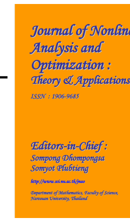
**J**ournal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 5, No. 2, (2014), 61-70
**ISSN : 1906-9685**
**http://www.math.sci.nu.ac.th**




# AUXILIARY PRINCIPLE TECHNIQUE AND ALGORITHM ASPECT FOR MIXED EQUILIBRIUM PROBLEMS

SUHEL AHMAD KHAN[1], FARHAT SUHEL[2,*] AND S. K. SRIVASTAVA[3]

[1]Department of Mathematics BITS-Pilani, Dubai Campus, Dubai-345055, UAE

[2]Department of Mathematics and Statistics DDU Gorakhpur University, Gorakhpur-273009, India

[3]Department of Mathematics and Statistics DDU Gorakhpur University, Gorakhpur-273009, India

**ABSTRACT.** In this paper, we consider a mixed equilibrium problem in real Hilbert space. By using the auxiliary principle technique, some new iterative algorithms for solving mixed equilibrium problems are suggested and analyzed. Further, we prove that the sequences generated by iterative algorithms converge weakly to a solution of mixed equilibrium problem.

**KEYWORDS** : Mixed equilibrium problems, nonexpansive mappings, mixed monotone mapping, regularized operator.
**AMS Subject Classification**: 47H06, 47H09, 65J15, 49J40, 90C33.

## 1. INTRODUCTION

Equilibrium problem theory is an important and interesting branch of applicable mathematics with a wide range of applications in pure and applied sciences. This theory has become a rich source of inspiration and motivation for the study of a large number of problems arising in economics, optimization, operation research in a general and unified way. There is a substantial number of papers on existence results for solving equilibrium problems based on different-relaxed monotonicity notions and various compactness assumptions. In 2002, Moudafi [9] considered a class of mixed equilibrium problems which includes variational inequalities as well as complementarity problems, convex optimization, saddle point problems, problems of finding a zero of a maximal monotone operator, and Nash equilibria problems as special cases. He studied sensitivity analysis and developed some iterative methods for mixed equilibrium problems. It is well-known that there are many numerical methods including projection methods, resolvent operator technique, Wiener-Hopf equations, extragradient and descent methods for

* *Corresponding author.*
*Email address* : khan.math@gmail.com, farhat.math@gmail.com and sudhirpr66@rediffmail.com.
*Article history* : Received February 17, 2014, Accepted September 08, 2014.

solving various variational inequality problems but there are no such methods for solving various equilibrium problems, since it is impossible to find the projection. To overcome this drawback, one uses usually the auxiliary principle technique. This technique deals with finding a suitable auxiliary problem and prove that the solution of an auxiliary problem is the solution of original problem by using fixed-point approach. Recently, Noor [11-13] and Ding [5] have used the auxiliary principle technique to suggest some iterative algorithms for solving generalized mixed variational inequality problems.

Inspired and motivated by the recent research work by [4,6,9,11,14-15], in this paper we study a new class of mixed equilibrium problems and by using the auxiliary principle, we define a class of resolvent mappings. Further, by using fixed point and resolvent methods, we give some iterative algorithms for solving mixed equilibrium problems and prove that the sequences generated by iterative algorithms converge weakly to the solution of mixed equilibrium problems. The auxiliary principle techniques and iterative methods presented in this paper generalize and improve the methods given in [11] for variational inequality problems and given in [9] for mixed equilibrium problems.

## 2. PROBLEM FORMULATION AND BASIC DEFINITIONS

Let $H$ be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let $K \subset H$ be nonempty, closed, convex set; $T, A : K \longrightarrow K$ be nonlinear mappings, and $N : K \times K \longrightarrow K$ be a nonlinear mapping. If $F : K \times K \longrightarrow \mathbb{R}$ and $\phi : H \times H \longrightarrow \mathbb{R}$ are nonlinear bi-mappings, then we consider the following *mixed equilibrium problem* (for short, MEP): Find $x \in K$ such that

$$F(x,y) + \langle N(Tx,Ax), y-x \rangle + \phi(x,y) - \phi(x,x) \geq 0, \ \forall y \in K. \tag{2.1}$$

This problem include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems and equilibrium problems as special cases (see for example, [2]).

*Some special cases:*

(I) If $N(Tx, Ax) = B(x)$, where $B : K \longrightarrow K$, then MEP (2.1) reduces to the mixed equilibrium problem of finding $x \in K$ such that

$$F(x,y) + \langle Bx, y-x \rangle + \phi(x,y) - \phi(x,x) \geq 0, \ \forall y \in K. \tag{2.2}$$

which has been studied in [6].

(II) If $N(Tx, Ax) = B(x)$, $\phi(x, y) = 0 \ \forall x, y \in K$, where $B : K \longrightarrow K$, then MEP (2.1) reduces to the mixed equilibrium problem of finding $x \in K$ such that

$$F(x,y) + \langle Bx, y-x \rangle \geq 0, \ \forall y \in K, \tag{2.3}$$

which has been studied in [9].

(III) If $N(Tx, Ax) = B(x)$, $F(x, y) = 0$ and $\phi(x, y) = \psi(y)$, $\forall x, y \in K$, where $\psi : K \longrightarrow \mathbb{R}$, then MEP (2.1) reduces to the variational inequality problem of finding $x \in K$ such that

$$\langle Bx, y-x \rangle + \psi(y) - \psi(x) \geq 0, \ \forall y \in K. \tag{2.4}$$

This problem has been studied in [11].

(IV) If $N(Tx, Ax) = 0, \ \forall x \in K$, then MEP (2.1) reduces to the equilibrium problem of finding $x \in K$ such that

$$F(x, y) + \phi(x, y) - \phi(x, x) \geq 0, \ \forall y \in K. \tag{2.5}$$

This problem has been studied in [13].

(V) If, in (IV), $\phi(x, y) = 0, \ \forall x, y \in K$, then (2.5) reduces to the equilibrium problem of finding $x \in K$ such that

$$F(x, y) \geq 0, \ \forall y \in K. \tag{2.6}$$

This problem has been studied in [2].

The following definitions and theorem will be needed in the sequel.

**Definition 2.1.** Let $N : K \times K \longrightarrow K$ be a nonlinear mapping. Then $N$ is said to be:

(a) *mixed monotone* w.r.t. $T$ and $A$, if

$$\langle N(Tx, Ax) - N(Ty, Ay), x - y \rangle \geq 0, \ \forall x, y \in K;$$

(b) *mixed pseudomonotone* w.r.t. $T$ and $A$, if

$$\langle N(Tx, Ax), y - x \rangle \geq 0 \text{ implies } \langle N(Ty, Ay), y - x \rangle \geq 0, \ \forall x, y \in K;$$

(c) $\theta$- *mixed pseudomonotone* w.r.t. $T$ and $A$, where $\theta$ is a real-valued multivariate function, if

$$\langle N(Tx, Ax), y - x \rangle + \theta(x, y) \geq 0 \text{ implies } \langle N(Ty, Ay), y - x \rangle + \theta(x, y) \geq 0, \ \forall x, y \in K;$$

(d) *mixed strongly monotone* w.r.t. $T$ and $A$, if

$$\langle N(Tx, Ax) - N(Ty, Ay), x - y \rangle \geq \|x - y\|^2, \ \forall x, y \in K;$$

(e) *inverse mixed strongly monotone* w.r.t. $T$ and $A$, if there exists a constant $\alpha > 0$ such that

$$\langle N(Tx, Ax) - N(Ty, Ay), x - y \rangle \geq \alpha \|N(Tx, Ax) - N(Ty, Ay)\|^2, \ \forall x, y \in K;$$

(f) *firmly nonexpansive* if it is inverse mixed strongly monotone with $\alpha = 1$;

(g) $\delta$-*mixed pseudo contactive* w.r.t. $T$ and $A$, if

$$\langle N(Tx, Ax) - N(Ty, Ay), x - y \rangle \geq \delta \|x - y\|^2;$$

(h) *k-Lipschitz continuous* w.r.t. $T$ and $A$, if there exists a constant $k > 0$ such that

$$\|N(Tx, Ax) - N(Ty, Ay)\| \leq k \|x - y\|, \ \forall x, y \in K;$$

(i) *nonexpansive* w.r.t. $T$ and $A$, if it is Lipschitz continuos with $k = 1$.

**Definition 2.2[1].** A bifunction $\phi : H \times H \longrightarrow \mathbb{R}$ is said to be skew-symmetric if

$$\phi(x, x) - \phi(x, y) - \phi(y, x) + \phi(y, y) \geq 0, \ \forall x, y \in H.$$

The skew-symmetric bifunctions have the properties which can be considered an analog of monotonicity of gradient and nonnegativity of second derivative for the convex function.

**Definition 2.3.** Let $K$ be a nonempty subset of a Hilbert space $H$ and let $\{x_n\}$ be a sequence in $H$. Then $\{x_n\}$ is Fejer monotone with respect to $K$ if

$$\|x_{n+1} - x\| \leq \|x_n - x\|, \ \forall x \in K.$$

**Definition 2.4.** Let $F : K \times K \longrightarrow \mathbb{R}$ be a real-valued function. Then $F$ is said to be:

(a) *monotone* if $F(x, y) + F(y, x) \leq 0$, for each $x, y \in K$;
(b) *strictly monotone* if $F(x, y) + F(y, x) < 0$, for each $x, y \in K$, with $x \neq y$;
(c) *upper-hemicontinuous*, if for all $x, y, z \in K$, $\lim_{t \longrightarrow 0^+} \sup F(tz + (1-t)x,\ y) \leq F(x, y)$.

The following Theorem is a special case of Theorem 3.9.3 of Chang [3].

**Theorem 2.1.** If the following conditions hold true for $F : K \times K \longrightarrow \mathbb{R}$:

(i) $F$ is monotone and upper-hemicontinuous;
(ii) $F(x, .)$ is convex and lower semicontinuous for each $x \in K$;
(iii) there exists a compact subset $B$ of $H$ and there exists $y_0 \in B \cap K$ such that $F(x, y_0) < 0$ for each $x \in K \backslash B$,

then the set of solutions to the following equilibrium problem of finding $x \in K$ such that $F(x, y) \geq 0,\ \forall y \in K$, is nonempty convex and compact.

Moreover, if $F$ is strictly monotone then the solution of equilibrium problem is unique.

**Lemma 2.1.** MEP (2.1) has a solution $x$ if and only if $x$ satisfies the equation

$$x = J_r^{F,\phi}(x - rN(Tx, Ax)),\ \text{for r} > 0. \tag{2.7}$$

We now define the residue vector $R(x)$ by the relation

$$R(x) = x - J_r^{F,\phi}[x - rN(Tx, Ax)]. \tag{2.8}$$

Invoking Lemma 2.1, one can observe that $x \in K$ is a solution of MEP (2.1) if and only if $x \in K$ is a zero of the equation

$$R(x) = 0. \tag{2.9}$$

## 3. AUXILIARY PROBLEMS AND ITERATIVE ALGORITHMS

We consider the following auxiliary problem (in short, AP) for MEP(2.1): For $r > 0$ and for each fixed $x \in H$, find $z \in K$ such that

$$F(z, y) + \phi(z, y) - \phi(z, z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in K. \tag{3.1}$$

We remarked that when $z = x$ then the solution sets of problem (2.5) and AP(3.1) are the same.

The following lemma which gives the existence and uniqueness of solution of AP(3.1) is a special case of Lemma 3.1 due to Ding [5].

**Lemma 3.1.** Let $K \subset H$ be a nonempty closed and convex subset of a real Hilbert space $H$. Let $F : K \times K \longrightarrow \mathbb{R}$ and $\phi : H \times H \longrightarrow \mathbb{R}$ be nonlinear bifunctions and let $r > 0$. Suppose that the following conditions are satisfied:

(i) $F$ satisfies conditions (i)-(ii) in Theorem 2.1;
(ii) $\phi$ is skew-symmetric, convex in second argument and continuous;

(iii) For each fixed $x \in H$, there exists a compact subset $D_x$ of $H$ and $y_0 \in K \cap D_x$ such that

$$F(z,y_0)+\phi(z,y_0)-\phi(z,z)+\frac{1}{r}\langle y_0-z,z-x\rangle<0,$$

for each $z \in K\backslash D_x$. Then for each fixed $x \in H$, AP(3.1) has a unique solution $z \in K$.

**Lemma 3.2.** It follows from Lemma 3.1 that for $r > 0$ and for each $x \in H$, we can write the unique solution of AP(3.1) as $z = J_r^{F,\phi}(x) \in K$. Then for all $y \in K$, we have

$$F(J_r^{F,\phi}(x),y)+\phi(J_r^{F,\phi}(x),y)-\phi(J_r^{F,\phi}(x),J_r^{F,\phi}(x))+\frac{1}{r}\langle J_r^{F,\phi}(x)-x,y-J_r^{F,\phi}(x)\rangle\geq 0. \tag{3.2}$$

Hence $x = J_r^{F,\phi} : H \longrightarrow K$ is well defined and single-valued mapping, which is called the resolvent mapping for MEP (2.1). We observe that $x = J_r^{F,\phi}(x)$ if and only if $x$ is a solution of problem (2.5). Further, Lemma 3.1 gives the strict proof of the assumption taken in [13] for the existence of solution of AP(3.1).

Throughout the rest of paper unless otherwise stated, we assume that the bifunctions $F, \phi$ satisfy all conditions of Lemma 3.1.

**Lemma 3.3.** The mapping $J_r^{F,\phi} : H \longrightarrow K$ is firmly nonexpansive.

**Proof.** Let us denote $u := J_r^{F,\phi}(x)$ and $v := J_r^{F,\phi}(y)$ for each $x, y \in H$. By Lemma 3.1 and Lemma 3.2 we have for each $x, y \in H$,

$$F(u,v)+\phi(u,v)-\phi(u,u)+\frac{1}{r}\langle v-u,u-x\rangle\geq 0,$$

$$F(v,u)+\phi(v,u)-\phi(v,v)+\frac{1}{r}\langle u-v,v-y\rangle\geq 0.$$

Adding above inequalities, we have

$$F(u,v)+F(v,u)-[\phi(u,u)-\phi(u,v)-\phi(v,u)+\phi(v,v)]+\frac{1}{r}\langle x-y,u-v\rangle\geq\frac{1}{r}\langle u-v,u-v\rangle.$$

Since $F$ is monotone and $\phi$ is skew-symmetric, above inequality reduces to

$$\langle u-v,x-y\rangle\geq\|u-v\|^2$$

because $r > 0$. This completes the proof.

**Remark 3.1.** Lemmas 3.1, 3.2 and Lemma 3.3 generalize Lemma 2.3 due to Peng and Yao [14].

The fixed point formulation given in Lemma 2.1 for MEP (2.1) is very useful from the numerical point of views. This fixed point formulation enables us to suggest and analyze the following iterative algorithm.

**Algorithm 3.1.** For a given $x_0 \in K$, compute the approximate solution $x_{n+1}$, by the iterative scheme

$$x_{n+1}=J_r^{F,\phi}[x_n-rN(Tx_n,Ax_n)],\ n=0,1,2,...$$

Rewrite equation (2.7) in the form

$$x=J_r^{F,\phi}\big[x-rN(TJ_r^{F,\phi}[x-rN(Tx,Ax)],AJ_r^{F,\phi}[x-rN(Tx,Ax)])\big]$$

by replacing the solution. This fixed point formulation allows us to suggest the following extragradient method.

**Algorithm 3.2.** For a given $x_0 \in K$, compute $x_{n+1}$, by the iterative scheme

$$x_{n+1} = J_r^{F,\phi}\big[x_n - rN(TJ_r^{F,\phi}[x_n - rN(Tx_n, Ax_n)], AJ_r^{F,\phi}[x_n - rN(Tx_n, Ax_n)])\big],$$

where $n = 0, 1, 2...$

If $F(x,y) = \delta_k(y) - \delta_k(x)$, and $\phi(x,y) = 0$ for all $x, y \in K$, then $J_r^{F,\phi} = P_k$, the projection of $H$ onto $K$ and have Algorithm 3.2 reduces the extragradient method of Korpelvich [7].

Now define the residue vector $R(x)$ by the relation

$$R(x) = x - J_r^{F,\phi}\big[x - rN(TJ_r^{F,\phi}[x - rN(Tx, Ax)], AJ_r^{F,\phi}[x - rN(Tx, Ax)])\big].$$

We can easily observe that $x \in K$ is a solution of MEP (2.1) if and only if $x \in K$ is a zero of the equation

$$R(x) = 0.$$

For a constant $\gamma \in (0, 2)$, equation (2.9) can be written as

$$x + rN(Tx, Ax) = x + rN(Tx, Ax) - \gamma R(x).$$

This formulation is used to suggest a new implicit method for solving MEP (2.1).

**Algorithm 3.3.** For a given $x_0 \in K$, compute $x_{n+1}$ by the iterative scheme

$$x_{n+1} = x_n + rN(Tx_n, Ax_n) - rN(Tx_{n+1}, Ax_{n+1}) - \gamma R(x_n),\ n = 0, 1, 2... \quad (3.3)$$

If $\gamma = 1$, then Algorithm 3.3 reduces to:

**Algorithm 3.4.** For a given $x_0 \in K$, compute $x_{n+1}$ by the iterative scheme

$$x_{n+1} = \big(I + rN(T(.), A(.))\big)^{-1}\big[J_r^{F,\phi}[I + rN(T(.), A(.))] + rN(T(.), A(.))\big]x_n,$$

where $n = 0, 1, 2...$ and $[N(T(.), A(.))x] = N(Tx, Ax),\ \forall x \in K$, which is a variant of the Douglas-Rachford splitting algorithm studied by Lions and Mercier [8], and appears to be new for MEP (2.1).

**Theorem 3.1.** Let $F$ satisfies the conditions of Theorem 2.1, and let $\bar{x} \in K$ be a solution of MEP (2.1). If $N$ is mixed monotone with respect to $T$ and $A$, then

$$\langle x - \bar{x} + r[N(Tx, Ax) - N(T\bar{x}, A\bar{x})], R(x)\rangle \geq \|R(x)\|^2,\ \forall x \in K,$$

where $R(x)$ is defined by equation (2.8).

**Proof.** Let $\bar{x} \in K$ be a solution of MEP (2.1), then

$$F(\bar{x}, y) + \langle N(T\bar{x}, A\bar{x}), y - \bar{x}\rangle + \phi(\bar{x}, y) - \phi(\bar{x}, \bar{x}) \geq 0,\ \forall y \in K. \quad (3.4)$$

Taking $y = x - R(x)$ in (3.4), we have

$$rF(\bar{x}, x - R(x)) + \langle rN(T\bar{x}, A\bar{x}), x - R(x) - \bar{x}\rangle + r\phi(\bar{x}, x - R(x)) - r\phi(\bar{x}, \bar{x}) \geq 0. \quad (3.5)$$

Setting: $y := \bar{x}$, $z = J_r^{F,\phi}(x) := J_r^{F,\phi}(x - rN(Tx, Ax)) = x - R(x)$ and $x := x - rN(Tx, Ax)$ in (3.2), we have

$$\begin{aligned} rF(x - R(x), \bar{x}) + r\phi(x - R(x), \bar{x}) - r\phi(x - R(x), x - R(x)) \\ + \langle x - R(x) - (x - rN(Tx, Ax)), \bar{x} - (x - R(x))\rangle \geq 0. \end{aligned} \quad (3.6)$$

Adding (3.5) and (3.6), we have

$$\begin{aligned} r[F(x - R(x), \bar{x}) + F(\bar{x}, (x - R(x))] + \langle rN(T\bar{x}, A\bar{x}) - rN(Tx, Ax) + R(x), x - R(x) - \bar{x}\rangle \\ - r[\phi(\bar{x}, \bar{x}) - \phi(\bar{x}, x - R(x)) - \phi(x - R(x), \bar{x}) + \phi(x - R(x), x - R(x))] \geq 0. \end{aligned} \quad (3.7)$$

Since $F$ is monotone and $\phi$ is skew-symmetric, equation (3.7) implies that

$$\langle rN(T\bar{x}, A\bar{x}) - rN(Tx, Ax) - R(x), (x - R(x)) - \bar{x}\rangle \geq 0. \tag{3.8}$$

Since $N$ is mixed monotone with respect to $T$ and $A$ from equation (3.8), we have

$$\begin{aligned}&\langle x - \bar{x} - r[N(Tx, Ax) - N(T\bar{x}, A\bar{x})], R(x)\rangle \\ &\quad = \langle R(x), R(x)\rangle + \langle R(x) - r[N(Tx, Ax) - N(T\bar{x}, A\bar{x}), x - \bar{x} - R(x)\rangle \\ &\quad\quad + r\langle N(Tx, Ax) - N(T\bar{x}, A\bar{x}), x - \bar{x}\rangle \geq \|R(x)\|^2.\end{aligned}$$

This completes the proof.

**Theorem 3.2.** Let $\bar{x} \in K$ be the solution of MEP (2.1). If the mapping $N$ is mixed monotone w.r.t. $T$ and $A$ then the iterative sequence $x_n$ generated by Algorithm 3.3 is bounded.

**Proof.** Since $\bar{x}$ is a solution of MEP (2.1) and $x_{n+1}$ satisfies (3.3), then using Theorem 3.1, we have

$$\begin{aligned}&\|x_{n+1} - \bar{x} + r[N(Tx_{n+1}, Ax_{n+1}) - N(T\bar{x}, A\bar{x})]\|^2 \\ &= \|x_n - \bar{x} + r[N(Tx_n, Ax_n) - N(T\bar{x}, A\bar{x})] - \gamma R(x_n)\|^2 \\ &\leq \|x_n - \bar{x} + r[N(Tx_n, Ax_n) - N(T\bar{x}, A\bar{x})]\|^2 - 2\gamma\|R(x_n)\|^2 + \gamma^2\|R(x_n)\|^2 \\ &= \|x_n - \bar{x} + r[N(Tx_n, Ax_n) - N(T\bar{x}, A\bar{x})]\|^2 - \gamma(2-\gamma)\|R(x_n)\|^2. \qquad (3.9)\\ &\leq \|x_n - \bar{x} + r[N(Tx_n, Ax_n) - N(T\bar{x}, A\bar{x})]\|^2 \qquad (3.10)\end{aligned}$$

because $\gamma \in (0, 2)$. Inequality (3.10) which gives the Fejers monotonicity of the sequence $\{(I + rN(T, A))x_n\}$ with respect to the solution set of MEP(2.1) and hence $\{(I + rN(T, A))x_n\}$ is bounded. Further it also follows from (3.10) that the sequence $\{\|(I + rN(T, A))x_n - (I + rN(T, A))\bar{x}\|^2$ is monotonically decreasing and therefore convergent.

Again since $N$ is mixed monotone w.r.t. $T$ and $A$, for any $x, y \in K$, we have

$$\begin{aligned}&\langle (I + rN(T, A))x - (I + rN(T, A))y, x - y\rangle \\ &= \|x - y\|^2 + r\langle N(Tx, Ax) - N(Ty, Ay), x - y\rangle \geq \|x - y\|^2\end{aligned}$$

which implies that the mapping $(I + rN(.,.))$ is 1-strongly monotone. Hence, we have

$$\|(I + rN(T, A))x_n - (I + rN(T, A))\bar{x}\| \geq \|x_n - \bar{x}\|.$$

This implies that the sequence $\{x_n\}$ is bounded.

**Theorem 3.3.** Let $H$ be a finite dimensional space. The approximate solution $x_{n+1}$ obtained from Algorithm 3.3 converges to a solution $\bar{x}$ of MEP (2.1).

**Proof.** Let $\bar{x} \in K$ be the solution of MEP (2.1). From Theorem 3.2, it follows that the sequence $\{x_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \gamma(2-\gamma)\|R(x_n)\|^2 \leq \|x_0 - \bar{x} + r[N(Tx_0, Ax_0) - N(T\bar{x}, A\bar{x})]\|^2,$$

and consequently

$$\lim_{n\longrightarrow\infty} R(x_n) = 0.$$

Let $\hat{x}$ be a limit point of $\{x_n\}$. A subsequence $\{x_{n_i}\}$ of $\{x_n\}$, which converges to $\hat{x}$. Since $R(x)$ is continuous, so

$$R(\hat{x}) = \lim_{i\longrightarrow\infty} R(x_{n_i}) = 0,$$

and hence $\hat{x}$ is the solution of MEP (2.1) and

$$\|x_{n+1} - \hat{x} + r[N(Tx_{n+1}, Ax_{n+1}) - N(T\hat{x}, A\hat{x})]\|^2$$
$$\leq \|x_n - \bar{x} + r[N(Tx_n, Ax_n) - N(T\hat{x}, A\hat{x})]\|^2.$$

It follows that the sequence $\{x_n\}$ has exactly one limit point and $\lim_{n\longrightarrow\infty} x_n = \hat{x} \in K$, satisfies the MEP (2.1).

## 4. RESOLVENT EQUATION TECHNIQUE

Now related to MEP (2.1), we consider the following resolvent equation (in short, RE): Find $z \in H$ such that for $x \in K$,

$$N(Tx, Ax) + A_r^{F,\phi}(z) = 0 \tag{4.1}$$

and

$$x = J_r^{F,\phi}(z), \text{ for r} > 0, \tag{4.2}$$

where $A_r^{F,\phi}$ is a regularized operator and is defined as $A_r^{F,\phi} = \frac{1}{r}(I - J_r^{F,\phi})$, $I$ is the identity operator on $H$.

**Lemma 4.1.** MEP (2.1) has a solution $x$ if and only if RE (4.1)-(4.2) has a solution $z \in H$ where

$$x = J_r^{F,\phi}(z) \tag{4.3}$$

and

$$z = x - rN(Tx, Ax), \text{ for r} > 0. \tag{4.4}$$

Lemma 4.1 shows that MEP (2.1) and RE (4.1)-(4.2) both have the same solution set.

Using the fact that $A_r^F = \frac{1}{r}(I - J_r^{F,\phi})$, RE (4.1)-(4.2) can be written as

$$z - J_r^{F,\phi}(z) + rN\big(TJ_r^{F,\phi}(z), AJ_r^{F,\phi}(z)\big) = 0.$$

For a step size $\gamma$, we can write above equation as

$$x = x - \gamma[z - J_r^{F,\phi}(z) + rN(TJ_r^{F,\phi}(z), AJ_r^{F,\phi}(z))] = 0.$$

This fixed point formulation allows us to suggest the following iterative algorithm for MEP (2.1).

**Algorithm 4.1.** For a given $x_0 \in K$, compute the approximate solution $x_{n+1}$ by the iterative schemes

$$z_n = x_n - rN(Tx_n, Ax_n)$$
$$w_n = z_n - J_r^{F,\phi} z_n + rN(TJ_r^{F,\phi} z_n, AJ_r^{F,\phi} z_n))$$
$$x_{n+1} = x_n - \gamma w_n$$

where $n = 0, 1, 2, ..., r > 0$ and $\gamma > 0$.

**Theorem 4.1.** Let $\bar{x} \in K$ be the solution of MEP (2.1) and let $N$ is $\theta$-mixed pseudomonotone w.r.t. $T$ and $A$, where $\theta(x, y) = F(x, y) + \phi(x, y) - \phi(x, x),\ \forall x, y \in K$ and $\delta$- mixed pseudo contractive. Then

$$\langle x - \bar{x}, R(x) - r[N(Tx, Ax) - N(Tz, Az)]\rangle \geq (1 - r\delta)\|R(x)\|^2;\ \forall x \in K,$$

where $z := x - rN(Tx, Ax)$ and $R(x)$ is defined by (2.8)

**Proof.** Since $N$ is $\theta$-mixed pseudomonotone w.r.t. $T$ and $A$, where $\theta(x,y) = F(x,y)+\phi(x,y)-\phi(x,x),\ \forall x,y\in K$, then for all $z,\bar{x}\in K$,

$$\langle N(T\bar{x},A\bar{x}),z-\bar{x}\rangle+\theta(x,y)\geq 0$$

implies

$$\langle N(Tz,Az),z-\bar{x}\rangle+\theta(x,y)\geq 0$$

$$i.e.,\ F(\bar{x},z)+\langle N(Tz,Az),z-\bar{x}\rangle+\phi(\bar{x},z)-\phi(\bar{x},\bar{x})\geq 0,\ \forall z\in K.$$

Since $F$ is monotone then above inequality implies that

$$-F(z,\bar{x})+\langle N(Tz,Az),z-\bar{x}\rangle+\phi(\bar{x},z)-\phi(\bar{x},\bar{x})\geq 0,\ \forall z\in K.$$

In particular for $z=x-R(x)$, we have

$$\begin{aligned}&-F(x-R(x),\bar{x})+\langle N(T(x-R(x)),A(x-R(x))),(x-R(x))-\bar{x}\rangle\\&+\phi(\bar{x},x-R(x))-\phi(\bar{x},\bar{x})\geq 0\end{aligned}\tag{4.5}$$

Adding (3.6) and (4.5), we have

$$\langle R(x)-rN(Tx,Ax)+rN(T(x-R(x)),A(x-R(x)),(x-R(x))-\bar{x}\rangle\geq 0,$$

where we have used skew-symmetricity of $\phi$. Since $N$ is mixed pseudo contractive w.r.t. $T$ and $A$, above inequality implies

$$\begin{aligned}&\langle R(x)-rN(Tx,Ax)+rN(T(x-R(x)),A(x-R(x)),x-\bar{x}\rangle\\&\quad\geq\langle R(x)-rN(Tx,Ax)+rN(T(x-R(x)),A(x-R(x)),R(x)\rangle\\&\quad\geq\|R(x)\|^2-r\langle N(Tx,Ax)-N(T(x-R(x)),A(x-R(x)),x-(x-R(x))\rangle\\&\quad\geq(1-r\delta)\|R(x)\|^2.\end{aligned}$$

This completes the proof.

**Theorem 4.2.** Let $\bar{x}\in K$ be the solution of MEP (2.1) and let $N$ is $\theta$-mixed pseudomonotone w.r.t. $T$ and $A$, where $\theta(x,y)=F(x,y)+\phi(x,y)-\phi(x,x),\ \forall x,y\in K$ and $\delta$-Lipschitz continuous w.r.t. $T$ and $A$. If $r\delta<1$ and $\gamma\in(0,2)$, then the iterative sequence $\{x_n\}$ generated by Algorithm (4.1) converges weakly to $\bar{x}$.

**Proof.** Let $\bar{x}\in K$ be the solution of MEP (2.1), using Algorthm (4.1), we have

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2&=\|x_n-\bar{x}-\gamma[x_n-rN(Tx_n,Ax_n)-J_r^{F,\phi}[x_n-rN(Tx_n,Ax_n]\\&\qquad+rN(TJ_r^{F,\phi})[x_n-rN(Tx_n,Ax_n),AJ_r^{F,\phi}[x_n-rN(Tx_n,Ax_n]]\|^2\\&=\|x_n-\bar{x}-\gamma[R(x_n)+rN(Tx_n-R(x_n))-rN(Tx_n,Ax_n)]\|^2\\&=\|x_n-\bar{x}\|^2-2\gamma\langle R(x_n)+rN(T(x_n-R(x_n),A(x_n-R(x_n))-rN(Tx_n,Ax_n),x_n-\bar{x}]\rangle\\&\quad+\gamma^2\|R(x_n)|+rN(T(x_n-R(x_n),A(x_n-R(x_n))-rN(Tx_n,Ax_n)\|^2\\&\leq\|x_n-\bar{x}\|^2-2\gamma\langle R(x_n)+rN(T(x_n-R(x_n),A(x_n-R(x_n))-rN(Tx_n,Ax_n),x_n-\bar{x}]\rangle\\&\quad+\gamma^2\{\|R(x_n)\|^2+\|N(T(x_n-R(x_n),A(x_n-R(x_n))-N(Tx_n,Ax_n)\|^2\}\\&\leq\|x_n-\bar{x}\|^2-\{2\gamma(1-r\delta)-\gamma^2(1+r^2\delta^2)\}\|R(x_n)\|^2\\&\leq\|x_n-\bar{x}\|^2,\end{aligned}\tag{4.6}$$

where we have used the Lipschitz continuity of $N$ and $r\delta<1$ and $\gamma\in(0,2)$.

Inequality (4.6) gives the Fejers monotonicity of the sequence $x_n$ with respect to the solution set of MEP(2.1) and hence $x_n$ is bounded. Further, we observe that the sequence $\|x_n-\bar{x}\|^2$ is monotonically decreasing and therefore convergent. This completes the proof.

We remark that the iterative methods presented in this paper improve and extend the iterative methods given in [11] for variational inequality problem (2.4) in finite dimensional space and given in [9] for problem (2.3).

**Acknowledgment.** Second author is thankful to NBHM, Department of Atomic Energy, India, Grant no. NBHM/PDF.2/2013/291 for supporting this research work.

## References

1. A.S. Antipin, Iterative gradient predictor type methods for computing fixed point of external mappings, In: J. Guddat, H.Th. Jonden, F. Nizicka, G. Still, F. Twitt (Eds.) Parametric Optimization and related topics IV [C], Peter Lang, Frankfurt Main, 1997, pp. 11-24.
2. E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Stud.* **63** (1994) 123-145.
3. S.S. Chang, Variational Inequalities and Complementarity Problems: Theory and Applications, Shanghai Scientific and Technical Press, Shanghai, 1991.
4. G. Cohen, Auxiliary problem principle extended to variational inequalities, *J. Optim. Theory Appl.* **59**(1988) 325-333.
5. X.P. Ding, Auxiliary principle and algorithm for mixed equilibrium problems and bilevel mixed equilibrium problems in Banach Spaces, *J. Optim. Theory Appl.* **146** (2010) 347-357.
6. K.R. Kazmi, S.H. Rizvi, Iterative algorithms for generalized mixed equilibrium problems, *J. Egypt. Math. Soc.* **21** (2013) 340-345
7. G.M. Korpelevich, An extregradient methods for finding saddle points and other problems, *Ekon. Mat. Metody* **12** (1976) 747-756.
8. P.-L. Lions, B. Mercier, Splitting algorithms for the sum of two nonlinear operators, *SIAM J. Num. Anal.* **16**(6) (1979) 964-979.
9. A. Moudafi, Mixed equilibrium problems: sensitivity analysis and algorithmic aspect, *Comput. Math. Appl.* **44** (2002) 1099-1108.
10. A. Moudafi, M. The´ra, Proximal and dynamical approaches to equilibrium problems, *Lecture Notes in Economics and Mathematical Systems*, *Vol. 477*, Springer-Verlag, New York, 1999.
11. M.A. Noor, Iterative schemes for quasimonotone mixed variational inequalities, *Optimization* **50** (2001) 29-44.
12. M.A. Noor, General algorithm for variational inequalities, *J. Optim. Theory Appl.* **73(2)**(1992) 409-412.
13. M.A. Noor, Auxiliary principle technique for equilibrium problems, *J. Optim. Theory Appl.* **122** (2004) 371-386.
14. J.W. Peng, J.C. Yao, A new hybrid-extragradient method for generalized mixed equilibrium problems: fixed point problems and variational inequality problems, *Taiwanese J. Math.* **12(6)** (2008) 1401-1432.
15. F. Suhel, S. K. Srivastava and S. A. Khan, A Wiener-Hopf dynamical system for mixed equilibrium problems, *Int. J. Math. and Mathemat. Sci.* Volume 2014, Article ID 102578, 8 pages, 2014.