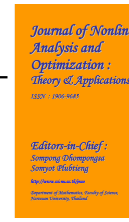

Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 5, No. 2, (2014), 81-90
ISSN : 1906-9685
http://www.math.sci.nu.ac.th

SOME DEVELOPMENT FOR NEWTON'S METHOD UNDER MILD DIFFERENTIABILITY CONDITIONS

I.K. ARGYROS* AND S.K. KHATTRI*

Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA.
Department of Engineering, Stord Haugesund University College, Norway.

ABSTRACT. We develop a semilocal convergence analysis for Newton's method under mild differentiability conditions. Our sufficient convergence conditions can be weaker than conditions found in earlier studies such as [11-14,16,18–21]. Numerical examples are also provided.

KEYWORDS : Newton's method; Banach space; Hölder continuity; Lipschitz continuity; semilocal convergence; Kantorovich hypothesis.
AMS Subject Classification: 47H17, 49M15, 65H10, 65G99, 65J15

1. INTRODUCTION

In this work, we study the problem of approximating a locally unique solution $x^\star$ of the nonlinear equation

$$\mathcal{F}(x) = 0 \tag{1.1}$$

where $\mathcal{F}$ is a Fréchet–differentiable operator defined on a convex subset $\mathbf{D}$ of a Banach space $\mathbf{X}$ with values in a Banach space $\mathbf{Y}$.

Many problems – from nonlinear convex analysis and other disciplines – can be formulated in the form of equation (1.1) using Mathematical Modelling [6]. The solution of these equations can rarely be found in closed form. That is why the solution methods for these equations are usually iterative. In particular, the practice of numerical analysis for finding such solutions is essentially connected to Newton-type methods [6, 13]. For iterattions $n = 0, 1, 2, \ldots$, we define Newton's method as

$$\begin{cases} \mathcal{F}'(\mathbf{x_n})\boldsymbol{\Delta}\mathbf{x_n} = -\mathcal{F}(\mathbf{x_n}) \\ \quad \mathbf{x_{n+1}} = \mathbf{x_n} + \boldsymbol{\Delta}\mathbf{x_n} \end{cases} \tag{1.2}$$

where x_0 is an initial guess. The study about convergence of iterative procedures is normally centered on two types: semilocal and local convergence analysis. The semilocal convergence analysis is based on the information around an initial point

I.K. Argyros Corresponding author.*
Email address : iargyros@cameron.edu(I.K. Argyros), sanjay.khattri@hsh.no(S.K. Khattri).
Article history : Received March 04, 2014 Accepted March 11, 2014.

to give criteria ensuring the convergence of iterative procedures. While the local analysis is based on the information around a solution to find estimates of the radii of convergence balls. There exist many studies which deals with the local and the semilocal convergence analysis of Newton-type methods such as [1-21].

In this work – particularly motivated by optimization considerations – we provide sufficient convergence conditions that can be weaker than the ones found in earlier studies such as [11-14,16,18-21] and our majorizing sequences can be finer as well. Moreover, the new advantages are obtained under the same computational cost.

The rest of the paper is organized as follows. In section 2, we present earlier results in order to make the study as self contained as possible. Section 3 contains the semilocal convergence of Newton's method. Section 4 reports numerical work.

2. Preliminaries

Let $\mathbb{U}(x_0, r)$ and $\overline{U}(x_0, r)$ stand, respectively, for the open and closed balls in $\mathbf{X}$ with center x_0 and radius $r > 0$. We denote by $L(\mathbf{X}, \mathbf{Y})$ the space of bounded linear operators from $\mathbf{X}$ into $\mathbf{Y}$. In this Section, we assume that $p \in (0, 1]$ unless stated otherwise. Let $x_0 \in \mathbf{D}$ be such that $\mathcal{F}'(x_0)^{-1} \in L(\mathbf{Y}, \mathbf{X})$. Assume $\mathcal{F}'$ satisfies a p-center–Hölder condition at some $p \in (0, 1]$

$$\left\| \mathcal{F}'(x_0)^{-1} \left(\mathcal{F}'(x) - \mathcal{F}'(x_0) \right) \right\| \leq L_0 \left\| x - x_0 \right\|^p \\ \text{for each } x \in \mathbb{U}(x_0, R) \subseteq \mathbf{D} \text{ and some } L_0 > 0, \quad R > 0 \tag{2.1}$$

and a p-Hölder condition at some $p \in (0, 1]$

$$\left\| \mathcal{F}'(x_0)^{-1} \left(\mathcal{F}'(x) - \mathcal{F}'(y) \right) \right\| \leq L \left\| x - y \right\|^p \\ \text{for each } x, y \in \mathbb{U}(x_0, R) \subseteq \mathbf{D} \text{ and for some } L > 0. \tag{2.2}$$

We have that

$$L_0 \leq L \tag{2.3}$$

holds in general and that L/L_0 can be arbitrarily large [3, 6] (see also the Examples in §4). Note that (2.2) always implies (2.1). In practice the computation of L requires computing L_0. Hence, (2.1) is not an additional requirement to the hypothesis (2.2). For earlier results on Newton's method under conditions (2.1) or (2.2), we refer interested reader to the interesting work: [2,4,6,10–18,21 and references therein]. In particular, a new result was given by Cianciaruso in [7] which improves earlier sufficient convergence conditions for $p \in (0, 1)$, but not necessarily the error bounds [8,10,14–19]. Consider

$$c_0 = \frac{L + \sqrt{L^2 + 4\, L_0\, L\, (1+p)^p\ p^{1-p}}}{2\, L} \tag{2.4}$$

and

$$h_0(t) = \left(1 - \frac{1}{t}\right)^p \frac{1+p}{\left((L_0\,(1+p))^{\dfrac{1}{1-p}} + (L\,t\,(t-1))^{\dfrac{1}{1-p}} \right)^{1-p}}. \tag{2.5}$$

Then, the convergence condition is given by

$$\eta^p \leq h_0(c_0). \tag{2.6}$$

3. Semilocal convergence analysis

First we present four auxiliary results on majorizing sequences for Newton's method.

Lemma 3.1. *Let $\ell_0 > 0$, $\ell > 0$, $\eta > 0$ and $p \in (0,1]$ be given parameters. Define function g_0 on $[0,1)$ by*

$$g_0(t) = \frac{\ell}{1+p}\eta^p(t-1) + \ell_0 t^{1/p}\Big[(1+t^{1/p})^p - 1\Big]\eta^p.$$

We denote by α the minimal zero of g_0 in the interval $(0,1)$. Suppose that

$$0 < \frac{\ell\eta^p}{(1+p)(1-\ell_0\eta^p)} \leq \alpha^{1/p} \leq 1 - \ell_0^{1/p}\eta. \tag{3.1}$$

Then, the scalar sequence $\{t_n\}$ defined by

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} + \frac{\ell(t_{n+1}-t_n)^{1+p}}{(1+p)(1-\ell_0 t_{n+1}^p)} \quad \text{for each} \quad n = 0,1,2,\ldots \tag{3.2}$$

is increasing, bounded from above by

$$t^{\star\star} = \frac{\eta}{1-\alpha^{1/p}} \tag{3.3}$$

and converges to its unique least upper bound denoted by $t^\star$ which satisfies $t^\star \in [\eta, t^{\star\star}]$. Moreover, the following estimates hold for each $n = 0,1,2,\ldots$

$$0 < t_{n+2} - t_{n+1} \leq \alpha^{1/p}(t_{n+1}-t_n) \leq \alpha^{(n+1)/p}\eta. \tag{3.4}$$

Proof. Since $g_0(1) = \ell_0\,\eta^p(2^p-1) > 0$, $g_0(0) = -\ell\,\eta^p/(1+p) < 0$ thus – it follows from the intermediate value theorem applied to the function g_0 on the interval $[0,1]$ that – the function g_0 has zeros in $(0,1)$. We denote the smallest such zero of function g_0– in the interval $(0,1)$ – by α.

Estimate (3.4) holds, if

$$0 < t_{k+2} - t_{k+1} = \frac{\ell\,(t_{k+1}-t_k)^p}{(1+p)(1-\ell_0\,t_{k+1}^p)} \leq \alpha^{1/p}. \tag{3.5}$$

Inequalities (3.4) and (3.5) hold for $k=0$ by (3.1). Let us assume that (3.4) and (3.5) hold for all $k \leq n$. Then, we obtain

$$\begin{aligned} t_{k+1} &\leq t_k + \alpha^{1/p}(t_k - t_{k-1}) \\ &\leq t_{k-1} + \alpha^{1/p}(t_{k-1}-t_{k-2}) + \alpha^{1/p}(t_k - t_{k-1}) \\ &\leq t_1 + \alpha^{1/p}(t_1-t_0) + \alpha^{1/p}(t_2-t_1) + \cdots + \alpha^{1/p}(t_k-t_{k-1}) \\ &\leq \eta + \alpha^{1/p}\eta + \alpha^{2/p}\eta + \cdots + \alpha^{k/p}\eta = \frac{1-\alpha^{k+1/p}}{1-\alpha^{1/p}}\eta < \frac{\eta}{1-\alpha^{1/p}} = t^{\star\star} \end{aligned} \tag{3.6}$$

and

$$t_{k+1} - t_k \leq \alpha^{1/p}(t_k - t_{k-1}) \leq \cdots \leq \alpha^{k/p}(t_1-t_0) = \alpha^{k/p}\eta. \tag{3.7}$$

In view of (3.6) and (3.7), estimate (3.5) holds, if

$$\ell\Big((\alpha^{1/p})^n\eta\Big)^p \leq (1+p)\alpha^{1/p}\Big[1 - \ell_0\Big(\frac{1-\alpha^{(n+1)/p}}{1-\alpha^{1/p}}\eta\Big)^p\Big]$$

or

$$\ell\alpha^n\eta^p + (1+p)\alpha^{1/p}\ell_0(1+\alpha^{1/p}+\alpha^{2/p}+\cdots+\alpha^{n/p})^p\eta^p - \alpha^{1/p}(1+p) \leq 0$$

or

$$\ell\alpha^n\eta^p + (1+p)\alpha^{1/p}\ell_0(1+\alpha^{1/p}+\alpha^{2/p}+\cdots+\alpha^{n/p})\eta - \alpha^{1/p}(1+p) \le 0. \tag{3.8}$$

Estimate (3.8) motivates us to introduce recurrent functions f_k on $(0,1)$ by

$$f_k(t) = \ell\eta^p t^k + (1+p)t^{1/p}\ell_0(1+t^{1/p}+t^{2/p}+\cdots+t^{k/p}) - t^{1/p}(1+p). \tag{3.9}$$

We need a relationship between two consecutive functions f_k. Using (3.9) we get that

$$f_{k+1} = f_k(t) + g_k(t) \tag{3.10}$$

where

$$g_k(t) = \frac{\ell}{1+p}\eta^p t^n(t-1) + \ell_0 t^{1/p}\eta^p\Big[(1+t^{1/p}+\cdots+t^{(n+1)/p})^p - (1+t^{1/p}+\cdots+t^{n/p})^p\Big]. \tag{3.11}$$

We have that

$$g_{k+1}(t) = g_k(t) + \frac{\ell}{1+p}\eta^p t^k(t-1)^2 + \ell_0 t^{1/p}\eta^p\Big[(1+t^{1/p}+\cdots+t^{(k+2)/p})^2 - 2(1+t^{1/p}+\cdots+t^{(k+1)/p})^p + (1+t^{1/p}+\cdots+t^{k/p})^p\Big] \ge g_k(t). \tag{3.12}$$

In particular, it follows from the definition of α, (3.10) and (3.12) that

$$f_{k+1}(\alpha) = g_k(\alpha) + f_k(\alpha) \ge g_{k-1}(\alpha) + f_k(\alpha) \ge \cdots \ge g_0(\alpha) + f_k(\alpha) = f_k(\alpha). \tag{3.13}$$

We define function f_∞ on $[0,1)$ by

$$f_\infty(t) = \lim_{t\longrightarrow\infty} f_k(t). \tag{3.14}$$

Then, using (3.8), (3.9) and (3.14) we get that

$$f_\infty(t) = (1+p)t^{1/p}\Big[\ell_0\Big(\frac{\eta}{1-\alpha^{1/p}}\Big)^p - 1\Big]. \tag{3.15}$$

Moreover, we have that

$$f_\infty(t) \ge f_k(t) \quad \text{for each} \quad k = 0,1,2,\ldots. \tag{3.16}$$

Hence, it follows from (3.8), (3.9), (3.15) and (3.16) that (3.8) holds if

$$f_\infty(\alpha) \le 0. \tag{3.17}$$

But (3.16) is implied by the right hand side of the hypothesis (3.1). The induction for (3.4) and (3.5) is complete. It follows that sequence $\{t_n\}$ is increasing, bounded from above by $t^{\star\star}$ and as such it converges to its unique least upper bound $t^\star \in [\eta, t^{\star\star}]$. □

Next, we present another auxiliary result of majorizing sequences for Newton's method.

Lemma 3.2. *Let $\ell_0 > 0$, $\ell > 0$, $\eta > 0$ and $p \in (0,1]$ be given parameters. Suppose that*

$$\ell_0\eta^p < 1 \tag{3.18}$$

and there exists $\alpha \in (0,1)$ such that

$$\min\Big\{\frac{\ell(s_2-s_1)^p}{(1+p)(1-\ell_0 s_2^p)}, \frac{\ell}{1+p}\Big(\frac{\ell_0\eta^{1+p}}{(1+p)(1-\ell_0\eta^p)}\Big)^p\alpha + \ell_0\alpha^{1/p}\Big(\eta + \frac{\ell_0\eta^{1+p}}{(1-\ell_0\eta^p)(1-\alpha^p)}\Big)^p\Big\} \le \alpha^{1/p}. \tag{3.19}$$

Then, the scalar sequence $\{s_n\}$ *defined by*

$$\begin{cases} s_0 = 0, \quad s_1 = \eta, \quad s_2 = s_1 + \dfrac{\ell_0(s_1 - s_0)^{1+p}}{(1+p)(1-\ell_0 s_1^p)}, \\ s_{n+2} = s_{n+1} + \dfrac{\ell(s_{n+1} - s_n)^{1+p}}{(1+p)(1-\ell_0 s_{n+1}^p)}, \quad \text{for each} \quad n = 1, 2, \ldots \end{cases} \tag{3.20}$$

is increasing, bounded from above by

$$s^{\star\star} = \left[1 + \frac{\ell_0 \eta^p}{(1-\alpha^{1/p})(1+p)(1-\ell_0\eta^p)}\right]\eta \tag{3.21}$$

and converges to its unique least upper bound $s^\star$ *which satisfies* $s^\star \in [s_2, s^{\star\star}]$. *Moreover, the following estimate holds for each* $n = 1, 2, 3, \ldots$

$$0 < s_{n+2} - s_{n+1} \leq \frac{\alpha^{n/p}\ell_0\eta^{1+p}}{(1+p)(1-\ell_0\eta^p)}. \tag{3.22}$$

Proof. Estimate (3.22) holds if

$$0 < \frac{\ell(s_{k+1} - s_k)^p}{(1+p)(1-\ell_0 s_{k+1}^p)} \leq \alpha^{1/p} \quad \text{for each} \quad k = 1, 2, \ldots. \tag{3.23}$$

Inequalities (3.22) and (3.23) hold for $k = 1$ by (3.19) and (3.20). From (3.23) and (3.22), we obtain

$$\begin{aligned} & s_3 - s_2 \leq \alpha^{1/p}(s_2 - s_1) \\ \implies \quad & s_3 \leq s_2 + \alpha^{1/p}(s_2 - s_1) \\ \implies \quad & s_3 \leq s_2 + (1 + \alpha^{1/p})(s_2 - s_1) - (s_2 - s_1) \\ \implies \quad & s_3 \leq s_1 + \frac{1-\alpha^{2/p}}{1-\alpha^{1/p}}(s_2 - s_1) < s_1 + \frac{1}{1-\alpha^{1/p}}(s_2 - s_1) = s^{\star\star} \end{aligned} \tag{3.24}$$

Suppose that (3.22) and (3.23) hold for all $k \leq n$. Then, we have

$$s_{k+2} - s_{k+1} \leq \alpha^{k/p}(s_2 - s_1) \tag{3.25}$$

and

$$s_{k+2} \leq s_1 + \frac{1-\alpha^{(k+1)/p}}{1-\alpha^{1/p}}(s_2 - s_1) < s^{\star\star}. \tag{3.26}$$

We need to show that (3.23) holds if k is replaced by $k+1$. Then – by (3.23), (3.25) and (3.26) – we must show that

$$\frac{\ell}{1+p}(s_{k+2} - s_{k+1})^p + \alpha^{1/p}\ell_0 s_{k+2}^p - \alpha^{1/p} \leq 0$$

or

$$\frac{\ell}{1+p}\Big(\alpha^{k/p}(s_2 - s_1)\Big)^p + \alpha^{1/p}\ell_0\Big(s_1 + \frac{1-\alpha^{(k+1)/p}}{1-\alpha^{1/p}}(s_2 - s_1)\Big)^p - \alpha^{1/p} \leq 0$$

or

$$\frac{\ell}{1+p}\alpha^k(s_2 - s_1)^p + \alpha^{1/p}\ell_0\Big(\eta + \frac{s_2 - s_1}{1-\alpha^{1/p}}\Big)^p - \alpha^{1/p} \leq 0$$

or since $\alpha \in (0, 1)$

$$\frac{\ell}{1+p}\alpha(s_2 - s_1)^p + \alpha^{1/p}\ell_0\Big(\eta + \frac{s_2 - s_1}{1-\alpha^{1/p}}\Big)^p \leq \alpha^{1/p}$$

which is true by (3.19). This completes the induction for (3.22) and (3.23). Hence, sequence $\{s_n\}$ is increasing, bounded from above by $s^{\star\star}$ and as such it converges to its unique least upper bound $s^\star \in [s_2, s^{\star\star}]$. $\square$

We extend Lemma 3.1 and Lemma 3.2 through the following two useful additions: Lemma 3.3 and Lemma 3.4, respectively

Lemma 3.3. *Let $\ell_0 > 0$, $\ell > 0$, $\eta > 0$ and $p \in (0, 1]$ be given parameters. Let $N = 0, 1, 2, \ldots$ be fixed. We define function g_0^N on $[0, 1)$ by*

$$g_0^N(t) = \frac{\ell}{1+p}(t_{N+1} - t_N)^p(t-1) + \ell_0 t^{1/p}\Big[(1 + t^{1/p})^p - 1\Big](t_{N+1} - t_N)^p.$$

Let α_N denotes the minimal zero of $g_0^N(t)$ in the interval $(0, 1)$. Suppose that

$$t_1 < t_2 < t_3 < \cdots < t_N < t_{N+1} < \ell_0^{-1/p} \tag{3.27}$$

and

$$0 < \frac{\ell(t_{N+1} - t_N)^p}{(1+p)(1 - \ell_0 t_{N+1}^p)} \leq \alpha_N^{1/p} \leq 1 - \ell_0^{1/p}(t_{N+1} - t_N).$$

Then, the sequence $\{t_n\}$ - defined by (3.2) - is increasing, bounded from above by $t_N^{\star\star} = (t_{N+1} - t_N)/(1 - \alpha^{1/p})$ and converges to its unique least upper bound $t_N^\star \in [t_{N+1}, t_N^{\star\star}]$. Moreover, for each $n = 1, 2, \ldots$, the following estimate

$$0 < t_{N+n} - t_{N+n-1} \leq \alpha^{1/p}(t_{N+n-1} - t_{N+n-2}) \quad \text{for each} \quad n = 0, 1, 2, \ldots$$

holds.

Lemma 3.4. *Let $\ell_0 > 0$, $\ell > 0$, $\eta > 0$ and $p \in (0, 1]$ be given parameters. Let $N = 0, 1, 2, \ldots$ be fixed. Suppose that*

$$s_1 < s_2 < s_3 < \cdots < s_N < s_{N+1} < \ell_0^{-1/p},$$
$$\ell_0(s_{N+1} - s_N)^p < 1$$

and (3.19) holds with $s_{N+1} - s_N$ replacing η. Then, scalar sequence $\{s_n\}$ - defined by (3.20) - is increasing, bounded from above by

$$s_N^{\star\star} = \Big[1 + \frac{\ell_0(s_{N+1} - s_N)^p}{(1 - \alpha^{1/p})(1+p)(1 - \ell_0 s_{N+1}^p)}\Big](s_{N+1} - s_N)$$

and converges to its unique least upper bound $s_N^\star \in [s_{N+1}, s_N^{\star\star}]$. Moreover, for each $n = 1, 2, \ldots$, the following estimate

$$0 < s_{N+n+2} - s_{N+n+1} \leq \frac{\alpha^{n/p}\ell_0(s_{N+1} - s_N)^{1+p}}{(1+p)(1 - \ell_0 s_{N+1}^p)} \quad \text{for each} \quad n = 0, 1, 2, \ldots$$

holds. Notice that, for $N = 0$, the Lemma 3.3 and the Lemma 3.4 reduce to the Lemma 3.1 and the Lemma 3.2, respectively.

We state the main semilocal convergence theorem for Newton's method. The proof of which is obtained from [4, Theorem 3.3] by replacing the hypotheses of Lemma 3.1 in [4] – involving the convergence of $\{t_n\}$ – with the new Lemma 3.3 or Lemma 3.4.

Theorem 3.5. *Let $\mathcal{F} : \mathbf{D} \subseteq \mathbf{X} \longrightarrow \mathbf{Y}$ be Fréchet-differentiable. Suppose there exist a point $x_0 \in \mathbf{D}$ and parameters $\eta > 0$, $\ell_0 > 0$, $\ell > 0$, $R > 0$, $p \in (0, 1]$, such that conditions (2.1), (2.2), hypotheses of Lemma 3.1 or Lemma 3.3 hold and $\overline{U}(x_0, t^\star) \subseteq U(x_0, R)$. Then, $\{x_n\}$ $(n \geq 0)$ generated by Newton's method is well* *defined, remains in $\overline{U}(x_0, t^\star)$ for all $n \geq 0$ and converges to a unique solution $x^\star \in \overline{U}(x_0, t^\star)$ of equation $\mathcal{F}(x) = 0$. Moreover, the following estimates*

$$\|x_{n+2} - x_{n+1}\| \leq \frac{\ell\,\|x_{n+1} - x_n\|^{1+p}}{(1+p)\,(1 - \ell_0\,\|x_{n+1} - x_0\|^p)} \leq t_{n+2} - t_{n+1} \tag{3.28}$$

and

$$\|x_n - x^\star\| \leq t^\star - t_n, \tag{3.29}$$

hold for each $n = 0, 1, \ldots$, where, iteration $\{t_n\}$ and $t^\star$ are given in Lemma 3.1. Furthermore, if there exists $R \geq t^\star$ such that

$$R_0 \leq R \quad \text{and} \quad \ell_0 \int_0^1 (\theta\, t^\star + (1-\theta)\, R)^p\, d\theta < 1,$$

then, the solution $x^\star$ is unique in $\overline{U}(x_0, R_0)$. If hypotheses of Lemma 3.2 (or Lemma 3.4) hold, instead of Lemma 3.1 (or Lemma 3.3), then the preceding conclusions hold with $s^\star$, $\{s_n\}$ (or $s^\star_N$, $\{s_N\}$) replacing $t^\star$, $\{t_N\}$ (or $t^\star_N$, $\{t_N\}$), respectively.

Remark 3.6. Until now we presented convergence criteria, that can be weaker than the ones reported in Section 2 (especially criterion (2.23)), and a majorizing sequence $\{s_n\}$ which is finer than the sequence $\{t_n\}$ [11]. Under the criterion (2.23), we notice that

$$s_n \leq t_n, \tag{3.30}$$

$$s_{n+1} - s_n \leq t_{n+1} - t_n, \tag{3.31}$$

$$s^\star \leq t^\star. \tag{3.32}$$

If $\ell_0 < \ell$ then for $n \geq 2$ we observe that strict inequality applies in (3.30) and (3.32) (also see the numerical examples).

However – along the lines of Lemma 3.3 and Lemma 3.4 – we can weaken the condition (2.23) in another way too as follows:

Lemma 3.7. *Let $\ell_0 > 0$, $\ell > 0$, $\eta > 0$ and $p \in (0,1)$. Let $N = 0, 1, 2, \ldots$ be fixed. Suppose that*

$$t_1 < t_2 < t_3 < \cdots < t_N < t_{N+1} < \ell_0^{1/p}$$

and

$$(t_{N+1} - t_N)^p \leq h_0(c_0)$$

where function h_0 is defined in (2.21). Then, sequence $\{t_n\}$ converges increasingly to $t^\star$.

Theorem 3.8. *Let $\mathcal{F} : \mathbf{D} \subseteq \mathbf{X} \to \mathbf{Y}$ be Fréchet differentiable. Suppose there exist a point $x_0 \in \mathbf{D}$ and parameters $\eta > 0$, $\ell_0 > 0$, $\ell > 0$, $p \in (0,1)$ such that conditions (2.2), (2.3), (2.4), hypotheses of Lemma 3.7 and $\overline{U}(x_0, t^\star) \subseteq \mathbb{U}(x_0, R)$. Then, the conclusions of Theorem 3.5 hold.*

Remark 3.9. If $N = 0$, the results in Lemma 3.7 and Theorem 3.8 reduce to the corresponding onces in [11]. Moreover, if $N \geq 1$, then two preceding results constitute an improvement of the results in [11]. Clearly, sequence $\{s_n\}$ can also replace $\{t_n\}$ in Lemma 3.7 and Theorem 3.8. In practice, we shall test existing criteria and use the onces that apply and also provide the best available error estimates and uniqueness result.

4. Numerical work

Example 4.1. Let $\mathbf{X} = \mathbf{Y} = \mathbb{R}^{n-1}$ for natural integer $n \geq 2$. $\mathbf{X}$ and $\mathbf{Y}$ are equipped with the max-norm $\|\mathbf{x}\| = \max_{1\leq i\leq n-1} \|x_i\|$. The corresponding matrix norm is

$$\|A\| = \max_{1\leq i\leq n-1} \sum_{j=1}^{j=n-1} |a_{ij}|$$

for $A = (a_{ij})_{1\leq i,j\leq n-1}$. On the interval $[0,1]$, we consider the following two point boundary value problem

$$\begin{cases} v'' + v^{3/2} = 0 \\ v(0) = v(1) = 0. \end{cases} \tag{4.1}$$

[**?**, see]]ah-WSPC,rokne. To discretize the above equation, we divide the interval $[0,1]$ into n equal parts with length of each part: $h = 1/n$ and coordinate of each point: $x_i = i\,h$ with $i = 0, 1, 2, \ldots, n$. A second-order finite difference discretization of equation (4.1) results in the following set of nonlinear equations

$$\mathcal{F}(\mathbf{v}) := \begin{cases} v_{i-1} + h^2\, v_i^{3/2} - 2v_i + v_{i+1} = 0 \\ \text{for} \quad i = 1, 2, \ldots, (n-1) \quad \text{and from (4.1)} \quad v_0 = v_n = 0 \end{cases} \tag{4.2}$$

where $\mathbf{v} = [v_1, v_2, \ldots, v_{(n-1)}]^{\mathrm{T}}$ For the above system-of-nonlinear-equations, we provide the Fréchet derivative

$$\mathcal{F}'(\mathbf{v}) = \begin{bmatrix} \frac{3v_1^{1/2}}{2n^2} - 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \frac{3v_2^{1/2}}{2n^2} - 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \frac{3v_3^{1/2}}{2n^2} - 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \frac{3v_{(n-1)}^{1/2}}{2n^2} - 2 \end{bmatrix}.$$

Let $n = 101$ and $x_0 = [50, 50, \ldots, 50]^{\mathrm{T}}$. To solve the linear systems (step 1 in the in Algorithm 1), we use MatLab routine "linsolve" which uses LU factorization with partial pivoting. Figure 1 plots our numerical solution. From the above expression and the inequalities (2.1), (2.2), (2.4), we obtain

$$p = \frac{1}{2}, \quad \eta = 9.15311 \times 10^{-5} \quad \text{and} \quad L = L_0 = 5.86207705 \times 10^{-4}.$$

From (2.20) and (2.21), we get

$$c_0 = 1.556421035 \quad \text{and} \quad h_0(c_0) = 883.3185142.$$

For the condition (2.23) we obtain: $0.009567188720 < 810.7294898$. Thus the condition (2.23) holds. For the Lemma 3.1, the condition (3.1) yields: $0 < 0.00008814005478 < 0.7422855001 < 0.9999971581$. Thus Lemma 3.1 is applicable. To verify the inequality (3.4) and to ascertain the properties of the sequence (3.2) we produce the Table 1: From the equation (3.3)

$$t^{\star\star} = 0.0002236156293.$$

From the Table 1 we see that the estimation (3.4) holds. Furthermore we notice that the sequence $\{t_n\}$ is increasing and bounded from above by $t^{\star\star}$.

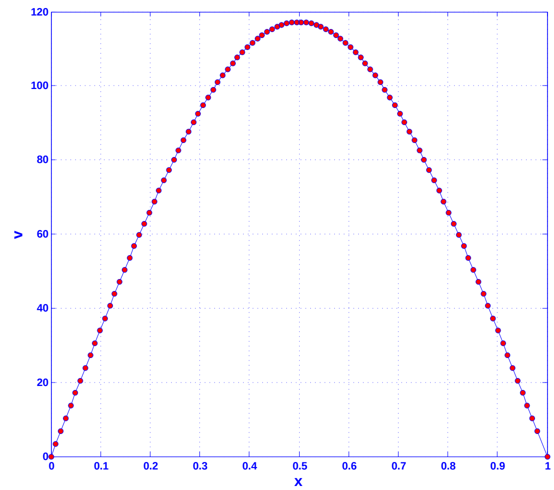

FIGURE 1. Solution of the boundary value problem (4.1).

n	t_n	$t_{n+2} - t_{n+1}$	$\alpha^{1/p}(t_{n+1} - t_n)$	$\alpha^{(n+1)/p}\eta$
0	$0.000000 \times 10^{+00}$	3.421069×10^{-10}	5.037923×10^{-05}	5.037923×10^{-05}
1	9.153110×10^{-05}	2.472017×10^{-18}	1.882976×10^{-10}	2.772901×10^{-05}
2	9.153144×10^{-05}	1.518400×10^{-30}	1.360612×10^{-18}	1.526221×10^{-05}
3	9.153144×10^{-05}	7.309504×10^{-49}	8.357357×10^{-31}	8.400404×10^{-06}
4	9.153144×10^{-05}	2.441409×10^{-76}	4.023192×10^{-49}	4.623630×10^{-06}
5	9.153144×10^{-05}	$1.490286 \times 10^{-117}$	1.343766×10^{-76}	2.544872×10^{-06}
6	9.153144×10^{-05}	$2.247571 \times 10^{-179}$	$8.202620 \times 10^{-118}$	1.400712×10^{-06}
7	9.153144×10^{-05}	$4.162736 \times 10^{-272}$	$1.237076 \times 10^{-179}$	7.709597×10^{-07}
8	9.153144×10^{-05}	$3.318005 \times 10^{-411}$	$2.291193 \times 10^{-272}$	4.243406×10^{-07}
9	9.153144×10^{-05}	$7.466626 \times 10^{-620}$	$1.826249 \times 10^{-411}$	2.335594×10^{-07}

TABLE 1. Numerical solution of (4.1) – after 6 nonlinear iterations – at Gauss-Legendre points.

REFERENCES

1. S. Amat, S. Busquier, Third–order iterative methods under Kantorovich conditions, J. Math. Anal. Appl., 336 (2007), 243–261.
2. J. Appell, E. De Pascale, J.V. Lysenko, P.P. Zabrejko, New results on Newton–Kantorovich approximations with applications to nonlinear integral equations, Numer. Funct. Anal. Optim., 18 (1997), 1–17.
3. I.K. Argyros, Concerning the "terra incognita" between convergence regions of two Newton methods, Nonlinear Analysis, 62 (2005), 179–194.
4. I.K. Argyros, S. Hilout, Weaker conditions for the convergence of Newton's method, J. Complexity, 28 (2012), 364–387.
5. I.K. Argyros, S. Hilout, Convergence of Directional Methods under mild differentiability and applications, Appl. Math. Comput., 217 (2011), 8731–8746.
6. I.K. Argyros, S. Hilout, Computational methods in nonlinear analysis, World Sci. Publishers, New Jersey, 2013.
7. F. Cianciaruso, A further journey in the "terra incognita" of the Newton–Kantorovich method, Nonlinear Funct. Anal. Appl., 15 (2010), 173–183.
8. F. Cianciaruso., E. De Pascale, Newton–Kantorovich aproximations when the derivative is Hölderian: Old and new results, Numer. Funct. Anal. Optim., 24 (2003), 713–723.

9. F. Cianciaruso, E. De Pascale, P.P. Zabrejko, Some remarks on the Newton-Kantorovich approximations, Atti Sem. Mat. Fis. Univ. Modena, 48 (2000), 207–215.
10. E. De Pascale, P.P. Zabrejko, Convergence of the Newton–Kantorovich method under Vertgeim conditions: a new improvement, Z. Anal. Anwendvugen, 17 (1998), 271–280.
11. J.A. Ezquerro, J.M. Gutiérrez, M.A. Hernández, N. Romero, M.J.Rubio, The Newton method: from Newton to Kantorovich. (Spanish), Gac. R. Soc. Mat. Esp., 13 (2010), 53–76.
12. J.A. Ezquerro, M.A. Hernández, On the R-order of convergence of Newton's method under mild differentiability conditions, J. Comput. Appl. Math., 197 (2006), 53–61.
13. L.V. Kantorovich, G.P. Akilov, Functional Analysis, Pergamon Press, Oxford, 1982.
14. J.V. Lysenko, Conditions for the convergence of the Newton–Kantorovich method for nonlinear equations with Hölder linearizations (in Russian), Dokl. Akad. Nauk BSSR, 38 (1994), 20–24.
15. P.D. Proinov, General local convergence theory for a class of iterative processes and its applications to Newton's method, J. Complexity, 25 (2009), 38–62.
16. P.D. Proinov, New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems, J. Complexity, 26 (2010), 3–42.
17. J. Rokne, Newton's method under mild differentiability conditions with error analysis, Numer. Math., 18 (1971/72), 401–412.
18. B.A. Vertgeim, On conditions for the applicability of Newton's method, (in Russian), Dokl. Akad. N., SSSR, 110 (1956), 719–722.
19. B.A. Vertgeim, On some methods for the approximate solution of nonlinear functional equations in Banach spaces, Uspekhi Mat. Nauk, 12 (1957), 166–169 (in Russian); English transl.: Amer. Math. Soc. Transl., 16, (1960), 378–382.
20. P.P. Zabrejko, D.F. Nguen, The majorant method in the theory of Newton-Kantorovich approximations and the Pták error estimates, Numer. Funct. Anal. Optim., 9 (1987), 671–684.
21. A.I. Zinčenko, Some approximate methods of solving equations with non-differentiable operators, (Ukrainian), Dopovidi Akad. Nauk Ukraïn. RSR, (1963), 156–161.