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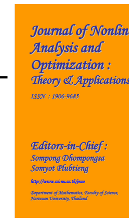

HARMONIC UNIVALENT FUNCTIONS WITH VARYING ARGUMENTS DEFINED BY USING SALAGEAN INTEGRAL OPERATOR

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ABSTRACT. In this paper we define and investigate a new class of harmonic functions defined by using Salagean integral operator with varying arguments. We obtain coefficient inequalities, extreme points and distortion bounds.

KEYWORDS : Harmonic functions; Analytic functions; Sense preserving; Salagean integral operator.
AMS Subject Classification: 30C45.

1. INTRODUCTION

A continuous complex-valued function $f = u + iv$ which is defined in a simply-connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply-connected domain we can write

$$f = h + \overline{g}, \tag{1.1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h^{'}(z)| > |g^{'}(z)|$, $z \in D$ (see [3]).
Denote by S_H the class of functions f of the form (1.1) that are harmonic univalent and sense-preserving in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = h(0) = f_z^{'}(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad , \quad g(z) = \sum_{k=1}^{\infty} b_k z^k \quad , |b_1| < 1. \tag{1.2}$$

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In 1984 Clunie and Shell-Small [3] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses.
Salagean integral operator I^n is defined as follows (see [6]):

(i) $I^0 f(z) = f(z)$;
(ii) $I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}dt$;
....

(iii) $I^n f(z) = I(I^{n-1}f(z))$ $(n \in \mathbb{N} = \{1, 2, 3, ...\})$.

In [4], Cotirla defined Salagean integral operator for harmonic univalent functions $f(z)$ such that $h(z)$ and $g(z)$ are given by (1.2) as follows:

$$I^n f(z) = I^n h(z) + (-1)^n \overline{I^n g(z)}, \tag{1.3}$$

where

$$I^n h(z) = z + \sum_{k=2}^{\infty} k^{-n} a_k z^k \text{ and } I^n g(z) = \sum_{k=1}^{\infty} k^{-n} b_k z^k.$$

With the help of the modified Salagean integral operator we let $E_H(m, n; \gamma, \rho)$ be the family of harmonic functions $f = h + \overline{g}$, which satisfy the condition

$$Re\left\{\left(1 + \rho e^{i\alpha}\right)\frac{I^n f(z)}{I^m f(z)} - \rho e^{i\alpha}\right\} \geq \gamma \tag{1.4}$$

$$(\alpha \in \mathbb{R}, 0 \leq \gamma < 1, \rho \geq 0, m \in \mathbb{N}, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, m > n, \text{ and } z \in U),$$

where $I^n f$ is defined by (1.3), we note that:
Taking $\alpha = 0$, $E_H(n+1, n; 2\beta - 1, 1) = H(n, \beta)$ $(0 \leq \beta < 1)$ (see Cotirla [4]).
Also we note that, by the special choices of α, γ, ρ, m and n, we obtain:

(i) Taking $\alpha = 0$, then $E_H(m, n, 2\beta - 1, 1) = H(m, n; \beta) = \Big\{ f \in S_H :$

$$Re\left\{\frac{I^n f(z)}{I^m f(z)}\right\} > \beta\,(0 \leq \beta < 1; m \in \mathbb{N}; n \in \mathbb{N}_0; m > n; z \in U)\Big\};$$

(ii) $E_H(n+1, n; \gamma, \rho) = E_H(n; \gamma, \rho) = \Big\{ f \in S_H : Re\Big\{\left(1 + \rho e^{i\alpha}\right)\frac{I^n f(z)}{I^{n+1} f(z)}$

$$-\rho e^{i\alpha}\Big\} \geq \gamma\,(\alpha \in \mathbb{R}; 0 \leq \gamma < 1; \rho \geq 0; n \in \mathbb{N}_0; z \in U)\Big\};$$

(iii) $E_H(1, 0; \gamma, \rho) = E_H(\gamma, \rho) = \Big\{ f \in S_H : Re\Big\{\left(1+\rho e^{i\alpha}\right)\frac{f(z)}{If(z)} - \rho e^{i\alpha}\Big\} \geq \gamma$

$$(\alpha \in \mathbb{R}; 0 \leq \gamma < 1; \rho \geq 0; z \in U)\Big\}.$$

Also we define the subclass $V_{\overline{H}}(m, n; \gamma, \rho)$ consists of harmonic functions $f_n = h + \overline{g}_n$ in $E_H(m, n; \gamma, \rho)$ such that h and g_n are the form:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad , \quad g_n(z) = \sum_{k=1}^{\infty} b_k z^k, \tag{1.5}$$

and there exist a real number ϕ such that, mod 2π,

$$\arg(a_k) + (k-1)\phi \equiv \pi,\ k \geq 2 \text{ and } \arg(b_k) + (k+1)\phi \equiv (n-1)\pi,\ k \geq 1. \tag{1.6}$$

Also we note that, by the special choices of α, γ, m and n, we obtain:

(i)Taking $\alpha = 0$, $V_{\overline{H}}(n+1, n; 2\beta - 1, 1) = V_{\overline{H}}(n, \beta)$;
(ii) Taking $\alpha = 0$, $V_{\overline{H}}(m, n, 2\beta - 1, 1) = V_{\overline{H}}(m, n; \beta)$;
(iii) $V_{\overline{H}}(n+1, n; \gamma, \rho) = V_{\overline{H}}(n; \gamma, \rho)$;
(iv) $V_{\overline{H}}(1, 0; \gamma, \rho) = V_{\overline{H}}(\gamma, \rho)$.

2. MAIN RESULTS

Unless otherwise mentioned, we assume in the reminder of this paper that, $\alpha \in \mathbb{R}$, $0 \leq \gamma < 1$, $\rho \geq 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$ and $z \in U$. We begin with a sufficient coefficient condition for functions in the class $E_H(m, n; \gamma, \rho)$.

Theorem 1. Let $f = h + \overline{g}$ be such that h and g are given by (1.2). Furthermore,

$$\sum_{k=1}^{\infty} \left[\tfrac{(1+\rho)k^{-n} - (\gamma+\rho)k^{-m}}{1-\gamma} |a_k| + \tfrac{(1+\rho)k^{-n} - (-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma} |b_k| \right] \leq 2, \tag{2.1}$$

where $a_1 = 1$. Then f is sense-preserving, harmonic univalent in U and $f \in E_H(m, n; \gamma, \rho)$.

Proof. If $z_1 \neq z_2$, then by using (2.1), we have

$$\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right|$$

$$\geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| = 1 - \left| \frac{\sum\limits_{k=1}^{\infty} b_k \left(z_1^k - z_2^k \right)}{(z_1 - z_2) + \sum\limits_{k=2}^{\infty} a_k \left(z_1^k - z_2^k \right)} \right|$$

$$\geq 1 - \frac{\sum\limits_{k=1}^{\infty} k |b_k|}{1 - \sum\limits_{k=2}^{\infty} k |a_k|} \geq 1 - \frac{\sum\limits_{k=1}^{\infty} \dfrac{(1+\rho)k^{-n} - (-1)^{m-n} (\gamma+\rho) k^{-m}}{1-\gamma} |b_k|}{1 - \sum\limits_{k=2}^{\infty} \dfrac{(1+\rho)k^{-n} - (\gamma+\rho) k^{-m}}{1-\gamma} |a_k|} \geq 0,$$

which proves univalence. Also f is sense-preserving in U since

$$\begin{aligned} |h'(z)| &\geq 1 - \sum_{k=2}^{\infty} k |a_k| |z|^{k-1} > 1 - \sum_{k=2}^{\infty} \frac{(1+\rho)k^{-n} - (\gamma+\rho) k^{-m}}{1-\gamma} |a_k| \\ &\geq \sum_{k=1}^{\infty} \tfrac{(1+\rho)k^{-n} - (-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma} |b_k| \geq \sum_{k=1}^{\infty} k |b_k| |z|^{k-1} \geq |g'(z)|. \end{aligned}$$

Now we show that $f \in E_H(m, n; \gamma, \rho)$. We only need to show that if (2.1) holds then the condition (1.4) is satisfied, then we want to prove that

$$Re \left\{ \frac{\left(1 + \rho e^{i\alpha}\right) I^n f(z) - \rho e^{i\alpha} I^m f(z)}{I^m f(z)} \right\} = Re \frac{A(z)}{B(z)} \geq \gamma \tag{2.2}$$

Using the fact that $Re\{w\} > \gamma$ if and only if $|1 - \gamma + w| > |1 + \gamma - w|$, it suffices to show that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0, \tag{2.3}$$

where $A(z) = \left(1 + \rho e^{i\alpha}\right) I^n f(z) - \rho e^{i\alpha} I^m f(z)$ and $B(z) = I^m f(z)$. Substituting for $A(z)$ and $B(z)$ in the left side of (2.3) we obtain

$$\begin{aligned}
&\left|\left(1+\rho e^{i\alpha}\right) I^n f(z) - \rho e^{i\alpha} I^m f(z) + (1-\gamma)\, I^m f(z)\right| \\
&- \left|\left(1+\rho e^{i\alpha}\right) I^n f(z) - \rho e^{i\alpha} I^m f(z) - (1+\gamma) I^m f(z)\right|
\end{aligned}$$

$$\begin{aligned}
= &\Bigg|(2-\gamma)\, z + \sum_{k=2}^{\infty}\left[\left(k^{-n} + (1-\gamma)\, k^{-m}\right) + \rho e^{i\alpha}\left(k^{-n} - k^{-m}\right)\right] a_k z^k \\
&- (-1)^m \sum_{k=1}^{\infty}\left[\left((\gamma-1)\, k^{-m} - (-1)^{m-n}\, k^{-n}\right) + \rho e^{i\alpha}\left(k^{-m} - (-1)^{m-n}\, k^{-n}\right)\right] \cdot \\
&\cdot \overline{b_k z^k}\Bigg| - \Bigg|\gamma z - \sum_{k=2}^{\infty}\left[\left(k^{-n} - (1+\gamma)\, k^{-m}\right) + \rho e^{i\alpha}\left(k^{-n} - k^{-m}\right)\right] a_k z^k + (-1)^m \cdot \\
&\sum_{k=1}^{\infty}\left[\left((1+\gamma)\, k^{-m} - (-1)^{m-n}\, k^{-n}\right) + \rho e^{i\alpha}\left(k^{-m} - (-1)^{m-n}\, k^{-n}\right)\right] \overline{b_k z^k}\Bigg|
\end{aligned}$$

$$\begin{aligned}
\geq\ & (2-\gamma)\, |z| - \sum_{k=2}^{\infty}\left[(1+\rho)k^{-n} - (\gamma+\rho-1)k^{-m}\right] |a_k|\, |z|^k \\
& - \sum_{k=1}^{\infty}\left|(1+\rho)k^{-n} - (-1)^{m-n}\,(\gamma+\rho-1)k^{-m}\right| |b_k|\, |z|^k \\
& - \gamma\, |z| - \sum_{k=2}^{\infty}\left[(1+\rho)k^{-n} - (\gamma+\rho+1)\, k^{-m}\right] |a_k|\, |z|^k \\
& - \sum_{k=1}^{\infty}\left|(1+\rho)k^{-n} - (-1)^{m-n}\,(\gamma+\rho+1)\, k^{-m}\right| |b_k|\, |z|^k
\end{aligned}$$

$$= \begin{cases}
2\,(1-\gamma)\, |z| - 2\sum_{k=2}^{\infty}\left[(1+\rho)k^{-n} - (\gamma+\rho)\, k^{-m}\right] |a_k|\, |z|^k \\
\qquad -2\sum_{k=1}^{\infty}\left[(1+\rho)k^{-n} + (\gamma+\rho)\, k^{-m}\right] |b_k|\, |z|^k, & \text{if n-m is odd;} \\
2\,(1-\gamma)\, |z| - 2\sum_{k=2}^{\infty}\left[(1+\rho)k^{-n} - (\gamma+\rho)\, k^{-m}\right] |a_k|\, |z|^k \\
\qquad -2\sum_{k=1}^{\infty}\left[(1+\rho)k^{-n} - (\gamma+\rho)\, k^{-m}\right] |b_k|\, |z|^k, & \text{if n-m is even.}
\end{cases}$$

$$\begin{aligned}
>\ & 2\Bigg\{(1-\gamma) - \Bigg[\sum_{k=2}^{\infty}\left[(1+\rho)k^{-n} - (\gamma+\rho)\, k^{-m}\right] |a_k| \\
& + \sum_{k=1}^{\infty}\left[(1+\rho)k^{-n} - (-1)^{m-n}\,(\gamma+\rho)\, k^{-m}\right] |b_k|\Bigg]\Bigg\} \\
\geq\ & 0, \text{ this by using (2.1).}
\end{aligned}$$

The harmonic univalent functions

$$f(z) = z + \sum_{k=2}^{\infty} \frac{1-\gamma}{(1+\rho)k^{-n} - (\gamma+\rho)k^{-m}} x_k z^k + \sum_{k=1}^{\infty} \frac{1-\gamma}{(1+\rho)k^{-n} - (-1)^{m-n}(\gamma+\rho)k^{-m}} \overline{y_k z^k}, \quad (2.4)$$

where $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, shows that the coefficient bound given by (2.1) is sharp. This completes the proof of Theorem 1.

In the following theorem, it is shown that the condition (2.1) is also necessary for function $f_n = h + g_n$, where h and g_n are of the form (1.5).

Theorem 2. Let $f_n = h + g_n$, where h and g_n are given by (1.5). Then $f_n \in V_{\overline{H}}(m,n;\gamma,\rho)$, if and only if the coefficient condition (2.1) holds.

Proof. Since $V_{\overline{H}}(m,n;\gamma,\rho) \subseteq E_H(m,n;\gamma,\rho)$, we only need to prove the \only if" part of the theorem. For functions $f_n = h + g_n$, where h and g_n are given by (1.5), the inequality (1.4) with $f = f_n$ is equivalent to

$$Re\left\{\frac{(1+\rho e^{i\alpha})[z+\sum\limits_{k=2}^{\infty}k^{-n}a_kz^k+(-1)^n\sum\limits_{k=1}^{\infty}k^{-n}\overline{b}_k\overline{z}^k]}{z+\sum\limits_{k=2}^{\infty}k^{-m}a_kz^k+(-1)^m\sum\limits_{k=1}^{\infty}k^{-m}\overline{b}_k\overline{z}^k}\right\}$$

$$-Re\left\{\frac{(\gamma+\rho e^{i\alpha})[z+\sum\limits_{k=2}^{\infty}k^{-m}a_kz^k+(-1)^m\sum\limits_{k=1}^{\infty}k^{-m}\overline{b}_k\overline{z}^k]}{z+\sum\limits_{k=2}^{\infty}k^{-m}a_kz^k+(-1)^m\sum\limits_{k=1}^{\infty}k^{-m}\overline{b}_k\overline{z}^k}\right\}>0.$$

The above condition holds for all values of $\alpha \in \mathbb{R}$ and $z \in U$. Upon choosing ϕ according (1.6) and substituting $\alpha = 0$ and $z = re^{i\phi}(0 < r < 1)$, we must have

$$\frac{E}{1-\left[\sum\limits_{k=2}^{\infty}k^{-m}|a_k|-(-1)^{m+n-1}\sum\limits_{k=1}^{\infty}k^{-m}|b_k|\right]r^{k-1}}>0, \qquad (2.5)$$

where

$$\begin{aligned} E &= (1-\gamma)-\left(\sum_{k=2}^{\infty}\left[(1+\rho)k^{-n}-(\gamma+\rho)\,k^{-m}\right]|a_k|\right)r^{k-1} \\ &\quad -\left(\sum_{k=1}^{\infty}\left[(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)\,k^{-m}\right]|b_k|\right)r^{k-1}. \end{aligned}$$

If the inequality (2.1) does not hold, then E is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0,1)$ for which the quotient in (2.5) is negative. But this is a contradiction, then the proof of Theorem 2 is completed.

We now obtain the distortion bounds for functions in $V_{\overline{H}}(m,n;\gamma,\rho)$.

Theorem 3. Let $f_n = h + g_n$, where h and g_n are given by (1.5) and $f_n \in V_{\overline{H}}(m,n;\gamma,\rho)$. Then for $|z| = r < 1$, we have

$$|f_n(z)| \le (1+|b_1|)\,r+\left[\tfrac{1-\gamma}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}-\tfrac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}\,|b_1|\right]r^2 \qquad (2.6)$$

and

$$|f_n(z)| \ge (1+|b_1|)\,r-\left[\tfrac{1-\gamma}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}-\tfrac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}\,|b_1|\right]r^2. \qquad (2.7)$$

Proof. We prove the first inequality.

Let $f_n \in V_{\overline{H}}(m,n;\gamma,\rho)$, we have

$$\begin{aligned} |f_n(z)| &\le (1+|b_1|)\,r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)\,r^k \le (1+|b_1|)\,r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)\,r^2 \\ &\le (1+|b_1|)\,r+\tfrac{1-\gamma}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}\sum_{k=2}^{\infty}\tfrac{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}{1-\gamma}(|a_k|+|b_k|)\,r^2 \\ &\le (1+|b_1|)\,r+\tfrac{1-\gamma}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}. \end{aligned}$$

$$\cdot \sum_{k=2}^{\infty}\left[\tfrac{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}{1-\gamma}|a_k|+\tfrac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}|b_k|\right]r^2$$
$$\leq (1+|b_1|)\,r+\tfrac{1-\gamma}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}\left[1-\tfrac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{1-\gamma}|b_1|\right]r^2$$
$$\leq (1+|b_1|)\,r+\left[\tfrac{1-\gamma}{(1+\rho)2^{-n}-2^{-m}(\gamma+\rho)}-\tfrac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{(1+\rho)2^{-n}-2^{-m}(\gamma+\rho)}|b_1|\right]r^2.$$

The proof of the second inequality is similar, thus it is left.
The bounds given in Theorem 3 for functions $f_n = h+\overline{g}_n$ such that h and g_n are given by (1.6) also hold for functions $f = h+\overline{g}$ such that h and g are given by (1.2) if the coefficient condition (2.1) is satisfied.
Using the same technique used earlier by Aghalary [1] we introduce the extreme points of the class $V_{\overline{H}}(m,n;\gamma,\rho)$.

Theorem 4. The closed convex hull of the class $V_{\overline{H}}(m,n;\gamma,\rho)$ (denoted by $clcoV_{\overline{H}}(m,n;\gamma,\rho)$) is

$$\left\{f(z)=z+\sum_{k=2}^{\infty}a_kz^k+\overline{\sum_{k=1}^{\infty}b_kz^k}\in E_H(m,n;\gamma,\rho):\right.$$
$$\left.\sum_{k=1}^{\infty}\left[\tfrac{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}{1-\gamma}|a_k|+\tfrac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}|b_k|\right]\leq 2\right\},$$

where a_1=1. Set λ_k=$\frac{1-\gamma}{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}$ and μ_k=$\frac{1-\gamma}{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}$. For b_1 fixed, $|b_1|\leq\frac{1-\gamma}{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}$, the extreme points of the class $V_{\overline{H}}(m,n;\gamma,\rho)$ are

$$\left\{z+\lambda_kxz^k+\overline{b_1z}\right\}\cup\left\{\overline{z+\mu_kxz^k+b_1z}\right\}, \qquad (2.8)$$

where $k\geq 2$ and $|x| = 1-\frac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{1-\gamma}$.

Proof. Any function $f\in V_{\overline{H}}(m,n;\gamma,\rho)$ may be expressed as

$$f(z)=z+\sum_{k=2}^{\infty}|a_k|\,e^{i\beta_k}z^k+\overline{b_1z}+\overline{\sum_{k=2}^{\infty}|b_k|\,e^{i\delta_k}z^k},$$

where the coefficients satisfy the inequality (2.1). Set

$$h_1(z)=z,\ g_1(z)=b_1z,\ h_k(z)=z+\lambda_ke^{i\beta_k}z^k,\ g_k(z)=b_1z+\mu_ke^{i\delta_k}z^k, k=2,3,\dots .$$

Writing $X_k=\frac{|a_k|}{\lambda_k}$, $Y_k=\frac{|b_k|}{\mu_k}$, $k=2,3,...$ and $X_1=1-\sum\limits_{k=2}^{\infty}X_k$, $Y_1=1-\sum\limits_{k=2}^{\infty}Y_k$, we have

$$f(z)=\sum_{k=1}^{\infty}\left(X_kh_k(z)+\overline{Y_kg_k(z)}\right).$$

In particular, setting

$$f_1(z)=z+\overline{b_1z},$$

and

$$f_k(z)=z+\lambda_kxz^k+\overline{b_1z}+\overline{\mu_kyz^k},$$
$$\left(k\geq 2,|x|+|y|=1-\frac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{1-\gamma}|b_1|\right),$$

we see that extreme points of the class $V_{\overline{H}}(m,n;\gamma,\rho)$ are contained in $\{f_k(z)\}$. To see that $f_1(z)$ is not an extreme point, note that $f_1(z)$ may be written as

$$f_1(z) = \frac{1}{2}\left\{f_1(z)+\lambda\left(1-\frac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}|b_1|\right)z^2\right\}$$

$$+\frac{1}{2}\left\{f_1(z)-\lambda\left(1-\frac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}|b_1|\right)z^2\right\},$$

a convex linear combination of functions in the class $V_{\overline{H}}(m,n;\gamma,\rho)$. Next we will show if both $|x|\neq 0$ and $|y|\neq 0$, then f_k is not an extreme point. Without loss of generality, assume $|x|\geq|y|$. Choose $\epsilon>0$ small enough so that $\epsilon<\frac{|x|}{|y|}$. Set $A=1+\epsilon$ and $B=1-\left|\frac{\epsilon x}{y}\right|$, we then see that both

$$t_1(z)=z+\lambda_k xAz^k+\overline{b_1z+\mu_k yBz^k}$$

and

$$t_2(z)=z+\lambda_k x\left(2-A\right)z^k+\overline{b_1z+\mu_k y\left(2-B\right)z^k},$$

are in the class $V_{\overline{H}}(m,n;\gamma,\rho)$ and note that

$$f_k(z)=\frac{1}{2}\left(t_1(z)+t_2(z)\right).$$

The extremal coefficient bounds shows that functions of the form (2.8) are the extreme points for the class$V_{\overline{H}}(m,n;\gamma,\rho)$, then the proof of Theorem 4 is completed.

Now we will examine the closure properties of the class $V_{\overline{H}}(m,n;\gamma,\rho)$ under the generalized Bernardi-Libera-Livingston integral operator (see [2, 5]) $L_c(f)$ which is defined by

$$L_c(f\left(z\right))=\frac{c+1}{z^c}\int\limits_0^z t^{c-1}f\left(t\right)dt\ \ (c>-1). \tag{2.9}$$

Theorem 5. Let $f_n=h+g_n\in V_{\overline{H}}(m,n;\gamma,\rho)$, where h and g_n are given by (1.5). Then $L_c(f_n(z))$ belongs to the class $V_{\overline{H}}(m,n;\gamma,\rho)$.
Proof. From the representation of $L_c(f_n(z))$, it follows that

$$\begin{aligned} L_c(f_n(z)) &= \frac{c+1}{z^c}\int\limits_0^z t^{c-1}\left(h\left(t\right)+\overline{g}_n(t)\right)dt \\ &= \frac{c+1}{z^c}\int\limits_0^z t^{c-1}\left\{t+\sum_{k=2}^{\infty}a_kt^k+\overline{\sum_{k=1}^{\infty}b_kt^k}\right\}dt \\ &= z+\sum_{k=2}^{\infty}A_kz^k+\overline{\sum_{k=1}^{\infty}B_kz^k}, \end{aligned}$$

where $A_k=\frac{c+1}{c+k}a_k$, $B_k=\frac{c+1}{c+k}b_k$. Therefore, we have,

$$\sum_{k=1}^{\infty}\left[\tfrac{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}{1-\gamma}\frac{c+1}{c+k}|a_k|+\tfrac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}\frac{c+1}{c+k}|b_k|\right]$$

$$\leq\sum_{k=1}^{\infty}\left[\tfrac{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}{1-\gamma}|a_k|+\tfrac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}|b_k|\right]\leq 2,$$

and the proof of Theorem 5 is completed.

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