabla f^L(x^*)^T (a(t) - x^*) + \| a(t) - x^* \| \epsilon(a(t), x^*) \\ &= f^L(x^*) + \nabla f^L(x^*)^T (a(t) - a(0)) + \| a(t) - a(0) \| \epsilon(a(t), a(0)) \\ &= f^L(x^*) + t \nabla f^L(x^*)^T \frac{(a(0+t) - a(0))}{t} + \| a(t) - a(0) \| \epsilon(a(t), a(0)) \end{aligned}$$

Where $\epsilon(a(t), a(0)) \longrightarrow 0$ as $\| a(t) - a(0) \| \longrightarrow 0$. Therefore, when $t \longrightarrow 0^+$, we have $\frac{a(0+t)-a(0)}{t} \longrightarrow a'_+(0) = \alpha d$, where $\alpha > 0$.

Since $\nabla f^L(x^*) d < 0$, we have $f^L(a(t_1)) < f^L(x^*)$ for a sufficiently small $t_1 > 0$. Similarly we have $f^U(a(t_2)) < f^U(x^*)$ for a sufficiently small $t_2 > 0$. Therefore we have $f^L(a(t)) < f^L(x^*)$ and $f^U(a(t)) < f^U(x^*)$ for a sufficiently small $t < min\{t_1, t_2\}$; consequently $f(a(t)) \prec_{LU} f(x^*)$ for a sufficiently small t, which contradicts the fact that x^* is type-I solution of $(IVP3)$. Therefore system (3.1) has no solution.

Now let A be the matrix whose rows are $\nabla f^L(x^*)^T$ and $\nabla f^U(x^*)^T$ and C be the matrix whose rows are $\nabla g_j^L(x^*)^T$ and $\nabla g_j^U(x^*)^T$ for $j \in J(x^*)$. According to Motzkins theorem of alternative, since the system (3.1) has no solution, there exist multipliers $\xi^L, \xi^U > 0$ and $\mu_j^L, \mu_j^U \geq 0$ in R for $j \in J(x^*)$, such that

$$\xi^L \nabla f^L(x^*) + \xi^U \nabla f^U(x^*) + \sum_{j \in J(x^*)} \{\mu_j^L \nabla g_j^L(x^*) + \mu_j^U \nabla g_j^U(x^*)\} = 0$$

Set $\mu_j^L = 0 = \mu_j^U$ for $j \in \{1, 2, \cdots, m\} \setminus J(x^*)$. Then we have

$$\xi^L \nabla f^L(x^*) + \xi^U \nabla f^U(x^*) + \sum_{j=1}^{m} \mu_j^L \nabla g_j^L(x^*) + \sum_{j=1}^{m} \mu_j^U \nabla g_j^U(x^*) = 0.$$
$$\mu_j^L g_j^L(x^*) = 0 = \mu_j^U g_j^U(x^*), \; for \; j = 1, 2, \cdots, m.$$

and the proof is completed.

□

The following theorem states some sufficient conditions for type-I optimality.

Theorem 3.4. *Suppose that the interval valued functions f and $g_j, j = 1, 2, \cdots, m$ are LU-convex and weakly continuously differentiable at $x^* \in X$. If there exist $\xi^L, \xi^U > 0$ and $\mu_j^L, \mu_j^U \geq 0, j = 1, 2, \cdots, m$ in R, such that*

(i) $\xi^L \nabla f^L(x^*) + \xi^U \nabla f^U(x^*) + \sum_{j=1}^{m} \mu_j^L \nabla g_j^L(x^*) + \sum_{j=1}^{m} \mu_j^U \nabla g_j^U(x^*) = 0$

(ii) $\mu_j^L g_j^L(x^*) = 0 = \mu_j^U g_j^U(x^*)$, *for* $j = 1, 2, \cdots, m$.

Then x^* is type-I and type-II solution of $(IVP3)$.

Proof. Since $g_j, j = 1, 2, ..., m$ are weakly continuously differentiable at x^*, by Definition 2.1 we see that the real valued functions g_j^L and $g_j^U, j = 1, 2, \cdots, m$ are continuously differentiable at x^*. Define the real valued function

$$\bar{g}_j(x) = \bar{\mu}_j^L g_j^L(x) + \bar{\mu}_j^U g_j^U(x), j = 1, 2, \cdots, m. \tag{3.2}$$

Where $\bar{\mu}_j^L, \bar{\mu}_j^U \geq 0, j = 1, 2, \cdots, m$. Since $g_j, j = 1, 2, \cdots, m$ are LU-convex at x^*, according to proposition 2.6, g_j^L and $g_j^U, j = 1, 2, \cdots, m$ are convex at x^*. Therefore $\bar{g}_j, j = 1, 2, \cdots, m$ are also convex and continuously differentiable at x^*.

Utilising (3.2), we see that

$$\begin{aligned} \mu_j^L \nabla g_j^L(x^*) + \mu_j^U \nabla g_j^U(x^*) &= \mu_j \{\bar{\mu}_j^L g_j^L(x) + \bar{\mu}_j^U g_j^U(x)\} \\ &= \mu_j \bar{g}_j(x), j = 1, 2, \cdots, m \end{aligned} \tag{3.3}$$

Where $\mu_j \bar{\mu}_j^L = \mu_j^L$ and $\mu_j \bar{\mu}_j^U = \mu_j^U$ for $j = 1, 2, \cdots, m$. Invoking (3.2) and (3.3) in (i) and (ii) of theorem we obtain.

(i) $\xi^L \nabla f^L(x^*) + \xi^U \nabla f^U(x^*) + \sum_{j=1}^{m} \mu_j \nabla \bar{g}_j(x^*) = 0$

(ii) $\mu_j \bar{g}_j(x^*) = 0$,for $j = 1, 2, \cdots, m$.

Therefore using Theorem 3.1, x^* is a type-I solution of the problem heaving interval valued objective function $f(x)$ subject to real valued constraints $\bar{g}_j(x^*), j = 1, 2, \cdots, m$, i.e.,

$$f(x^*) \prec_{LU} f(\bar{x}), for\ each\ \bar{x}(\neq x^*) \in X. \tag{3.4}$$

Now suppose that x^* is not a type-I solution of problem $(IVP3)$. Then, based on Definition 2.2, there exists $\bar{x} \in X$, such that

$$\begin{cases} f^L(\bar{x}) \leq f^U(x^*) \\ f^L(\bar{x}) < f^U(x^*); \end{cases} or \begin{cases} f^L(\bar{x}) < f^U(x^*) \\ f^L(\bar{x}) < f^U(x^*); \end{cases} or \begin{cases} f^L(\bar{x}) < f^U(x^*) \\ f^L(\bar{x}) \leq f^U(x^*). \end{cases}$$

Therefore we see that, $f(\bar{x}) \prec_{LU} f(x^*)$. Which contradicts (3.4). This shows that x^* is type-I solution of problem $(IVP3)$ and hence by Remark 2.4, x^* is also type-II solution of problem $(IVP3)$. This proves the theorem.

□

Example 3.5. Consider the following programming problem with interval valued objective and constraint functions:

$$\begin{array}{c} \text{Min } f(x) = [f^L(x), f^U(x)] = [2x_1^2 + 2x_2^2 + 3, 2x_1^2 + 2x_2^2 + 4] \\ \text{Subject to } g_1 = [g_1^L, g_1^U] = [1 - x_1 - x_2, 6 - 3x_1 - x_2] \preceq [0, 0] \\ x_1 \geq 0, x_2 \geq 0 \end{array}$$

Then we have

$$\begin{array}{c} f^L(x_1, x_2) = 2x_1^2 + 2x_2^2 + 3, f^U(x_1, x_2) = 2x_1^2 + 2x_2^2 + 4 \\ g_1^L(x_1, x_2) = 1 - x_1 - x_2, g_1^U(x_1, x_2) = 6 - 3x_1 - x_2 \end{array}$$

It is easy to see that the above functions satisfy the assumptions of Theorem 3.4. We have to find x_1, x_2 and ξ^L, ξ^U and μ_1^L, μ_1^U , such that:

$$\xi^L \begin{bmatrix} 4x_1 \\ 4x_2 \end{bmatrix} + \xi^U \begin{bmatrix} 4x_1 \\ 4x_2 \end{bmatrix} + \mu_1^L \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \mu_1^U \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{cases} 1 - x_1 - x_2 \leq 0, \\ 6 - 3x_1 - x_2 \leq 0, \\ \mu_1^L(1 - x_1 - x_2) = 0, \\ \mu_1^U(6 - 3x_1 - x_2) = 0, \\ \xi^L, \xi^U > 0, \mu_1^L, \mu_1^U \geq 0, x_i \geq 0, i = 1, 2. \end{cases} \tag{3.5}$$

That is, we have to find a solution for the following simultaneous equations which satisfy the conditions (3.5).

$$4\xi^L x_1 + 4\xi^U x_1 - \mu_1^L - 3\mu_1^U = 0;$$

$$4\xi^L x_2 + 4\xi^U x_2 - \mu_1^L - \mu_1^U = 0.$$

After some algebraic calculations, we obtain

$(x_1^*, x_2^*) = (\frac{9}{5}, \frac{3}{5}), \xi^L = \frac{1}{4}, \xi^U = \frac{1}{4}, \mu_1^L = 0$ and $\mu_1^U = \frac{6}{5}$.

Since $g_1^U(\frac{9}{5}, \frac{3}{5}) = 0$, condition (ii) in Theorem 3.4 is satisfied. Therefore $(x_1^*, x_2^*) = (\frac{9}{5}, \frac{3}{5})$ is type-I solution. In view of the Remark 2.4, type-II solution is obvious.

Further let k is non zero integer such that $1 < k < m$, the sufficient conditions can be resorted as:

Theorem 3.6. *Under the same assumption of Theorem 3.4, let k be any integer with $1 < k < m$. If there exist (Lagrange) multipliers $\mu_j^L, \mu_j^U \geq 0, j = 1, 2, \cdots, m$ s.t,*

(i) $\nabla f^L(x^*) + \sum\limits_{j=1}^{k} \mu_j^L \nabla g_j^L(x^*) + \sum\limits_{j=1}^{k} \mu_j^U \nabla g_j^U(x^*) = 0$

(ii) $\nabla f^U(x^*) + \sum\limits_{j=k}^{m} \mu_j^L \nabla g_j^L(x^*) + \sum\limits_{j=k}^{m} \mu_j^U \nabla g_j^U(x^*) = 0$

(iii) $\mu_j^L g_j^L(x^*) = 0 = \mu_j^U g_j^U(x^*), for j = 1, 2, \cdots, m.$

Then x^* is type-I and type-II solution of $(IVP3)$.

Proof. Multiplying (i) by $\xi^L > 0$ and (ii) by $\xi^U > 0$, and adding then we get.

(i) $\xi^L \nabla f^L(x^*) + \xi^U \nabla f^U(x^*) + \sum\limits_{j=1}^{k} \hat{\mu}_j^L \nabla g_j^L(x^*) + \sum\limits_{j=1}^{k} \hat{\mu}_j^U \nabla g_j^U(x^*) = 0$

(ii) $\hat{\mu}_j^L g_j^L(x^*) = 0 = \hat{\mu}_j^U g_j^U(x^*), for j = 1, 2, \cdots, m.$

Where $\hat{\mu}_j^L = \xi^L \mu_j^L, \hat{\mu}_j^U = \xi^L \mu_j^U$, for $j = 1, 2, \cdots, k$ and $\hat{\mu}_j^L = \xi^U \mu_j^L, \hat{\mu}_j^U = \xi^U \mu_j^U$, for $j = k+1, \cdots, m$ Therefore by Theorem 3.4, we see that x^* is type-I and type-II solution of $(IVP3)$.

□

Next we introduce centre function $f^c = \frac{1}{2}[f^L + f^U]$ and then resort the conditions as:

Theorem 3.7. *Under the same assumption of Theorem 3.4, Let $f^C = \frac{1}{2}[f^L + f^U]$. If there exist $\xi^C, \xi^U > 0$ and $\mu_j^L, \mu_j^U \geq 0, j = 1, 2, \cdots, m.$ s.t,*

(i) $\xi^U \nabla f^U(x^*) + \xi^C \nabla f^C(x^*) + \sum\limits_{j=1}^{m} \mu_j^L \nabla g_j^L(x^*) + \sum\limits_{j=1}^{m} \mu_j^U \nabla g_j^U(x^*) = 0$

(ii) $\mu_j^L g_j^L(x^*) = 0 = \mu_j^U g_j^U(x^*), for j = 1, 2, \cdots, m.$

Then x^* is type-I and type-II solution of $(IVP3)$.

Proof. Using (3.2) and (3.3) in (i) and (ii) of this theorem we obtain.

(i) $\xi^U \nabla f^U(x^*) + \xi^C \nabla f^C(x^*) + \sum\limits_{j=1}^{m} \mu_j \nabla \bar{g}_j(x^*) = 0$

(ii) $\mu_j^L \bar{g}_j^L(x^*) = 0, for j = 1, 2, \cdots, m.$

Also since f is LU-convex and weakly continuously differentiable at x^*, therefore $f^C = \frac{1}{2}[f^L + f^U]$ is convex and continuously differentiable at x^* (by Proposition 2.6 and Definition 2.1). Now suppose x^* is not type-II solution, using similar arguments as in Theorem 3.4, we conclude that x^* is a type-II solution of problem $(IVP3)$. Since

$$\begin{aligned}\xi^U \nabla f^U(x^*) + \xi^C \nabla f^C(x^*) &= \xi^U \nabla f^U(x^*) + \frac{1}{2}\xi^C[\nabla f^L(x^*) + \nabla f^U(x^*)] \\ &= \frac{1}{2}\xi^C \nabla f^L(x^*) + \left(\frac{1}{2}\xi^C + \xi^U\right)\nabla f^U(x^*),\end{aligned}$$

We conclude that x^* is also a type-I solution by using Theorem 3.4.

□

Next we present KKT conditions for type-II solution.

Theorem 3.8. *Suppose that the interval valued functions $g_j, j = 1, 2, ..., m$ are LU-convex and weakly continuously differentiable at $x^* \in X$. Also suppose that the interval valued function f is UC-convex and weakly continuously differentiable at x^*. If there exist (Lagrange) multipliers $\xi^L, \xi^C > 0$ and $\mu_j^L, \mu_j^U \geq 0, j = 1, 2, ..., m$ in R, such that*

(i) $\xi^U \nabla f^U(x^*) + \xi^C \nabla f^C(x^*) + \sum_{j=1}^{m} \mu_j^L \nabla g_j^L(x^*) + \sum_{j=1}^{m} \mu_j^U \nabla g_j^U(x^*) = 0$

(ii) $\mu_j^L g_j^L(x^*) = 0 = \mu_j^U g_j^U(x^*), for j = 1, 2, \cdots, m$

Then x^* is type-II solution of $(IVP3)$.

Proof. Using (3.2) and (3.3) in (i) and (ii) of this theorem we obtain.

(i) $\xi^U \nabla f^U(x^*) + \xi^C \nabla f^C(x^*) + \sum_{j=1}^{m} \mu_j \nabla \bar{g}_j(x^*) = 0$

(ii) $\mu_j \bar{g}_j(x^*) = 0, for\ j = 1, 2, \cdots, m$

Since f is UC-convex and weakly continuously differentiable at x^*, we see that f^U and f^C are convex and continuously differentiable at x^* (by Proposition 2.6 and Definition 2.1). Using similar arguments as in Theorem 3.4, we conclude that x^* is a type-II solution of $(IVP3)$. Despite of the fact that

$$\xi^U \nabla f^U(x^*) + \xi^C \nabla f^C(x^*) = \frac{1}{2}\xi^C \nabla f^L(x^*) + \left(\frac{1}{2}\xi^C + \xi^U\right)\nabla f^U(x^*).$$

We cannot conclude that x^* is also a type-I solution in view of Theorem 3.4, since the assumptions in this theorem is different from that of Theorem 3.4 and the UC-convexity does not imply LU-convexity in general.

□

Next let us consider pseudoconvexity in order to relax the convexity assumptions of interval valued objective function.

Definition 3.9. [10] Let f be differentiable real valued function defined on non empty convex set X of R, then f is said to be pseudoconvex at x^* if $f(x) < f(x^*)$ then $\nabla f(x^*)^T(x - x^*) < 0$ for $x \in X$ and f is strictly pseudoconvex at x^* if $f(x) \leq f(x^*)$ then $\nabla f(x^*)^T(x - x^*) < 0$ for $x \in X$.

Definition 3.10. [10] Let $f(x) = [f^L(x), f^U(x)]$ be an interval valued function defined on convex set $X \subseteq R^n$. We say that f is pseudoconvex at x^* if the real valued functions f^L and f^U are pseudoconvex at x^*.

Let X be a nonempty feasible set and $x^* \in cl(X)$ (the closure of X). The cone of feasible directions of X at x^*, denoted by D, is defined by

$$D = \{d \in R^n : d \neq 0, \text{there exists a } \delta > 0 \text{ such that } x^* + \tau d \in X \text{ for all } \tau \in (0, \delta)\}.$$

Each d of D is called a feasible direction of X.

Proposition 3.11. [19] *Let $X = \{x \in R^n : g_j(x) \leq 0, j = 1, 2, ..., m\}$ be a feasible set and a point $x^* \in X$. Assume that g_j are differentiable at x^* for all $j = 1, 2, ..., m$. Let $J = \{j : g_j(x^*) = 0\}$ be the index set for the active constraints. Then*

$$D \subseteq \{d \in R^n : \nabla g_j(x^*)^T d \leq 0 \textit{ for each } j \in J\}.$$

(Note that this proposition still hold true if we just assume that g_j are continuous at x^* instead of differentiable at x^* for $j \notin J$).

In the proof of the following theorem the Tuckers theorem of the alternative is needed. It states that, given matrices A and C, exactly one of the following system has a solution:

System 1: $Ax \leq 0, Ax \neq 0, Cx \leq 0$ for some $x \in R^n$;
System 2: $A^T\lambda + C^T\mu = 0$, for some $\lambda > 0$ and $\mu \geq 0$.

Theorem 3.12. *Suppose that the interval valued functions $g_j, j = 1, 2, ..., m$ are LU-convex and weakly continuously differentiable at $x^* \in X$. and the interval valued objective function f is weakly differentiable and pseudoconvex at x^*. If there exist (Lagrange) multipliers $\mu_j^L, \mu_j^U \geq 0, j = 1, 2, ..., m$ in R, such that,*

(i) $\nabla f^L(x^*) + \sum_{j=1}^{m} \mu_j^L \nabla g_j^L(x^*) = 0$

(ii) $\nabla f^U(x^*) + \sum_{j=1}^{m} \mu_j^U \nabla g_j^U(x^*) = 0$

(iii) $\mu_j^L g_j^L(x^*) = 0 = \mu_j^U g_j^U(x^*), for\ j = 1, 2, \cdots, m$

Then x^* is type-I and type-II solution of $(IVP3)$.

Proof. We shell prove this result by contradiction. Suppose that x^* is not a type-I solution. Then by definition there exists an $\bar{x} \neq x^*$ such that $f(\bar{x}) \prec_{LU} f(x^*)$, which implies that either $f^L(\bar{x}) < f^L(x^*)$ or $f^U(\bar{x}) < f^U(x^*)$. Since f is weakly differentiable and pseudoconvex at x^*, by Definition 2.1 and 3.10, f^L and f^U are differentiable and pseudoconvex at x^*, we have either $\nabla f^L(x^*)^T(\bar{x} - x^*) < 0$ or $\nabla f^U(x^*)^T(\bar{x} - x^*) < 0$. Let us consider the case

$$\nabla f^L(x^*)^T(\bar{x} - x^*) < 0. \tag{3.6}$$

Let $d = \bar{x} - x^*$. Then $x = x^* + \tau d = \tau\bar{x} + (1 - \tau)x^* \in X$ for $\tau \in (0, 1)$. Since X is a convex set and $\bar{x}, x^* \in X$. This shows that $d \in D$. From Proposition 3.11, we conclude that

$$\nabla g_j^L(x^*)d \leq 0 \; for \; each \; j \in J(x^*). \tag{3.7}$$

Let A be the matrix whose rows are $\nabla f^L(x^*)^T$ and C be the matrix whose rows are $\nabla g_j^L(x^*)^T$ for $j \in J$. Therefore by using Tuckers theorem of the alternative, since System 1 has a solution $d = \bar{x} - x^*$ (see (3.6) and (3.7)), there exist no multipliers $0 < \lambda \in R$ and $0 \leq \mu_j \in R, j \in J(x^*)$, such that $\lambda\nabla f^L(x^*) + \sum_{j \in J(x^*)} \mu_j \nabla g_j^L(x^*) = 0$, or equivalently, there exist no multipliers $0 \leq \mu_j^L \in R$, $j \in J(x^*)$, such that $\nabla f^L(x^*) + \sum_{j \in J(x^*)} \mu_j^L \nabla g_j^L(x^*) = 0$, where $\mu_j^L = \frac{\mu_j}{\lambda}$. Setting $\mu_j^L = 0, j \notin J$, we get a contradiction to (i) and (iii). Similarly, If $\nabla f^U(x^*)^T(\bar{x} - x^*) < 0$, then conditions (ii) and (iii) will be violated. This shows that x^* is a type-I solution. From Remark 2.4, the proof is complete. □

4. CONCLUSIONS

Interval programming is one of the approaches to handle the uncertain optimization, in which an interval is used to model the uncertainty of variables. Most of the recent work has been done by considering interval coefficients of objective function. Although the same uncertainty is also likely to be imposed on constraints. In this paper we have successfully derived the KKT optimality conditions for programming problems with interval valued objective and interval valued constraint functions. Although the interval valued equality constraints are not considered in this paper, the similar methodology proposed in this paper can also be used to handle the interval valued equality constraints. Future research is oriented to consider the uncertain environment in order to study the optimality conditions involving Fuzzy parameters.

REFERENCES

1. A. Ben-Israel, P.D. Robers, A decomposition method for interval linear programming, Manag. Sci. 16 (1970), 374-387.
2. A. Ben-Tal, L. El Ghaoui, A. Nemirovski, Robust optimization. Princeton Series in Applied Mathematics, Princeton University Press, USA, 2009.
3. A. Charnes, F. Granot, E. Phillips. An algorithm for solving interval linear programming problems, Oper. Res. 21 (1977), 688-695.
4. A. Prekopa, Stochastic programming. Kluwer Academic Publishers, Boston, 1995.
5. C. Oliveria, C.H. Antunes, Multiple objective linear programming models with interval coefficients-an illustrated review, Eur. J. Oper. Res. 181 (2007), 1434-1463.
6. F. Marz, Calculating the exact bounds of optimal values in LP with interval coefficients, Annal. oper. res. 81 (1998), 51-62.

7. H. Ishibuchi, H. Tanaka, Multiobjective programming in optimization of interval valued objective functions, Eur. J. Oper. Res. 48 (1990), 219-225.
8. H. Rommelfanger, R. Hanuscheck, J. Wolf, Linear programming with fuzzy objectives, Fuzzy sets syst. 29(1989), 31-48.
9. H.C. Wu, On Interval valued Nonlinear Programming Problems, J. Math. Anal. Appl. 338 (2008), 299-316.
10. H.C. Wu, The karush Kuhn tuker optimality conditions in an optimization problem with interval valued objective functions, Eur. J. Oper. Res. 176 (2007), 46-59.
11. H.C. Wu, The karush Kuhn tuker optimality conditions in multiobjective programming problems with interval valued objective functions, Eur. J. Oper. Res. 196 (2009), 49-60.
12. I.M. Stanchu-Minasian, Stochastic programming with multiple objective functions, D. Reidei Publishing Company. Bordrecht, 1984.
13. I.M. Stanchu-Minasian, St. Tigan, Multiobjective mathematical programming with inexact data.in: R. Slowinski. J. Teghem (Eds), stochastic verses fuzzy approaches to multiobjctive mathematical programming under uncertainty. Kluwer Academic Publishers, Boston, pp (1990), 395-418.
14. J.R. Birg, F. Luoveaux, Introduction to Stochastic programming, Physica-Veriag, NY, 1997.
15. J. Zhang, S. Liu, L. Li, Q. Feng, The KKT optimality conditions in a class of generalized convex optimization problems with an interval-valued objective function, Optim Lett (2012), DOI 10.1007/s11590-012-0601-6.
16. M. Delgado, J. Kacprzyk, J.L. Verdegay, M.A. Vila, Fuzzy Optimization: Recent Advances, Physica-Verlag. NY, 1994.
17. M. Innuguchi, J. Ramik, Possibilistic linear programming: A brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem. Fuzzy sets Syst. 111 (2000), 3-28.
18. M.R. Benjamin, The interval programming model for multiobjective decision making. Computer Science and Artificial Intelligence Labratory (2004), 320-330.
19. M.S. Bazarra, H.D. Sherali and C.M. Shetty, Nonlinear programming, Wiley, YN, 1993.
20. R.E. Steuer, Algorithms for linear programming problems with interval objective function coefficients, Math. Oper. Res. 6 (1981), 333-348.
21. R. Slowinski, Fuzzy sets in decision analysis, Operation research and statistics, Kluwerv Academic Publishers, Boston, 1998.
22. R. Slowinski, J. Teghem, Stochastic verses fuzzy approaches to multiobjective mathematical programming, Kluwerv Academic Publishers, Boston, 1990.
23. S.Chanas, D. Kuchta, Multiobjective programming in optimization of interval objective functions-A generalized approach, Euro. J. of Oper. Res. 94 (1996), 594-598.
24. S. Vajda, Probabilistic programming, Academic press, NY, 1972.
25. Y.J. Lai, C.L. Hwang, Fuzzy mathematical programming: Methods and Applications, Lecture notes in economics and mathematical systems 394. Springer-Verlag, NY,1992.
26. Y.J. Lai, C.L. Hwang, Fuzzy multiple objective decision making: methods 2 and applications, Lecture notes in Economics and mathematical systems 404. Spriger-Verlag, NY, 1994.
27. Y. Chalco-Cano, W.A. Lodwick, A. Rufian-Lizana, Optimality conditions of type KKT for optimization problem with interval-valued objective function via generalized derivative, Fuzzy optim Decis Making (2013) DOI 10.1007/s10700-013-9156-y.