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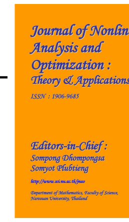

HYBRID FIXED POINT THEORY FOR NONINCREASING MAPPINGS IN PARTIALLY ORDERED METRIC SPACES AND APPLICATIONS

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ABSTRACT. We present some hybrid fixed point theorems for nonincreasing mappings in a partially ordered complete metric space and apply to prove the existence as well as an algorithm for the solutions of initial value problems of nonlinear first order ordinary differential equations. An example is also provided to illustrate the abstract theory developed in this paper.

KEYWORDS : Hybrid fixed point theorem; Nonlinear differential equation; Existence of solutions.
AMS Subject Classification: 34A12, 34A38

1. INTRODUCTION

It is well-known that the hybrid fixed point theorems which are obtained using the mixed arguments from different branches of mathematics are very rich in applications to allied areas of mathematics, particularly to the theory of nonlinear differential and integral equations (see Heikkilä and Lakshmikantham [6], Zeidler [9] and Dhage [2, 3, 5]). Recently, Ran and Reurings [8] initiated the study of hybrid fixed point theorems in partially ordered sets which is further continued in Nieto and Rodriguez-Lopez [7] and proved the hybrid fixed point theorems for the monotone mappings in partially ordered metric spaces using the mixed arguments from algebra, analysis and geometry. Monotone mappings include both nondecreasing and nonincreasing mappings on ordered sets. The monotone nondecreasing mappings are frequently used in nonlinear analysis whereas nincreasing mappings are rare. Very recently, a different approach for nondecreasing mappings in partially ordered sets is established in Dhage [5] which is very much useful in the existence

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theory for nonlinear equations. In this paper we follow the same approach and obtain some hybrid fixed point theorems for noninreasing operators in metric spaces. The following two notions of regularity and monotone mappings are fundamental for the fixed point theory in ordered spaces.

Definition 1.1. A partially ordered metric space $(X, \preceq, d)$ is called **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in X such that $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$.

Definition 1.2. A mapping $\mathcal{T} : X \longrightarrow X$ is called **monotone nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in X$. Similarly, $\mathcal{T}$ is called **monotone nonincreasing** if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in X$. A **monotone** mapping $\mathcal{T}$ is one which is either monotone nondecreasing or monotone nonincreasing on X.

Nieto and Lopez [7] introduced the following definition.

Condition (NL): A partially ordered metric space X with metric d is said to satisfy Condition(NL) if for every convergent sequence $\{x_n\}$ in X to the point x^* whose consecutive terms are comparable then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that every term is is comparable to the limit x^*.

The following hybrid fixed point theorem for nonincreasing mappings is proved in Nieto and Lopez [7].

Theorem 1.3 (Nieto and Rodriguez-Lopez [7]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d in X such that (X, d) is a complete metric space. Let $T : X \longrightarrow X$ be a monotone nonincreasing mapping such that there exists a constant $k \in [0, 1)$ such that*

$$d(\mathcal{T}x, \mathcal{T}y) \leq k\, d(x, y) \tag{1.1}$$

for all elements $x, y \in X$, $x \geq y$. Assume that either $\mathcal{T}$ is continuous or X satisfies Condition (NL). Further if there is an element $x_0 \in X$ satisfying $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point which is further unique if "every pair of elements in X has a lower and an upper bound."

Note that Condition (NL) of Theorem 1.3 is very difficult to verify in actual practice and only continuous case has been applied in Nieto and Lopez [7] to periodic BVP of first order differential equations for proving the existence of a unique solution, wherein the nonlinearity is a nonincrasing function in the unknown variable. In this paper we generalize Theorem 1.3 under a condition which is more general than Condition (NL) for the self-mappings of a partially ordered metric space satisfying a condition of nonlinear contraction which is again more general than (1.1). Our abstract result is applied to a nonlinear first order ordinary differential equations for proving the existence of unique solution under partially Lipschitz condition.

2. Hybrid Fixed Point Theory

We consider the following definition in what follows.

Condition (D): A partially ordered metric space X with metric d is said to satisfy Condition (D) if every sequence $\{x_n\}$ in X whose consecutive terms are comparable has a monotone, i.e. nondecreasing or nonincreasing subsequence.

There do exist sequences in X with Condition (D). For example, if we consider $X = \mathbb{R}$, then the sequence $\{x_n\}$ in $\mathbb{R}$ defined by $x_n = (-1)^{n+1}\dfrac{1}{n}$ has two subsequences, one is nondecreasing another is nonincreasing. Again, the sequence $\{1, -\frac{1}{2}, 3, -\frac{1}{4}, \ldots\}$ satisfies the Condition (D) but not Condition (NL).

Note that Condition (D) is more general than Condition (NL) in the sense that Condition (D) implies Condition (NL), however converse may no be true. Indeed if the Condition (D) holds and if $\{x_n\}$ is any sequence in X converging to x^* whose consecutive terms are comparable, then there is a monotone subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which also converges to x^*. By regularity of X, $x_{n_k} \preceq x^*$ or $x_{n_k} \succeq x^*$ for all $k \in \mathbb{N}$, that is, every term of $\{x_{n_k}\}$ is comparable to the limit x^*.

Let (X, d) be a metric space and let $\mathcal{T} : X \longrightarrow X$ be a mapping. Given an element $x \in X$, we define an orbit $\mathcal{O}(x; \mathcal{T})$ of $\mathcal{T}$ at x by

$$\mathcal{O}(x; \mathcal{T}) = \big\{x, \mathcal{T}x, \mathcal{T}^2x, ..., \mathcal{T}^nx, \ldots\big\}.$$

Then $\mathcal{T}$ is called T-orbitally continuous on X if for any sequence $\big\{x_n\big\} \subseteq \mathcal{O}(x; \mathcal{T})$, we have that $x_n \longrightarrow x^*$ implies $\mathcal{T}x_n \longrightarrow \mathcal{T}x^*$ for each $x \in X$. The metric space X is called $\mathcal{T}$-orbitally complete if every Cauchy sequence $\big\{x_n\big\} \subseteq \mathcal{O}(x; \mathcal{T})$ converses to a point x^* in X. Notice that continuity implies that $\mathcal{T}$-orbitally continuity and completeness implies $\mathcal{T}$-orbitally completeness of a metric space X, but the converse may not be true.

Definition 2.1 (Dhage [5]). A mapping $\mathcal{T} : X \longrightarrow X$ is called **partially continuous** at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on X if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on X, then it is continuous on every chain C contained in X.

We frequently need a fundamental result concerning Cauchy sequence in what follows. For, we need the following definition.

Definition 2.2 (Dhage [4]). A mapping $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called a **dominating function** or, in short, $\mathcal{D}$**-function** if it is an upper semi-continuous and monotonic nondecreasing function satisfying $\psi(0) = 0$.

There do exist $\mathcal{D}$-functions and commonly used $\mathcal{D}$-functions are

$$\begin{aligned}
&\psi(r) = k\,r, \quad \text{for some constant } k > 0,\\
&\psi(r) = \frac{L\,r}{K + r}, \quad \text{for some constants } L > 0, K > 0,\\
&\psi(r) = \tan^{-1} r,\\
&\psi(r) = \log(1 + r),\\
&\psi(r) = e^r - 1.
\end{aligned}$$

The above defined $\mathcal{D}$-functions have been widely used in the existence theory of nonlinear differential and integral equations.

Remark 2.3. If $\phi, \psi\ \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are two $\mathcal{D}$-functions, then i) $\phi + \psi$, ii) $\lambda\,\phi$, $\lambda > 0$, and iii) $\phi \circ \psi$ are also $\mathcal{D}$-functions on $\mathbb{R}_+$. The class of $\mathcal{D}$-functions on $\mathbb{R}_+$ is denoted by $\mathfrak{D}$.

Lemma 2.4 (Dhage [4]). *Let* $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *be a* $\mathcal{D}$*-function satisfying* $\psi(r) < r$ *for* $r > 0$. *Then* $\lim_{n \longrightarrow \infty} \psi^n(t) = 0$ *for each* $t \in \mathbb{R}_+$ *and vice versa.*

Now we are ready to state a key result in terms of $\mathcal{D}$-function characterizing the Cauchy sequences in a metric space X.

Lemma 2.5. *If $\{x_n\}$ is a sequence in a metric space (X, d) satisfying*

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \tag{2.1}$$

for all $n \in \mathbb{N}$, where ψ is a $\mathcal{D}$-function such that $\psi(r) < r$, $r > 0$, then $\{x_n\}$ is Cauchy.

Proof. The proof is well-known and may found in Dhage [5]. So we omit the details. □

Theorem 2.6. *Let $(X, \preceq, d)$ be a partially ordered metric space. Let $\mathcal{T} : X \longrightarrow X$ be a monotone nonincreasing mapping such that there exists a $\mathcal{D}$-function such that*

$$d(\mathcal{T}x, \mathcal{T}^2 x) \leq \psi(d(x, \mathcal{T}x)) \tag{2.2}$$

for all elements $x \in X$ comparable to $\mathcal{T}x$, where $\psi(r) < r$, $r > 0$. Suppose that either X is $\mathcal{T}$-orbitally complete and $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous or $\mathcal{T}$ is partially $\mathcal{T}$-orbitally continuous and X is regular and satisfies Condition (D). Further if there is an element $x_0 \in X$ satisfying $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{T^n x_0\}$ of iterations converges to x^*.

Proof. Define a sequence $\{x_n\}$ of successive iterations of $\mathcal{T}$ at x_0 as

$$x_{n+1} = \mathcal{T}x_n, \; n = 0, 1, \ldots. \tag{2.3}$$

By nonincreasing nature of $\mathcal{T}$, $\{x_n\}$ is a sequence in X whose consecutive terms are comparable. If $x_n = x_{n+1}$ for some $\in \mathbb{N}$, then $u = x_n$ is a fixed point of T. Therefore, we assume that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$. If $x = x_{n-1}$ and $y = x_n$, then by condition (2.2), we obtain

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \tag{2.4}$$

for each $n = 1, 2, \ldots$. Now, an application of Lemma 2.5, $\{x_n\}$ is Cauchy. Since the metric space X is $\mathcal{T}$-orbitally complete, $\{x_n\}$ converges to a unique limit x^*. If $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous, x^* is a fixed point of $\mathcal{T}$ and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^*. Next, suppose that X satisfies Condition (D). By hypothesis, the sequence $\{\mathcal{T}^n x_0\}$ of iterates of $\mathcal{T}$ at x_0 has a monotone subsequence, say $\{x_{n_k}\}$. Then $\{x_{n_k}\}$ also converges to x^* and $x_{n_k} \leq x^*$ for all $k \in \mathbb{N}$. By the partial $\mathcal{T}$-orbitally continuity of $\mathcal{T}$, we obtain

$$\mathcal{T}x^* = \mathcal{T}\left(\lim_{k \longrightarrow \infty} x_k\right) = \lim_{k \longrightarrow \infty} \mathcal{T}(\mathcal{T}^{n_k} x_0) = \lim_{k \longrightarrow \infty} x_{n_k+1} = x^*.$$

This completes the proof.‘ □

Theorem 2.7. *Let $(X, \preceq, d)$ be a partially ordered metric space. Let $\mathcal{T} : X \longrightarrow X$ be a monotone nonincreasing mapping such that there exists a $\mathcal{D}$-function such that*

$$d(\mathcal{T}x, \mathcal{T}y) \leq \psi(d(x, y)) \tag{2.5}$$

for all comparable elements $x, y \in X$, where $\psi(r) < r$, $r > 0$. Suppose that either X is $\mathcal{T}$-orbitally complete and $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous or $\mathcal{T}$ is partially $\mathcal{T}$-orbitally continuous and X is regular and satisfies Condition (D). Further if there is an element $x_0 \in X$ satisfying $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of iterations converges to x^* which is further unique if "every pair of elements in X has a lower and an upper bound."

Proof. Now the inequality (2.5) implies that the mapping $\mathcal{T}$ is partially $\mathcal{T}$-orbitally continuous on X. If we let $y = \mathcal{T}x$ in (2.5), then it reduces to (2.2). Therefore, by Theorem 2.6, $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^*. The uniqueness of fixed point is proved using arguments given in Nieto and Lopez [7]. □

It is known that 'every pair of elements in X has a lower and an upper bound if it is a lattice (cf. Birkhoff [1]). As every monotone nondecreasing or monotone nonincreasing sequence always has a monotone subsequence and the limit of the sequence is the limit of the subsequence, we obtain the following general fixed point theorems for both nondecreasing as well as nonincreasing mappings on partially ordered metric spaces.

Theorem 2.8. *Let $(X, \preceq, d)$ be a partially ordered metric space. Let $\mathcal{T} : X \longrightarrow X$ be a monotone mapping (monotone nonincreasing or monotone nonincreasing) satisfying (2.2). Suppose that either X is $\mathcal{T}$-orbitally complete and $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous or $\mathcal{T}$ is partially $\mathcal{T}$-orbitally continuous and X is regular and satisfies Condition (D). If there exists an $x_0 \in X$ with $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of iterations converges to x^*.*

Theorem 2.9. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $\mathcal{T} : X \longrightarrow X$ be a monotone mapping (monotone nonincreasing or monotone nonincreasing) satisfying (2.5). Suppose that either X is $\mathcal{T}$-orbitally complete and $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous or $\mathcal{T}$ is partially $\mathcal{T}$-orbitally continuous and X is regular and satisfies Condition (D). If there exists an $x_0 \in X$ with $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of iterations of $\mathcal{T}$ at x_0 converges to x^* which is further unique if "every pair of elements in X has a lower and an upper bound."*

Finally, we mention that the claim made in Nieto and Lopez [7] that the continuity of the mapping $\mathcal{T}$ is not required to guarantee the existence of unique fixed point is not true. Actually we need certain kind of continuity, namely, the partial continuity of the mapping $\mathcal{T}$ which follows directly from the condition of partial contraction on X. However, the monotonicity of $\mathcal{T}$ is not essential for the existence of the fixed points, so in this context we replace this monotonicity condition by preservation of comparable elements, that is transformation of comparable elements into comparable elements. A couple of fixed point results in this direction are as follows.

Theorem 2.10. *Let $(X, \preceq, d)$ be a partially ordered metric space. Let $\mathcal{T} : X \longrightarrow X$ be a mapping satisfying (2.2) and maps comparable elements into comparable elements, that is,*

$$x, y \in X, x \preceq y \Rightarrow \begin{cases} \mathcal{T}x \preceq \mathcal{T}y \\ \quad or \\ \mathcal{T}x \succeq \mathcal{T}y. \end{cases}$$

Suppose that X is $\mathcal{T}$-orbitally complete and $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous or $\mathcal{T}$ is partially $\mathcal{T}$-orbitally continuous and X is regular and satisfies Condition (D). If there exists an $x_0 \in X$ with x_0 is comparable to $\mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of iterations converges to x^*.

Theorem 2.11. *Let $(X, \preceq, d)$ be a partially ordered metric space. Let $\mathcal{T} : X \longrightarrow X$ be a mapping satisfying (2.5) and maps comparable elements into comparable* *elements, that is,*

$$x, y \in X, x \preceq y \Rightarrow \begin{cases} \mathcal{T}x \preceq \mathcal{T}y \\ \quad or \\ \mathcal{T}x \succeq \mathcal{T}y. \end{cases}$$

Suppose that either X is $\mathcal{T}$-orbitally complete and $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous or $\mathcal{T}$ is partially $\mathcal{T}$-orbitally continuous and X is regular and satisfies Condition (D). If there exists an $x_0 \in X$ with x_0 is comparable to $\mathcal{T}x_0$, then $\mathcal{T}$ has a unique fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of iterations converges to x^*.

The proofs of Theorems 2.10 and 2.11 are similar to Theorems 2.6 and 2.7 and so we omit the details.

3. Applications to Hybrid Differential Equations

Given a closed and bounded interval $J = [t_0, t_0 + a]$ of the real line $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $a > 0$, consider the initial value problem (in short IVP) of first order ordinary nonlinear hybrid differential equation (in short HDE)

$$\left.\begin{aligned} x'(t) &= f(t, x(t)), \ \ t \in J, \\ x(t_0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{3.1}$$

where $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous function.

By a solution of the HDE (3.1) we mean a function $x \in C^1(J, \mathbb{R})$ that satisfies equation (1.1), where $C^1(J, \mathbb{R})$ is the space of continuously differentiable real-valued functions defined on J.

The HDE (3.1) is well-known in the literature and discussed at length for existence as well as other aspects of the solutions. The HDE (3.1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.2}$$

and

$$x \leq y \iff x(t) \leq y(t) \tag{3.3}$$

for all $t \in J$. Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\leq$ in it. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ is regular as well as a lattice.

We need the following definition in what follows.

Definition 3.1. A function $u \in C^1(J, \mathbb{R})$ is said to be a lower solution of the HDE (1.1) if it satisfies

$$\left.\begin{aligned} u'(t) &\leq f(t, u(t)), \\ u(t_0) &\leq x_0, \end{aligned}\right\} \tag{$*$}$$

for all $t \in J$.

We consider the following set of assumptions in what follows:

(A_1) There exist constants $\lambda > 0$ and $\mu > 0$, with $\lambda \geq \mu$, such that
$$\frac{-\mu(x-y)}{1+(x-y)} \leq [f(t,x) + \lambda x] - [f(t,y) + \lambda y] \leq 0,$$
for all $t \in J$ and $x, y \in \mathbb{R}$, $x \geq y$.

(A_2) The HDE (1.1) has a lower solution $u \in C^1(J, \mathbb{R})$.

Consider the IVP of the HDE

$$\left.\begin{aligned} x'(t) + \lambda x(t) &= \tilde{f}(t, x(t)), \\ x(t_0) &= x_0, \end{aligned}\right\} \tag{3.4}$$

for all $t \in J$, where $\tilde{f}, g : J \times \mathbb{R} \longrightarrow \mathbb{R}$ and

$$\tilde{f}(t, x) = f(t, x) + \lambda x. \tag{3.5}$$

Remark 3.2. Note that the function $\tilde{f}$ is continuous on $J \times \mathbb{R}$, and so the associated superposition Nymetski operator (Fx) is integrable on J. Again, a function $u \in C^1(J, \mathbb{R})$ is a solution of the HDE (3.4) if and only if it is a solution of the HDE (1.1) on J.

Lemma 3.3. *A function $u \in C^1(J, \mathbb{R})$ is a solution of the HDE (3.4) if and only if it is a solution of the nonlinear integral equation,*

$$x(t) = c\, e^{-\lambda t} + e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} \tilde{f}(s, x(s))\, ds \tag{3.6}$$

for all $t \in J$ where c is a real number defined by $c = x_0\, e^{t_0}$.

Theorem 3.4. *Assume that hypotheses (A_1) and (A_2) hold. Then the HDE (1.1) has a unique solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_{n+1}(t) = c\, e^{-\lambda t} + e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} \tilde{f}(s, x_n(s))\, ds \tag{3.7}$$

where $x_0 = u$, converges to x^*.

Proof. Set $E = C(J, \mathbb{R})$ and define two operators $\mathcal{A}$ on E by

$$\mathcal{A}x(t) = c\, e^{-\lambda t} + e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} \tilde{f}(s, x(s))\, ds, \quad t \in J. \tag{3.8}$$

From the continuity of the integral, it follows that $\mathcal{A}$ defines the map $\mathcal{A} : E \longrightarrow E$. Now by Lemma 3.3, the HDE (3.1) is equivalent to the operator equation

$$\mathcal{A}x(t) = x(t), \quad t \in J. \tag{3.9}$$

We shall show that the operator $\mathcal{A}$ satisfies all the conditions of Theorem 2.6.

First we show that $\mathcal{A}$ is monotone nonincreasing on E. Let $x, y \in E$ be such that $x \geq y$. Then by hypothesis (A_1), we obtain

$$\begin{aligned} \mathcal{A}x(t) &= c\, e^{-\lambda t} + e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} \tilde{f}(s, x(s))\, ds \\ &\leq c\, e^{-\lambda t} + e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} \tilde{f}(s, y(s))\, ds \\ &= \mathcal{A}y(t), \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{A}$ is nonincreasing operator on E into E.

Next, let $x, y \in E$ be such that $x \geq y$. Then,

$$\begin{aligned} |\mathcal{A}x(t) - \mathcal{A}y(t)| &= \left| e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} [\tilde{f}(s, x(s)) - \tilde{f}(s, y(s))]\, ds \right| \\ &\leq e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} \frac{\mu(x(s) - y(s))}{1 + (x(s) - y(s))}\, ds \\ &\leq e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} \lambda \frac{|x(s) - y(s)|}{1 + |x(s) - y(s)|}\, ds \end{aligned}$$

$$\begin{aligned}
&\leq e^{-\lambda t}\int_{t_0}^{t}\frac{d}{ds}e^{\lambda s}\frac{\|x-y\|}{1+\|x-y\|}\,ds\\
&\leq \left[1-e^{-\lambda(t-t_0)}\right]\frac{\|x-y\|}{1+\|x-y\|}\\
&\leq \frac{\|x-y\|}{1+\|x-y\|},
\end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain

$$\|\mathcal{A}x-\mathcal{A}y\| \leq \psi(\|x-y\|),$$

for all $x, y \in E$ with $x \geq y$, where ψ is a $\mathcal{D}$-function defined by $\psi(r) = \dfrac{r}{1+r} < r$, $r > 0$. Hence $\mathcal{A}$ satisfies the contraction condition (2.5) on E which further implies that $\mathcal{A}$ is a partially continuous and consequently partially $\mathcal{T}$-orbitally continuous on E.

Next, we show that u satisfies the operator inequality $u \leq \mathcal{A}u$. By hypothesis (A_2), the HDE (1.1) has a lower solution u. Then we have

$$\left.\begin{aligned} u'(t) &\leq f(t,u(t)),\\ u(t_0) &\leq x_0, \end{aligned}\right\} \tag{3.10}$$

for all $t \in J$. Adding $\lambda u(t)$ on both sides of the first inequality in (3.10), we obtain

$$u'(t)+\lambda u(t) \leq f(t,u(t))+\lambda u(t), \quad t \in J. \tag{3.11}$$

Again, multiplying the above inequality (3.11) by $e^{\lambda t}$,

$$\left(e^{\lambda t}u(t)\right)' \leq e^{\lambda t}\tilde{f}(t,u(t)). \tag{3.12}$$

A direct integration of (3.12) from t_0 to t yields

$$u(t) \leq c\,e^{-\lambda t}+e^{-\lambda t}\int_{t_0}^{t}e^{\lambda s}\tilde{f}(s,u(s))\,ds \tag{3.13}$$

for all $t \in J$. From definition of the operator $\mathcal{A}$ it follows that $u(t) \leq \mathcal{A}u(t)$ for all $t \in J$. Hence $u \leq \mathcal{A}u$. Thus $\mathcal{A}$ satisfies all the conditions of Theorem 2.7 and we apply it to conclude that the operator equation $\mathcal{A}x = x$ has a solution. Consequently the integral equation and the HDE (1.1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (3.7) converges to x^*. This completes the proof. □

Remark 3.5. The conclusion of Theorem 3.4 also remains true if we replace the hypothesis (A_1) with the following one:

(A_1') There exist a continuous and nondecreasing function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and the constants $\lambda > 0$ and $\mu > 0$, with $\lambda \geq \mu$, such that

$$-\mu\phi(x-y) \leq [f(t,x)+\lambda x]-[f(t,y)+\lambda y] \leq 0,$$

for all $t \in J$ and $x, y \in \mathbb{R}$, $x \geq y$, where $\phi(r) < r$, $r > 0$.

Finally, we give a numerical example to show the realization of the abstract theory in this section.

Example 3.6. Given a closed and bounded interval $J = [0,1]$, consider the IVP of HDE,

$$\left.\begin{aligned} x'(t) &= -\tan^{-1}x(t)-x(t),\\ x(0) &= 1 \in \mathbb{R}, \end{aligned}\right\} \tag{3.14}$$

for all $t \in J$.

Here, $f(t,x) = -\tan^{-1} x - x$. Clearly, the functions f is continuous on $J \times \mathbb{R}$. The function f satisfies the hypothesis (A_1) with $\lambda = 1 = \mu$. To see this, we have

$$0 \leq \tan^{-1} x - \tan^{-1} y \leq \frac{1}{1+\xi^2}(x-y)$$

for all $x, y \in \mathbb{R}$, $x \geq y$, where $x > \xi > y$. Therefore, $\lambda = 1 = \mu$, and $\psi(r) = \dfrac{r}{1+\xi^2}$, $0 < \xi < r$, so the hypothesis (A_1') is satisfied. Finally, the HDE (3.15) has a lower solution $u(t) = -2$ defined on J and so (A_2) is held. Thus all the hypotheses of Theorem 3.4 are satisfied. Hence we apply Theorem 3.4 and conclude that the HDE (3.15) has a solution x^* defined on J and the sequence $\{x_n\}$ defined by

$$x_{n+1}(t) = e^{-t} + e^{-t} \int_0^t e^s \tan^{-1} x_n(s)\, ds, \quad t \in J, \tag{3.15}$$

where $x_0 = -2$, converges to x^*.

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