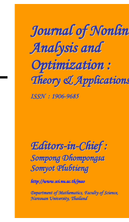

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AN EXISTENCE THEOREM FOR A CLASS OF DIFFERENTIAL INCLUSIONS WITH A PRECISE ESTIMATE ON THE GRADIENT OF A SOLUTION

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ABSTRACT. In this paper, by using nonsmooth version of very recently theorem of Ricceri relating to continuously functionals, we get a new class of nonlinearities for which the Dirichlet problem has a solution, with a precise estimate on the gradient.

KEYWORDS : locally Lipschitz functions; differential inclusions; uniqueness; local minimum.
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1. INTRODUCTION

Ricceri in [9] established a new result for sequentially weakly lower semicontinuous functionals and in [10, Theorem 1.1], applied this result for a class of continuous functions with a certain assumption and proved that problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = h(x)f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $h \in L^\infty(\Omega)$ and f is continuous function, has a weak solution and gradient of solution is lower than an estimate depends on value of a positive constant r.

In the present paper, we apply this result of Ricceri [9, Theorem 1], for a class of discontinuous functions, which obtains a new existence theorem for Dirichlet problems involving this type of functions. It is essential, in view of this study, that our nonlinearity is an upper-semicontinuous multifunction with compact convex

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values. Our main theorem (Theorem 3.2) has a few assumptions on nonlinearity for which the existence of a solution is obtained with a precise estimate on the gradient of solution.

The existence of solution is proved by using variational method, following the ideas of Chang [1] relating to partial differential inclusions. we will prove that our problem admits a nonzero solution under some technical assumptions on nonlinearity.

Partial differential inclusions involving a multifunction have been studied by some authors using nonsmooth critical point theory introduced by Clark [2]. Among others, we refer the reader to [3], [4], [7] and [13].

2. PRELIMINARIES AND NOTATIONS

Now, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and also $p > 1$. On the Soboleve space $W_0^{1,p}(\Omega)$, consider the norm

$$||u|| = (\int_\Omega |\nabla u(x)|^p dx)^{\frac{1}{p}}.$$

Our assumptions on the multifunction F defined on $\mathbb{R}$ are the following:

(F_1) $F : \mathbb{R} \to 2^{\mathbb{R}}$ is upper semicontinuous with compact convex values;

(F_2) $|\xi| \leqslant e(1 + |s|^q)$ for all $s \in \mathbb{R}, \xi \in F(s)$ $(e > 0)$ such that $0 < q \leq \dfrac{pn}{n-p}$ for $n > p$ and $0 < q < +\infty$ for $n \leq p$.

Suppose that $\gamma \in L^\infty(\Omega)_+ \backslash \{0\}$ and F satisfies (F_1),(F_2), consider the following Dirichlet problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) \in \gamma(x)F(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

Our aim is to prove the existence of a non-zero solution for problem (2.1). In follow-up, this section is devoted to the statement some lemmas and results of nonsmooth analysis.

Let X be a Banach space whose dual is denoted by X^*. We recall that the generalized directional derivative $\Phi^\circ(u; v)$ of a locally Lipschitz function $\Phi : X \to \mathbb{R}$ at a point $u \in X$ and in the direction $v \in X$ is defined by

$$\Phi^\circ(u; v) = \limsup_{\substack{w \to u \\ \tau \to 0^+}} \frac{\Phi(w + \tau v) - \Phi(w)}{\tau}.$$

The set $\partial\Phi(u) := \{u^* \in X^* :< u^*, v >\leq \Phi^\circ(u; v) \ for\ all\ v \in X\}$ denotes the generalized gradient of the function Φ.

Lemma 2.1. *([6, Proposition 1.1]). Let $\Phi \in C^1(X)$ be a functional. Then Φ is locally Lipschitz and*

(1) $\Phi^\circ(u; v) =< \Phi'(u), v >$ *for all* $u, v \in X$;

(2) $\partial\Phi(u) = \{\Phi'(u)\}$ *for all* $u \in X$.

Lemma 2.2. *([2, Proposition 2.2.9]). Let Φ be Lipschitz near each point of an open convex subset U of X. Then Φ is convex on U if and only if the multifunction $\partial\Phi(u)$ is monotone on U, that is, if and only if*

$$< u_1^* - u_2^*, u_1 - u_2 >\geqslant 0 \quad \forall u_i \in U,\ \forall u_i^* \in \partial\Phi(u_i)\ (i = 1, 2).$$

Lemma 2.3. *([6, Proposition 1.6]). Let $\Phi, \Psi : X \to \mathbb{R}$ be a locally Lipschitz functionals. Then*

(1) $\partial(\lambda\Phi)(u) = \lambda\partial\Phi(u)$ *for all* $u \in X, \lambda \in \mathbb{R}$;

(2) $\partial(\Phi + \Psi)(u) \subseteq \partial\Phi(u) + \partial\Psi(u)$ *for all* $u \in X$.

The following lemma helps us to relate locally Lipschitz functions to lower semicontinuous functions.

Lemma 2.4. *(*[5, Lemma 6]*). Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional with compact gradient. Then f is sequentially weakly continuous.*

Proposition 2.5. *(*[6, Corollary 1.1]*). If $u \in U$ is a local minimum or maximum of the locally Lipschitz function $f : U \to \mathbb{R}$ on an open set a Banach space X, then $0 \in \partial f(u)$.*
If, in addition, f is convex, then the above condition is also sufficient for u to be a global minimum.

The next theorem has proved by Ricceri[11], recall a consequence of the variational principle, and is a technical tool which obtains the estimate on solution.

Theorem 2.1. *Let X be a reflexive real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous functionals, with Ψ also coercive and $\Phi(0) = \Psi(0) = 0$.*

Then, for each $\sigma > \inf_X \Psi$ and each λ satisfying

$$\lambda > -\frac{\inf\limits_{\Psi^{-1}(]-\infty,\sigma])} \Phi}{\sigma},$$

the restriction of $\lambda\Psi + \Phi$ to $\Psi^{-1}(]-\infty, \sigma[)$ has a global minimum.

3. MAIN RESULTS

In the first of this section, we collect some basic notations that used in our main result and some especially results about our nonlinearity in form of some lemmas.

For each $\lambda \in [0, +\infty]$, we denote by M_λ the set of all global minima of $\lambda\psi - \varphi$ or the empty set according to whether $\lambda < +\infty$ or $\lambda = +\infty$. We adopt the conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$.

Moreover, for a, b that are two fixed number in $[0, +\infty]$, with $a < b$, we put

$$\alpha := \max\{\inf_X \psi, \sup_{M_b} \psi\}$$

and

$$\beta := \min\{\sup_X \psi, \inf_{M_a} \psi\}.$$

From [7], by standard results of setvalued analysis, for a F satisfies (F_1) and (F_2), the mapping $\min F : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous and $\max F : \mathbb{R} \to \mathbb{R}$ is upper semicontinuous.
Put

$$f(s) = \begin{cases} \max F(s) & \text{if } s < 0 \\ \min F(s) & \text{if } s \geqslant 0 \end{cases}$$

then $f : \mathbb{R} \to \mathbb{R}$ is a measurable selection of F. Moreover, f is lower semicontinuous on $[0, +\infty[$ and upper semicontinuous on $]-\infty, 0[$, so it is measurable in $\mathbb{R}$.

Now, we set

$$H(s) = \int_0^s f(t)dt \;\; \text{for all } s \in \mathbb{R},$$

that the convexity of $F(s)$ (see (F_1)) implies the convexity of $H(s)$ for every $x \in \mathbb{R}$.
Finally, set

$$J(u) := \int_\Omega \gamma(x)H(u)dx \;\; \text{for all } u \in L^p(\Omega).$$

The growth condition (F_2) implies that J is well defined on $L^p(\Omega)$, because for all $u \in L^p(\Omega)$ we have

$$\int_\Omega \gamma(x)\Big|\int_0^u f(s)ds\Big|dx \leq ||\gamma||_\infty \int_\Omega e(|u| + \frac{|u|^p}{p})dx \leq c||u||_p^p \ (c > 0).$$

Lemma 3.1. *([4, Lemma 3.2]). The functional $J : L^p(\Omega) \to \mathbb{R}$ is Lipschitz on any bounded subset of L^p. Moreover, for all $u \in L^p$ and $u^* \in \partial J(u)$, we have $u^*(x) \in \gamma(x)F(u(x))$ for a.a. $x \in \Omega$.*

Now, for the Banach space X defined before, we have the following lemma.

Lemma 3.2. *([4, Lemma 3.3]). The functional $J : X \to \mathbb{R}$ is locally Lipschitz and its gradient $\partial J : X \to 2^{X^*}$ is compact.*

Proof. Since the space $X = W_0^{1,p}(\Omega)$ is compactly embedded into $L^p(\Omega)$, so proof is similar to the proof of [4, Lemma 3.3] and we do not repeat it. □

Ricceri in [9, Theorem 1], proved the next theorem in a measurable space where φ and ψ were sequentially weakly lower semicontinuous. Here, by applying Lemma 2.4, we denote this theorem for locally Lipschitz functions.

Theorem 3.1. *Let X be a reflexive real Banach space, and $\psi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous functional and $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional with compact gradient such that $\sup_X \psi > 0$ and*

$$\inf_{x\in X} \frac{\psi(x)}{1 + ||x||^p} > -\infty,$$

for some $p > 0$. Moreover, assume that the functional $\lambda\psi - \varphi$ is coercive and has a unique global minimum for each $\lambda \in]a, b[$. Suppose also that $\alpha < \beta$.

Then, for each $\gamma \in L^\infty(\Omega)_+\backslash\{0\}$, and for each $r \in]\alpha, \beta[$ if we put

$$V_{\gamma,r} := \{u \in L^p(\Omega) : \int_\Omega \gamma(x)\psi(u(x))dx \leqslant r \int_\Omega \gamma(x)dx\},$$

we have

$$\sup_{u\in V_{\gamma,r}} \int_\Omega \gamma(x)\varphi(u(x))dx \leqslant \sup_{\psi^{-1}(r)} \varphi \int_\Omega \gamma(x)dx. \tag{3.1}$$

Proof. First of all by Lemma 2.4, functional φ is sequentially weakly continuous, and the rest of proof is noting else than a very particular case of [9, Theorem 1]. □

The following lemma is a particular case of [12, Theorem 1].

Lemma 3.3. *Let $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ be two functions such that, for each $\lambda \in]a, b[$, the function $\lambda\psi - \varphi$ is lower semicontinuous, coercive and has a unique global minimum in $\mathbb{R}$. Assume that $\alpha < \beta$, then, for each $r \in]\alpha, \beta[$, there exists $\lambda_r \in]a, b[$, such that the unique global minimum of the function $\lambda_r\psi - \varphi$ lies in $\psi^{-1}(r)$.*

Definition 3.4. A function $u \in X$ is a weak solution of problem (2.1) if there exists $u^* \in L^q(\Omega)$ ($\frac{1}{p} + \frac{1}{q} = 1$) such that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x) - u^* v dx = 0 \quad \text{for all } v \in X,$$

such that $u^*(x) \in \gamma(x)F(u(x))$ for a.a. $x \in \Omega$.

Moreover, if $\lambda_{1,p}$ denotes the principal eigenvalue of the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

we obtain

$$\lambda_{1,p} = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u(x)|^p dx}{\int_\Omega |u(x)|^p dx}.$$

We now are ready to state our main theorem.

Theorem 3.2. *Let $\gamma \in L^\infty(\Omega)_+ \setminus \{0\}$ and $(F_1), (F_2)$ hold. Furthermore, we assume*

(i) $H(s) \leqslant m(1 + |s|^l)$ *for* $s \in \mathbb{R}, 1 < l < p,\ m > 0$;

(ii) $\liminf_{s \to 0^+} \frac{H(s)}{s^p} > \frac{\lambda_{1,p}}{p \operatorname{ess\,inf} \gamma}$, *where* $\operatorname{ess\,inf} \gamma(x) > 0$;

(iii) *for all* $\lambda > 0$, *function* $s \to \lambda |s|^p - H(s)$ *has a unique global minimum in* $\mathbb{R}$;

(iv) *there is* $r > 0$ *satisfying* $\alpha < r < \beta$ *such that*

$$\sup_{|s|^p < r} H(s) < r\,(\frac{\lambda_{1,p}}{p \operatorname{ess\,sup} \gamma}), \tag{3.2}$$

then, the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) \in \gamma(x) F(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.3}$$

has a non-zero weak solution satisfying

$$\int_\Omega |\nabla u(x)|^p dx < r\,(\frac{\lambda_{1,p} \int_\Omega \gamma(x) dx}{\operatorname{ess\,sup} \gamma}).$$

Proof. We are going to apply Lemma 3.3 by taking $\varphi(s) = H(s)$, $\psi(s) = |s|^p$. According to definition of $H(s)$, φ is lower semicontinuous and therefore, $\lambda\psi - \varphi$ is lower semicontinuous. Also, by hypothesis (F_2), f is bounded on any bounded subset of $\mathbb{R}$, hence H is Lipshitz on any such set with constant $L > 0$, in particular, H is a locally Lipschitz. Set $a := 0, b := +\infty$. So, let $\lambda \in]a, b[$ and from (i) one can conclude that

$$\lambda |s|^p - H(s) \geqslant \lambda |s|^p - m(1 + |s|^l),$$

since $1 < l < p$, it follows that

$$\lim_{|s| \to +\infty} (\lambda |s|^p - H(s)) = +\infty,$$

this means that $\lambda\psi - \varphi$ is coercive.

Also by (iii) for all $\lambda > 0$ the function $\lambda\psi - \varphi$ has a unique global minimum. Now, we are allowed to apply Lemma 3.3, note that $\alpha = 0$ and $\beta = \inf_{s \in A} |s|^p$, where $A = \{s \in \mathbb{R} : 0 \in -\partial H(s)\}$. Set $r := \frac{1}{p}\beta$, thus, if we put

$$V_{\gamma,r} := \{u \in L^p(\Omega) : \int_\Omega \gamma(x)\psi(u(x))dx \leqslant \frac{\beta}{p} \int_\Omega \gamma(x)dx\},$$

Theorem 3.1, ensures that

$$\sup_{u \in V_{\gamma,r}} \int_\Omega \gamma(x) H(u(x)) dx \leqslant \sup_{|s|^p < r} H \int_\Omega \gamma(x) dx. \tag{3.4}$$

Also, by definition of function ψ, it follows that

$$\int_\Omega \gamma(x)\psi(u(x))dx \leqslant \operatorname{ess\,sup}\gamma \int_\Omega |u(x)|^p dx, \tag{3.5}$$

and, according to the sobolev embedding theorem, one can conclude that

$$\{u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u(x)|^p dx \leqslant \frac{\beta}{p}\Big(\frac{\lambda_{1,p}\int_\Omega \gamma(x)dx}{\operatorname{ess\,sup}\gamma}\Big)\}$$

$$\subseteq \{u \in L^p(\Omega) : \int_\Omega |u(x)|^p dx \leqslant \frac{\beta}{p}\Big(\frac{\int_\Omega \gamma(x)dx}{\operatorname{ess\,sup}\gamma}\Big)\}. \tag{3.6}$$

By setting $B := \{u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u(x)|^p dx \leqslant \frac{\beta}{p}\Big(\frac{\lambda_{1,p}\int_\Omega \gamma(x)dx}{\operatorname{ess\,sup}\gamma}\Big)\}$, and due to (3.5), (3.6) and definition of ψ, one can get for $u \in B$,

$$\begin{aligned}\int_\Omega \gamma(x)\psi(u(x))dx &\leqslant \operatorname{ess\,sup}\gamma \int_\Omega |u(x)|^p dx \\ &\leqslant \operatorname{ess\,sup}\gamma \frac{\beta\int_\Omega \gamma(x)dx}{p\operatorname{ess\,sup}\gamma} \\ &= \frac{\beta}{p}\int_\Omega \gamma(x)dx,\end{aligned}$$

hence $B \subseteq V_{\gamma,r}$. Consequently

$$\sup_{u\in B}\int_\Omega \gamma(x)H(u(x))dx \leqslant \sup_{u\in V_{\gamma,r}}\int_\Omega \gamma(x)H(u(x))dx. \tag{3.7}$$

Accordingly, if put $\sigma = \frac{\beta}{p}\Big(\frac{\lambda_{1,p}\int_\Omega \gamma(x)dx}{\operatorname{ess\,sup}\gamma}\Big)$ in view of (3.2), (3.4) and (3.7) admits

$$\begin{aligned}\sup_{u\in B}\int_\Omega \gamma(x)H(u(x))dx &\leqslant \sup_{|s|^p<r} H\int_\Omega \gamma(x)dx \\ &\leqslant \frac{\beta\lambda_{1,p}}{p^2\operatorname{ess\,sup}\gamma}\int_\Omega \gamma(x)dx \\ &= \frac{1}{p}\sigma.\end{aligned}$$

At this point, by applying Theorem 2.1 and taking $X = W_0^{1,p}(\Omega)$, $\Psi(u) = \int_\Omega |\nabla u(x)|^p dx$ and $\Phi(u) = -J(u)$, problem has a local minimum u which is weak solution for problem (3.3) such that $\int_\Omega |\nabla u(x)|^p dx < \frac{\beta}{p}\Big(\frac{\lambda_{1,p}\int_\Omega \gamma(x)dx}{\operatorname{ess\,sup}\gamma}\Big)$.

We finally remark that 0 is not a local minimum of the energy functional. Indeed, by a classical result, there is a bounded and positive $v \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla v(x)|^p dx = \lambda_{1,p} \int_\Omega |v(x)|^p dx. \tag{3.8}$$

On the other hand, the assumption (ii) implies that there exists an element $k > 0$ such that for every $s \in]0, k[$, it follows that

$$H(s) > \frac{\lambda_{1,p}\ s^p}{p \text{ ess inf } \gamma}. \tag{3.9}$$

We deduce that for each $\eta \in]0, \dfrac{k}{\sup_\Omega v}[$ and (3.8) and (3.9),

$$\begin{aligned} I(\eta v(x)) &= \frac{1}{p}\int_\Omega \Big(|\nabla \eta v(x)|^p dx - \int_\Omega \gamma(x) H(\eta v(x))\Big) dx \\ &< \frac{1}{p}\int_\Omega \Big(|\nabla \eta v(x)|^p - \text{ess inf } \gamma(x) \frac{\lambda_{1,p}\ (\eta v(x))^p}{p \text{ ess inf } \gamma}\Big) dx \\ &= \frac{1}{p}\int_\Omega \Big(\lambda_{1,p}|\eta v(x)|^p - \text{ess inf } \gamma(x) \frac{\lambda_{1,p}\ (\eta v(x))^p}{p \text{ ess inf } \gamma}\Big) dx. \end{aligned}$$

We then get $I(\eta v(x)) < 0$. This implies that the energy functional takes negative values in each ball of $W_0^{1,p}$ centered at 0, and so 0 is not a local minimum for it, and the proof is complete. □

Corollary 3.5. *Let* $\Omega =]0, 1[, \gamma = \dfrac{1}{4}$, *by applying Theorem 3.2, the only positive solution of the problem*

$$\begin{cases} -u'' \in \gamma(x) F(u) & \text{in }]0, 1[\\ u(0) = u(1) = 0 \end{cases} \tag{3.10}$$

that for each $s \in \mathbb{R}$ *when* $u(x) = s$, *function*

$$F(s) = \begin{cases} \{1\} & s < 2 \\ [1, \frac{3}{2}] & s = 2 \\ \{2s - 3\} & s > 2 \end{cases}$$

satisfies the inequality

$$\int_\Omega |u'(x)|^2 dx \leqslant r\Big(\frac{\lambda_{1,2} \int_\Omega \gamma(x) dx}{\text{ess sup } \gamma}\Big). \tag{3.11}$$

In fact, for $p = 2$ *clearly the assumptions* $(F_1), (F_2)$ *and (i) and (iii) in Theorem 3.2 are verified. On the other hand,* $\liminf_{s \to 0^+} \dfrac{H(s)}{s^2} \geqslant \dfrac{\lambda_{1,2}}{2 \text{ ess inf } \gamma}$.

Also, for $\beta = \inf_{s = \frac{3}{2}} |s|^2 = \dfrac{9}{4}$, *suppose that there is* $0 < r < \dfrac{9}{4}$ *such that*

$$\sup_{|s|^2 < r} H(s) < r\Big(\frac{\lambda_{1,2}}{2 \text{ ess sup } \gamma}\Big),$$

this implies that the only weak solution of problem (3.10) satisfies (3.11).

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